Flow rate measurement in a heterogeneous fluid with acoustic waves

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Outline

Ultrasonic flowmeters

Context:
Ultrasonic flowmeters are a non-intrusive non-invasive tool for measuring the flow rate of fluids through a pipe.

Problem:
However, water extracted from natural bodies of water contains particles which disturb the propagation of the ultrasonic waves.

Contribution:
To construct a mathematical model for the propagation of ultrasonic waves through a fluid carrying particles.

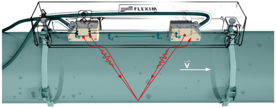
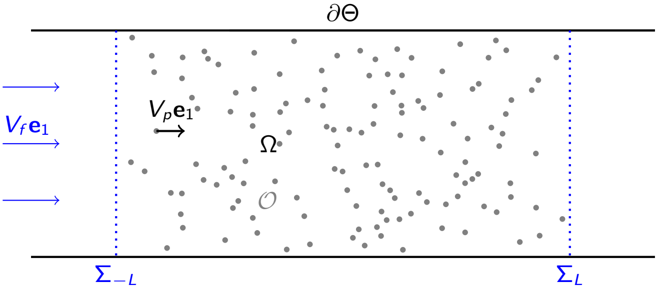


Figure: Ultrasonic flowmeter

Outline

A simplified model: the domain

Mathematical model for understanding the problem.



- Obstacles $\mathcal{O} \subset\subset \{x_1 \in (-L, L)\}$.
- Fluid Ω, truncated domain $\Omega_L = \Omega \cap \{x_1 \in (-L, L)\}$.
- Pipe wall $\partial\Theta$.
- V_f - "free" speed of the fluid.
- All particles move with velocity $V_p\mathbf{e}_1$.

A simplified model: the acoustic field equations

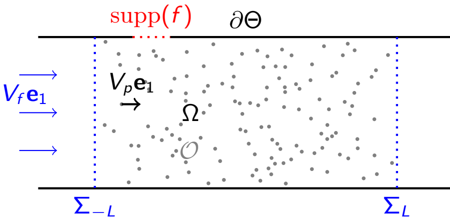
- **Convected Wave equation** where $D_t = \partial_t + \mathbf{V}_f \cdot \nabla$, $\mathbf{V}_f$ fluid velocity: .

$$\rho_0 D_t(c_0^{-2} D_t\phi) - \nabla \cdot (\rho_0 \nabla \phi) = 0$$

- Change inertial frames so that particles are *stationary*.
- Replace $\mathbf{V}_f$ by $\mathbf{V} = \mathbf{V}_f - V_p \mathbf{e}_1$, *relative* fluid velocity.
- Assume $\mathbf{V}$ is *steady*,
- **Convected Helmholtz equation** where $D_\omega = -i\omega + \mathbf{V} \cdot \nabla$:

$$\rho_0 D_\omega(c_0^{-2} D_\omega \varphi) - \nabla \cdot (\rho_0 \nabla \varphi) = 0$$

- Ultrasonic flowmeter source: $\partial_{\mathbf{n}} \varphi = f$.



The relative fluid flow $\mathbf{V}$ depends implicitely on $\mathcal{O}$ and Ω. Two sequential problems:

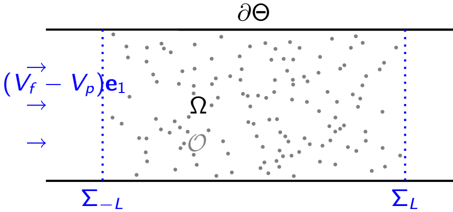
1. Create a realisation $\mathcal{O}$.
2. Solve for $\mathbf{V}$
3. Solve for φ, with D_ω given from $\mathbf{V}$.

The flow model

- Non-dimensionalise the problem, $\mathbf{V} = c_0 \mathbf{M}$.
- Assume that $\mathbf{V}$ is steady irrotational incompressible potential flow.

$$\mathbf{V} = c_0 \mathbf{M} = c_0 \nabla U = c_0 \nabla u^{\mathrm{inc}} + c_0 \nabla u$$

where $u^{\mathrm{inc}}(x) = \tilde{M} x_1 = \frac{V_f - V_p}{c_0} x_1$.



- $\mathcal{O}, \partial\Theta$ are impenetrable.

$$\begin{cases} -\Delta u & = 0, \text{ in } \Omega_L \\ \partial_{\mathbf{n}} u & = -\partial_{\mathbf{n}} u^{\mathrm{inc}}, \text{ on } \partial\mathcal{O} \\ \partial_{\mathbf{n}} u & = 0, \text{ on } \partial\Theta \\ \text{R.C. on} & \Sigma_{\pm L} \end{cases}$$

Mach flow given by $\mathbf{M} = \tilde{M}\mathbf{e}_1 + \nabla u$.

Fluid potential flow

$$\begin{cases} -\Delta u & = 0, \text{ in } \Omega_L \\ \partial_{\mathbf{n}} u & = -\partial_{\mathbf{n}} u^{\mathrm{inc}}, \text{ on } \partial\mathcal{O} \\ \partial_{\mathbf{n}} u & = 0, \text{ on } \partial\Theta \\ \text{R.C. on} & \Sigma_{\pm L} \end{cases} \quad (1)$$

where $u^{\mathrm{inc}}(x) = \tilde{M} x_1$.

- Well-posed in $H^1(\Omega_L)/\mathbb{R}$.
- Solutions are smooth $||\nabla u||_{L^\infty(\Omega)} \leq C(\Omega, \mathcal{O})\tilde{M}$.
- Convected Helmoltz needs $\mathbf{M} = \tilde{M}\mathbf{e}_1 + \nabla u$ subsonic.

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Theorem

Assume that all particles $\{\mathcal{O}_i\}_i$ are smooth and define:

$$r = \min_i r_i, \qquad d = \min \left(\mathrm{dist}(\partial\mathcal{O}, \partial\Theta), \min_{i \neq j} \mathrm{dist}(\mathcal{O}_i, \mathcal{O}_j) \right)$$

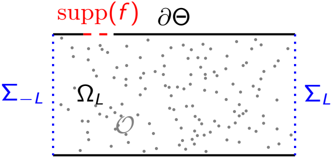
where r_i denotes the radius of the ith particle. Then for $u^{\mathrm{inc}}(x) = \tilde{M} x_1$ there is a unique solution $u \in H^1(\Omega_L)/\mathbb{R}$ for Problem (1) and there exist a constant $C(d/r)$ such that:

$$||\nabla u||_{L^\infty(\Omega_L)} \leq C(d/r)\frac{\tilde{M}}{r}.$$

Acoustic potential problem

Let **M** be an incompressible uniformly subsonic vector field with:

$$\begin{cases} ||\mathbf{M}||_{L^\infty(\Omega_L)} & < 1 \\ \mathbf{M}\cdot\mathbf{n} & = 0, \text{ on } \partial\Theta \\ \mathbf{M}\cdot\mathbf{n} & = 0, \text{ on } \partial\mathcal{O} \end{cases}$$



Let $f \in H^{-1/2}(\partial\Theta)$ have a compact support in $\partial\Theta \cap \overline{\Omega_L}$. Denote by $D_k = -ik + \mathbf{M}\cdot\nabla$. Find $\varphi \in H^1(\Omega_L)$ such that:

$$\begin{cases} -\nabla\cdot(\nabla\varphi - D_k\varphi\mathbf{M}) - ikD_k\varphi = 0 \text{ in } \Omega_L \\ \partial_{\mathbf{n}}\varphi = 0 \text{ on } \partial\mathcal{O} \\ \partial_{\mathbf{n}}\varphi = f \text{ on } \partial\Theta \\ (\nabla\varphi - D_k\varphi\mathbf{M})\cdot\mathbf{n} = -T_\pm\varphi \text{ on } \Sigma_{\pm L} \end{cases} \tag{2}$$

where $T_\pm$ are DtN operators derived from *assuming* uniform flow of Mach Speed $\tilde{M}$ in $\Omega \setminus \Omega_L$.

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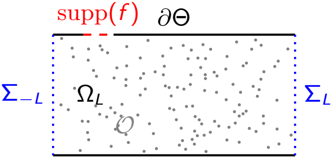

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Theorem

Let $f, \mathbf{M}$ be as above. Then Problem (2) is of Fredholm type and is well-posed except for a countable collection of wavenumbers.

No particle case: $\mathcal{O} = \emptyset$

The problem for the incident wave φ^{inc} is given by setting $D_k = -ik + \tilde{M}\partial_{x_1}$:

$$\begin{cases} (D_k D_k - \Delta)\varphi^{\mathrm{inc}} = 0 \text{ in } \Omega \\ \partial_{\mathbf{n}}\varphi^{\mathrm{inc}} = f \text{ on } \partial\Theta \\ (\nabla\varphi^{\mathrm{inc}} - D_k\varphi\tilde{M}\mathbf{e}_1)\cdot\mathbf{n} = -T_{\pm}\varphi \text{ on } \Sigma_{\pm L} \end{cases}$$

We solve this by a Green's function G.

$$\varphi^{\mathrm{inc}}(x) = \int_{\partial\Theta} G(x,y) f(y)\,\mathrm{d}\sigma(y)$$

$$G(x,y) = \sum_n \frac{c_n(y_2)}{2i\gamma_n} e^{i\gamma_n|x_1-y_1|+i\kappa(x_1-y_1)} c_n(x_2)$$

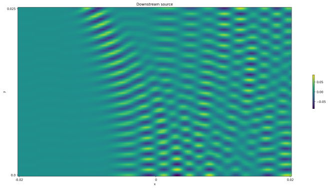


Figure: φ^{inc} in frequency domain.

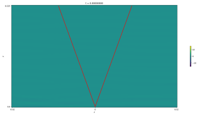
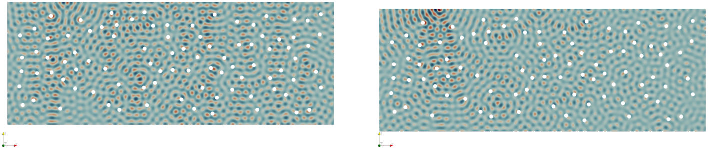


Figure: Time domain.

$$\varphi^{\mathrm{inc}}(x_1,x_2) = \sum_n \frac{c_n(h)}{2i\gamma_n} I_n(x_1) c_n(x_2), \quad I_n(x_1) = \int_{\partial\Theta} e^{i\gamma_n|x_1-y_1|+i\kappa(x_1-y_1)} f(y)\,\mathrm{d}\sigma(y).$$

Finite Element Implementation



(a) $k = 4000$ diffracted field

(b) $k = 4000$ total field

Figure: Frequency domain of problem with particles.

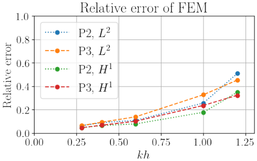


Figure: Convergence of FEM, relative error to $kh = 0.2$

Waveguide is 0.1m × 0.05 m with particle radius $r = 0.5$mm and 3% particle volume density. FreeFem++ with Domain Decomposition and Petsc.

Outline

1 Introduction

2 General Model

3 Homogenisation

4 Perspectives

Periodic Homogenisation: formal expansion

Periodic distribution of particles, $\varepsilon =$ size of cell.

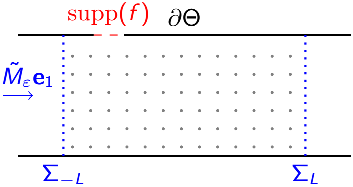


Figure: $\Omega_{L,\varepsilon}$ the perforated domain

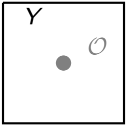
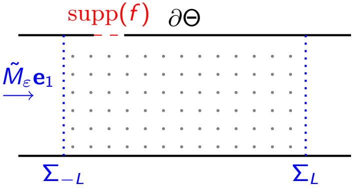


Figure: Y the periodic cell

Periodic Homogenisation: formal expansion

Periodic distribution of particles, ε = size of cell. $V_p(\varepsilon)$ **depends** on ε.



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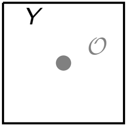

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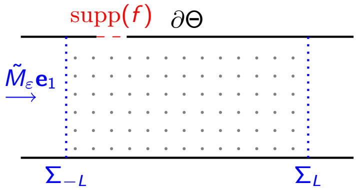

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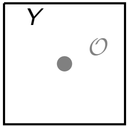

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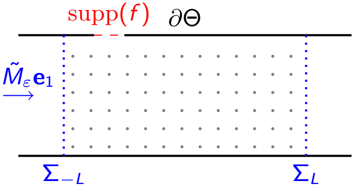

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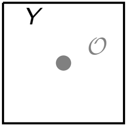

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Write the Convected Helmholtz equation as:

$$-\nabla \cdot \big(\underbrace{(\mathrm{Id} - \mathbf{M}^\varepsilon \otimes \mathbf{M}^\varepsilon)}_{=:\mathbf{A}^\varepsilon} \nabla\varphi^\varepsilon\big) - 2ik\mathbf{M}^\varepsilon \cdot \nabla\varphi^\varepsilon - k^2\varphi^\varepsilon = 0, \quad \text{in } \Omega_{L,\varepsilon}$$

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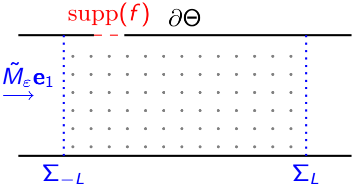

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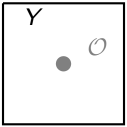

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Periodic distribution of particles, ε = size of cell. $V_p(\varepsilon)$ **depends** on ε.

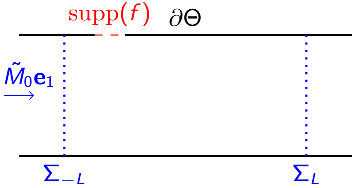
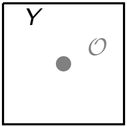


Figure: $\Omega_{L,*}$ the homogenised domain



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Formal expansion gives the following homogenised equation:

$$-\mathrm{div}_x(\mathbf{A}^* \nabla_x \varphi_0) - 2ik\mathbf{M}^* \cdot \nabla_x \varphi_0 - k^2 \rho^* \varphi_0 = 0, \quad \text{in } \Omega_{L,*}$$

Formal calculation of Tensors

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Expand out the various tensors:

$$\mathbf{A}^\varepsilon = \sum_{j\geq 0}\varepsilon^j\mathbf{A}_j\left(x, \frac{x}{\varepsilon}\right) \quad , \mathbf{M}^\varepsilon = \tilde{M}_\varepsilon\mathbf{e}_1 + \nabla u^\varepsilon = \sum_{j\geq 0}\varepsilon^j\mathbf{M}_j\left(x, \frac{x}{\varepsilon}\right).$$

Can show through $\mathbf{A}^\varepsilon = \mathrm{Id} - \mathbf{M}^\varepsilon\otimes\mathbf{M}^\varepsilon$ that:

$$\mathbf{A}_0(x,y) = \mathrm{Id} - \mathbf{M}_0(x,y)\otimes\mathbf{M}_0(x,y)$$

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Corresponding cell problem to find $w_i : \Omega_{L,*} \to H^1_\#(Y)$ with:

$$\begin{cases} \mathrm{div}_y(\mathbf{A}_0(x,y)(\nabla_y w_i(x,y) + \mathbf{e}_i)) & = 0 \quad \text{in } Y \\ \mathbf{A}(x,y)w_i(x,y)\cdot\mathbf{n} & = -\mathbf{n}\cdot\mathbf{e}_i \quad \text{on } \partial\mathcal{O} \end{cases}$$

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$$\mathbf{A}^*_{ij}(x) = \int_Y e_j \cdot \mathbf{A}_0(x,y)(\nabla_y w_i(x,y) + e_i)\,\mathrm{d}y, \quad \mathbf{M}^*(x) = \int_Y \mathbf{M}_0(x,y)\,\mathrm{d}y, \quad \rho^* = \int_Y \mathrm{d}y$$

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Remark

The effective tensors $\mathbf{A}^*,\mathbf{M}^*,\rho^*$ depend only on Y and $\mathbf{M}_0$.

Weak convergence of the flow problem

$$\begin{cases} -\Delta u_\varepsilon & = 0, \quad \text{in } \Omega_{L,\varepsilon} \\ \partial_{\mathbf{n}} u_\varepsilon & = -\tilde{M}_\varepsilon n_1, \quad \text{on } \partial\Theta \cup \partial\mathcal{O}_\varepsilon \\ R.C. & \text{on } \Sigma_{\pm L} \end{cases} \tag{3}$$

Theorem

Let u^ε solve Problem (3) and suppose $\tilde{M}_\varepsilon \to \tilde{M}_0$. Then (up to a subsequence):

$$\mathcal{T}_\varepsilon u^\varepsilon \rightharpoonup u_0, \text{ in } L^2(\Omega_{L,*}, H^1_\#(Y)), \qquad \mathcal{T}_\varepsilon \nabla u^\varepsilon \rightharpoonup \nabla u_0 + \nabla_y u_1, \text{ in } L^2(\Omega_{L,*} \times Y),$$

where $(u_0, u_1) \in V := H^1(\Omega_{L,*})/\mathbb{R} \times L^2(\Omega_{L,*}; H^1_\#(Y)/\mathbb{R})$ solve the following variational problem:

$$a^*_\Delta((u_0, u_1), (\psi_0, \psi_1)) = \tilde{M}_0 F((\psi_0, \psi_1)), \qquad \text{for all } (\psi_0, \psi_1) \in V,$$

where a^*_Δ is symmetric positive definite bilinear form on V, F is a bounded linear form on V and neither depend on $\tilde{M}_0$.

Uniform bounds for flow

Let $(u_0, u_1) \in V := H^1(\Omega_{L,*})/\mathbb{R} \times L^2(\Omega_{L,*}; H^1_\#(Y)/\mathbb{R})$ solve the following variational problem:

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Recall:

$$\mathbf{M}^\varepsilon(x) = \tilde{M}_\varepsilon \mathbf{e}_1 + \nabla u^\varepsilon(x), \qquad \mathbf{M}_0(x, y) = \tilde{M}_0 \mathbf{e}_1 + \nabla u_0(x) + \nabla_y u_1(x, y).$$

Remark

If $\tilde{M}_0 = 0$ *then it follows that* $\mathbf{M}_0 = 0$.

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Corollary

In the case where $\tilde{M}_\varepsilon = \varepsilon \tilde{M}_1 + o(\varepsilon)$ *there exists a* $C(Y)$ *dependent only on the unit perioidic cell* Y *such that:*

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Theorem

Let u^ε *solve Problem* (3). *Suppose that* $\tilde{M}_\varepsilon = \varepsilon \tilde{M}_1 + o(\varepsilon)$. *Then (up to a subsequence):*

$$||\mathbf{M}^\varepsilon||_{L^2(\Omega_{L,\varepsilon})} \to 0$$

Periodic Homogenisation: rigorous framework for convected Helmholtz

$$\begin{cases} -\nabla\cdot(\mathbf{A}^{\varepsilon}\nabla\varphi^{\varepsilon}) - 2ik\mathbf{M}^{\varepsilon}\cdot\nabla\varphi^{\varepsilon} - k^{2}\varphi^{\varepsilon} & = 0, \text{ in } \Omega_{L,\varepsilon} \\ \mathbf{n}\cdot\mathbf{A}^{\varepsilon}\nabla\varphi^{\varepsilon} & = 0, \text{ on } \partial\mathcal{O}^{\varepsilon} \\ \mathbf{n}\cdot\mathbf{A}^{\varepsilon}\nabla\varphi^{\varepsilon} & = f^{\varepsilon}, \text{ on } \partial\Theta \\ \mathbf{n}\cdot\mathbf{A}^{\varepsilon}\nabla\varphi^{\varepsilon} - ik\varphi^{\varepsilon}\mathbf{M}^{\varepsilon}\cdot\mathbf{n} & = -T^{\varepsilon}_{\pm}\varphi^{\varepsilon}, \text{ on } \Sigma_{\pm L} \end{cases} \tag{4}$$

Theorem

Define the homogenised system with $\mathbf{A}^{*}$ *corresponding to* $\mathbf{M}_0 = 0$*:*

$$\begin{cases} -\nabla_x\cdot(\mathbf{A}^{*}\nabla_x\varphi_0) - k^{2}\rho^{*}\varphi_0 & = 0, \textit{ in } \Omega_{L,*} \\ \mathbf{n}\cdot\mathbf{A}^{*}\nabla_x\varphi_0 & = f^{*}, \textit{ on } \partial\Theta \\ \mathbf{n}\cdot\mathbf{A}^{*}\nabla_x\varphi_0 & = -T^{*}_{\pm}\varphi_0, \textit{ on } \Sigma_{\pm L} \end{cases} \tag{5}$$

Take φ^{ε} *solves* (4) *and assume that* k *is such that Problem* (5) *is well-posed. Suppose that* $\tilde{M}_{\varepsilon} = \varepsilon\tilde{M}_1 + o(\varepsilon)$*, then for* $\tilde{M}_1$ *sufficiently small:*

$$\mathcal{T}_{\varepsilon}\varphi^{\varepsilon} \rightharpoonup \varphi_0, \textit{ in } L^{2}(\Omega_{L,*}; H^{1}_{\#}(Y)), \quad \mathcal{T}_{\varepsilon}\nabla\varphi^{\varepsilon} \rightharpoonup \nabla\varphi_0 + \nabla_y\varphi_1, \textit{ in } L^{2}(\Omega_{L,*}\times Y)$$

where φ_0 *solves Problem* (5) *(convergence up to a subsequence).*

Numerical tests

Take $\tilde{M}_\varepsilon = \frac{\varepsilon}{1500}$. Corresponding $(c_0, V_f, V_p) = (1500, 1.0, 1.0 - \varepsilon)$, Test with $\varepsilon = 0.001$, $f^\varepsilon = f$, $\omega = 2\pi \times 10^6$.

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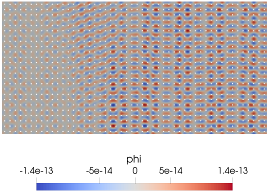

Figure: φ^ε for $\varepsilon = 0.001$, $\omega = 2\pi \times 10^6$

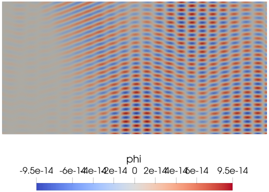

Figure: φ_0, for $\omega = 2\pi \times 10^6$

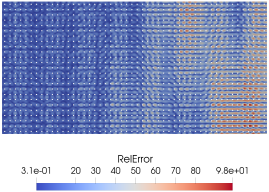

Figure: $\frac{100|\varphi_0 - \varphi^\varepsilon|}{||\varphi^\varepsilon||_{L^2(\Omega_{L,\varepsilon})}}$

Relative L^2 difference $=$92.2044%, Relative H^1 difference $=$ 94.4024% for $\omega = 2\pi \times 10^6$.

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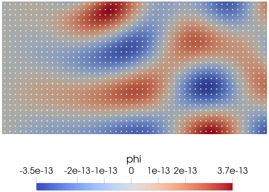

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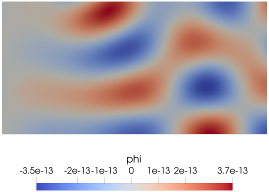


Figure: φ_0, for $\omega = 2\pi \times 10^5$

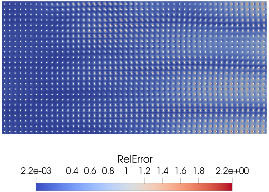


Figure: $\frac{100|\varphi_0 - \varphi^\varepsilon|}{||\varphi^\varepsilon||_{L^2(\Omega_{L,\varepsilon})}}$

Relative L^2 difference $=1.41168\%$, Relative H^1 difference $= 22.1114\%$ for $\omega = 2\pi \times 10^5$.

Numerical tests: Convergence curves

Take $\tilde{M}_\varepsilon = \frac{\varepsilon}{1500}$. Corresponding $(c_0, V_f, V_p) = (1500, 1.0, 1.0 - \varepsilon)$.
$\omega = 2\pi \times 10^6, 2\pi \times 10^5$

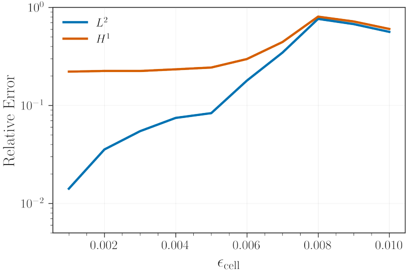
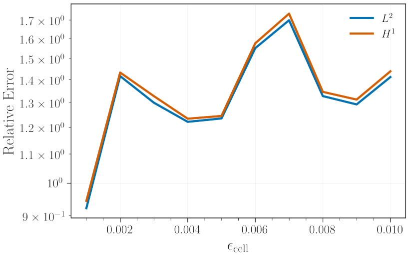


Figure: Relative error of φ_0 to φ^ε for $\omega = 2\pi \times 10^5$.



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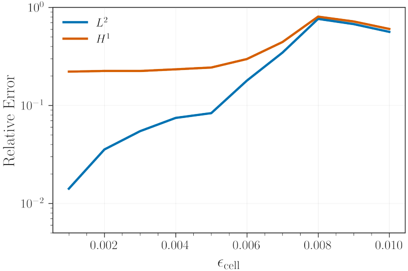

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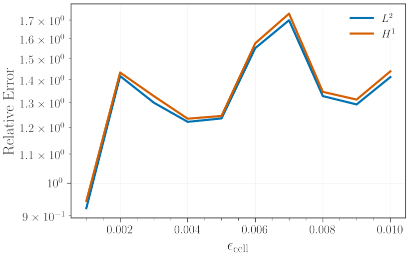

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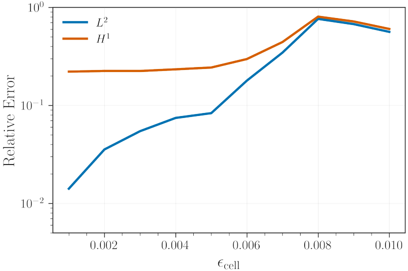

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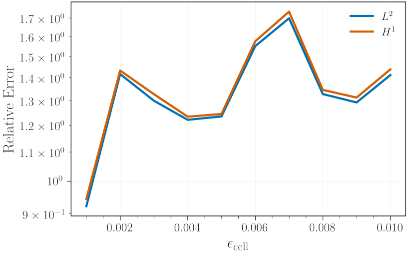

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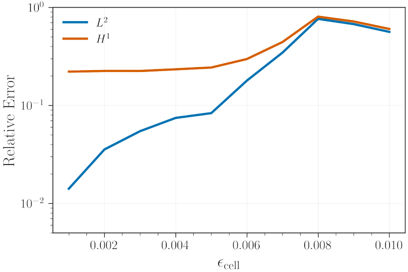

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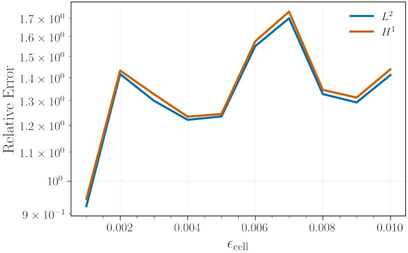

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MHz signals: $\varepsilon = 10^{-4}\mathrm{m}$.
Low density particles: $r = 10^{-5}\mathrm{m}$.

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Proof only valid for $\tilde{M}_\varepsilon = \varepsilon \tilde{M}_1 + o(\varepsilon)$.

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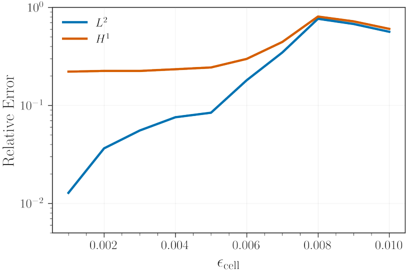

Figure: Convergence of fem solution to homogenised field for 100kHz frequency for $\tilde{M}_\varepsilon = \tilde{M}_0$

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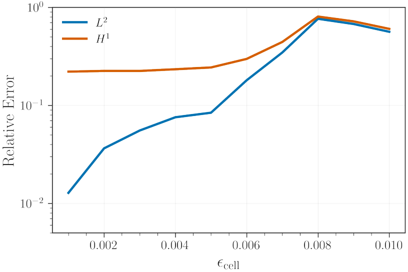

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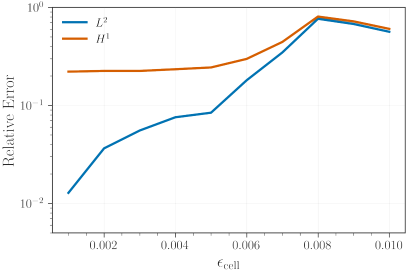

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What does this mean? I am not sure.

Outline

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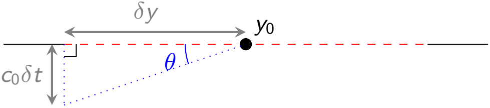
Thank you for your attention.

Modelling the Ultrasonic Flowmeter

Pulse: $F(t) = \sin(\omega_0(t - t_0))e^{-\frac{(t-t_0)^2}{2\sigma^2}}$
In the stationary frame:

$$\partial_{\mathbf{n}}\Phi(y,t) = F(t)g(y)$$



$$\partial_{\mathbf{n}}\Phi(y,t) = F\left(t - \frac{\tan\theta}{c_0}(y - y_0)\right)g(y).$$

In the moving frame, corresponding time domain boudnary condition is:

$$\partial_{\mathbf{n}}\Phi(y,t) = F\left(t - \frac{\tan\theta}{c_0}(y - y_0)\right)g\left(y + V_p t\right).$$

Corresponding frequency domain boundary condition:

$$\partial_{\mathbf{n}}\varphi(y,\omega) = \int e^{i\omega t}F\left(t - \frac{\tan\theta}{c_0}(y - y_0)\right)g\left(y + V_p t\right)\,\mathrm{d}t$$

How can we compute this?

Explicit formula in Gaussian case

Proposition

Let $F(t) = e^{-\frac{(t-t_0)^2}{2\sigma^2}} \sin(\omega_0(t-t_0))$, $g(y) := \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(y-y_0)^2}{2\sigma_g^2}}$ *for* $\sigma_0, \sigma_g, \omega_0 > 0$. *Then for* $V_p > 0$ *and* $\omega \in \mathbb{R}$*:*

$$\int e^{i\omega t} F\left(t - \frac{\tan\theta}{c_0}(y - y_0)\right) g(y+V_p t)\,\mathrm{d}t = e^{i\mu_+(y-y_0)} f_0^+(y) - e^{i\mu_-(y-y_0)} f_0^-(y)$$

Where:

$$f_0^\pm(y) = \frac{\sigma_0}{2V_p i\sqrt{a_2+a_3}} \exp\left(\frac{4i(\mp a_2\omega_0 + a_3\omega)t_0 - a_1(y)^2 - 4a_2a_3(\omega \pm \omega_0)^2}{4(a_2+a_3)}\right)$$

$$\mu_\pm = -\frac{\omega}{V_p} + \frac{a_3\omega \mp a_2\omega_0}{a_2+a_3}\left(\frac{\tan\theta}{c_0} + \frac{1}{V_p}\right)$$

$$a_1(y) := t_0 + \frac{y + y_0\tan\theta}{c_0} + \frac{y - y_0}{V_p}, \quad a_2 = \frac{\sigma_0^2}{2}, \quad a_3 = \frac{\sigma_g^2}{2V_p^2}$$

Numerical Convergence of uniform flow

Source term:

$$f(y,\omega) = e^{i\omega t_0}\left[e^{\frac{-\sigma_0(\omega+\omega_0)^2}{2}} - e^{\frac{-\sigma_0(\omega-\omega_0)^2}{2}}\right] g(y) e^{-i\frac{\omega}{c}\tan\theta(y-y_0)}$$

(h, L)	$(0.025, 0.025)$
Inertial Frame	Pipe
Obstacles	None
k	$4188.79 (= \omega_0 / c_0)$
$(\omega_0, \sigma_0, t_0,)$	$(2\pi \times 10^6, 1 \times 10^{-6}, 0.0)$
$(\sigma_\Theta, \theta_{\text{inc}})$	$(0.002, 20^\circ)$
(c_0, V_f, V_p)	$(1500, 6.0, 0.0)$
$\varepsilon_{\text{mesh}}$	0.00015
FE	P1
$\mathbf{N}_{\text{modes}}$	$\mathbf{33} + (\mathbf{0}, \mathbf{10}, \mathbf{20}, \ldots, \mathbf{140})$
$(N_\Theta, L_{\text{sup}})$	$(500, 0.02 = (10\sigma_\Theta))$

Table: Table of paramters used to investivate modal convergence of the analytical solution

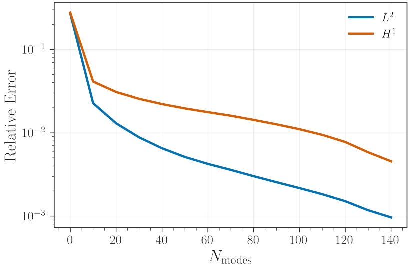


Figure: Convergence of incidence field against $N_{\text{modes}} = 233$