A nonsmooth extension of the Brezzi-Rappaz-Raviart approximation theorem via metric regularity techniques

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The Brezzi-Rappaz-Raviart (BRR for short) theorem [2] can be seen as quantitative version of the inverse function theorem tailored for the numerical analysis of nonlinear PDEs. More precisely, given Banach spaces X and Y, a mapping $F\colon X \to Y$, approximations $F_h\colon X \to Y$ of F, and a solution $\bar{x} \in X$ to $F(\bar{x}) = 0$, we are interested in finding solutions $\bar{x}_h \in X$ to

$$F_h(\bar{x}_h) = 0 \tag{1}$$

and to quantify the error $\|\bar{x} - \bar{x}_h\|_X$. In its simplest form, the BRR theorem roughly states that, if the mappings F and F_h are of class C^1, with $dF[\bar{x}]$ invertible, if

$$\lim_{h\to 0} F_h(\bar{x}) = 0 \tag{2}$$

and if

$$\lim_{h\to 0} \|dF[\bar{x}] - dF_h[\bar{x}]\|_{\mathcal{L}(X,Y)} = 0, \tag{3}$$

then, for all h small enough, there exists $\bar{x}_h \in X$ solving (1) and we have the error estimate

$$\|\bar{x} - \bar{x}_h\|_X \leq 2\|dF[\bar{x}]^{-1}\|_{\mathcal{L}(X,Y)}\|F_h(\bar{x})\|_Y. \tag{4}$$

In other words, under the consistency assumption (2) and the stability assumption (3), there exists a solution to the approximate problems and the error estimate (4) is proportional to the consistency error.

We propose a generalization of this result to the case where the mappings F and F_h are merely Lipschitz continuous. In this case the condition (3) is replaced by a smallness condition on the Lipschitz constant of the difference $F - F_h$. In order to find a substitute for the invertibility of $dF[\bar{x}]$ in this nondifferentiable setting, we make use of the theory of *metrically regular mappings*, a theory that has found significant applications in the context of variational analysis as a replacement for the inverse function theorem. We also discuss applications of our result to finite element approximations of solutions to Hamilton-Jacobi equations and mean field games. Indeed, in this context, the nonlinearities are typically not smooth enough to ensure that F and F_h are of class C^1.

[1] J. Berry, O. Ley, F. J. Silva. *A nonsmooth extension of the Brezzi-Rappaz-Raviart approximation theorem via metric regularity techniques and applications to nonlinear PDEs.* Preprint, arXiv :2507.05774 [math.NA] (2025), 2025.

[2] F. Brezzi, J. Rappaz, P. A. Raviart. *Finite dimensional approximation of nonlinear problems. I : Branches of nonsingular solutions.* Numer. Math., **36**, 1–25, 1980. doi :10.1007/BF01395985.

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