Space-time hybrid methods for collisional kinetic equations

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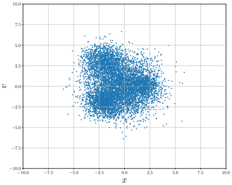
System of particles in interactions

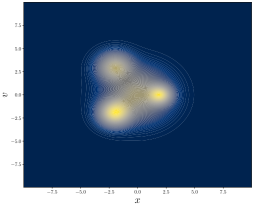
→ **Multiscale systems**

Microscopic



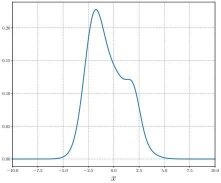
- Exact description
- Too costly simulations

Mesoscopic



- Microscopic effects
- Heavy simulations

Macroscopic



- Cheaper simulations
- Loss of microscopic effects

The Vlasov-BGK equation

$f(t,x,v)$ distribution function, $t > 0$, $x \in \Omega_x \subset \mathbb{R}^{d_x}$ and $v \in \Omega_v \subset \mathbb{R}^{d_v}$

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \mathcal{M}_{\rho,u,\theta} - f$$

Maxwellian and moments:

$$\mathcal{M}_{\rho,u,\theta} = \frac{\rho}{(2\pi\theta)^{d_v/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right)$$

$$\rho = \int_{\mathbb{R}^{d_v}} f \,\mathrm{d}v; \quad u = \frac{1}{\rho}\int_{\mathbb{R}^{d_v}} vf \,\mathrm{d}v; \quad \theta = \frac{1}{d_v \rho}\int_{\mathbb{R}^{d_v}} |v-u|^2 f \,\mathrm{d}v;$$

$$\mathcal{E} = \int_{\mathbb{R}^{d_v}} \frac{|v|^2}{2} f \,\mathrm{d}v$$

From kinetic to fluid: a hydrodynamic limit

Knudsen Number $\varepsilon = \frac{\text{Mean free path}}{\text{Characteristic length}}$

Hydrodynamic scaling

$t \rightsquigarrow \frac{t}{\varepsilon}$ and $x \rightsquigarrow \frac{x}{\varepsilon}$

$$\partial_t f^\varepsilon + v \nabla_x f^\varepsilon + E \nabla_v f^\varepsilon = \frac{1}{\varepsilon} \left(\mathcal{M}_{\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon} - f^\varepsilon \right)$$

$$f^\varepsilon(t, x, v) \underset{\varepsilon \longrightarrow 0}{\longrightarrow} \mathcal{M}_{\rho, u, \theta}$$

Euler with force:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + pI) - \rho E = 0 \\ \partial_t \mathcal{E} + \nabla_x \cdot ((\mathcal{E} + p) u) - \rho u E = 0 \end{cases}$$

$p = (\gamma - 1) \left(\mathcal{E} - \frac{\rho |u|^2}{2} \right)$, $\gamma = \frac{d_v + 2}{d_v}$

From kinetic to fluid: a diffusive limit

Knudsen Number $\varepsilon = \frac{\text{Mean free path}}{\text{Characteristic length}}$

Diffusive scaling

$t \rightsquigarrow \frac{t}{\varepsilon^2}$ et $x \rightsquigarrow \frac{x}{\varepsilon}$

$$\varepsilon^2 \partial_t f + \varepsilon \left(v \cdot \nabla_x f^\varepsilon + E \cdot \nabla_v f^\varepsilon \right) = \rho M - f$$

$$f(t,x,v) \underset{\varepsilon \longrightarrow 0}{\rightarrow} \rho(t,x) M(v), \quad M(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d_v/2}}$$

Drift diffusion:

$$\partial_t \rho - \nabla_x \cdot (\nabla_x \rho - E\rho) = 0$$

Kinetic simulations, a heavy numerical cost

- high-dimensional system : $d_x = d_v = 3 \Rightarrow 7$ dimensions !
- Collision operators, small parameters

→ Hybrid methods

Space

- Filbet, Rey (2015)

→ **Diffusive regime**
→ **Domaine indicators**
→ **Interface conditions**

Time

- Grigori, Hirstoaga, Salomon (2023)

→ **Collisions + hyperbolic**
→ **Multiscale Pred.-corr.**
→ **Parallel efficiency**

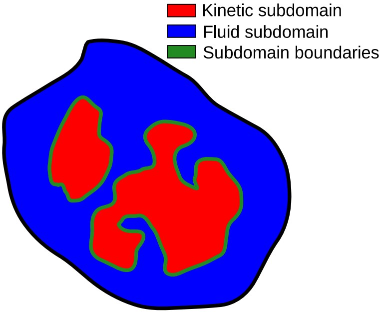
Space & Time

- AP scheme → **Stability and consistance**
- Spatial adaptation → **Fast phase-space solver**
- Multiscale parareal → **Parallel efficiency**

Outline

Hybridization in space (diffusive)

Domain adaptation[1]



- Maximize the size of the fluid domain
- Limit memory access
- Cell-wise decomposition
- Multiscale communications at interfaces

Should I Kinetic or Should I Fluid?

[1] Laidin 2023; Laidin and Rey 2023.

Kinetic criterion

Diffusive scaling: $\varepsilon^2 \partial_t f + \varepsilon (v \cdot \partial_x f^\varepsilon + E \cdot \partial_v f^\varepsilon) = \rho M - f$
Fluid limit: $\partial_t \rho - \partial_x \cdot (\partial_x \rho - E\rho) = 0$

Micro-macro decomposition:

$$f^\varepsilon(t,x,v) = \underbrace{\rho^\varepsilon(t,x)M(v)}_{\text{Local equilibria in velocity}} + \underbrace{g^\varepsilon(t,x,v)}_{\text{Perturbation}}$$

where $M(v)$ is known.

$$||g^\varepsilon(t,x,\cdot)||_2 \leq \delta_0 \iff \text{locally at equilibrium}$$

Macroscopic criterion

Truncated Chapman-Enskog expansion:

$$f^\varepsilon(t,x,v) = \rho^\varepsilon(t,x)M(v) + \sum_{k=1}^{K} \varepsilon^k h^{(k)}(t,x,v)$$

$$\begin{aligned} k = 0: &\qquad h^{(1)} = -\,\mathsf{T}(\rho^\varepsilon M) \\ k = 1: &\qquad h^{(2)} = -\,\partial_t\,(\rho^\varepsilon M) - \mathsf{T}(h^{(1)}) \\ 2 \le k \le K-1: &\qquad h^{(k+1)} = -\,\partial_t\, h^{(k-1)} - \mathsf{T}(h^{(k)}) \end{aligned}$$

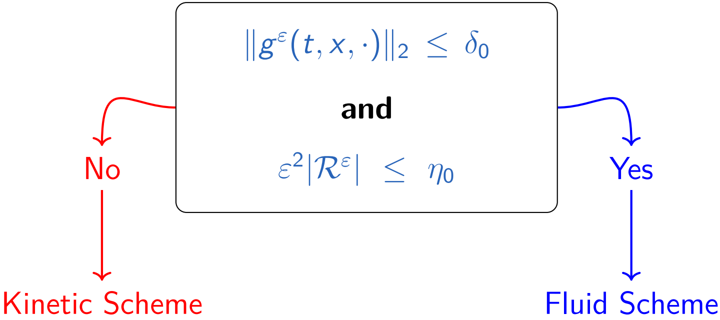
A hierarchy of fluid models:

$$\begin{aligned} K = 1 &\quad \partial_t \rho \,-\, \partial_x\,(\partial_x\,\rho - E\rho) = 0 \\ K = 3 &\quad \partial_t \rho^\varepsilon - \partial_x\,(\partial_x\,\rho^\varepsilon - E^\varepsilon \rho^\varepsilon) = \varepsilon^2 \mathcal{R}^\varepsilon \end{aligned}$$

where $\mathcal{R}^\varepsilon$ depends only on $\rho^\varepsilon, E^\varepsilon$ and moments $m_k = \int v^k M \,\mathrm{d}v$

$\varepsilon^2 |\mathcal{R}^\varepsilon| \le \eta_0 \iff$ the solution is locally close to the limit model

At each time step and for each position:



Micro-macro model & finite volume discretization

Continuous micro-macro model:

$$\begin{cases} \partial_t g^\varepsilon + \dfrac{1}{\varepsilon^2} g^\varepsilon + \dfrac{1}{\varepsilon}\left(\mathsf{T} g^\varepsilon - \partial_x \langle v g^\varepsilon \rangle M + v M J_x^\varepsilon\right) = 0 \\ \partial_t \rho^\varepsilon + \dfrac{1}{\varepsilon} \partial_x \langle v g^\varepsilon \rangle = 0 \end{cases}$$

Continuous limit model ($\varepsilon \to 0$):

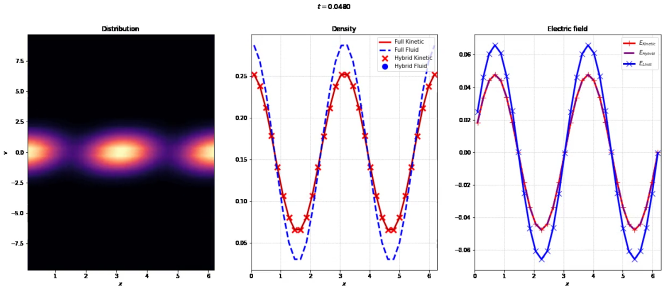
$$\partial_t \rho = \partial_x \left(\partial_x \rho - E\rho \right)$$

Discretization ($\Delta x, \Delta v, \Delta t > 0$):

- Primal cells $\mathcal{X}_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, staggered cells $\mathcal{X}_{i+\frac{1}{2}} = (x_i, x_{i+1})$
- Velocity cells $\mathcal{V}_j = (v_{j-\frac{1}{2}}, v_{j+\frac{1}{2}})$, control volumes $K_{i+\frac{1}{2},j} = \mathcal{X}_{i+\frac{1}{2}} \times \mathcal{V}_j$
- Unknowns:

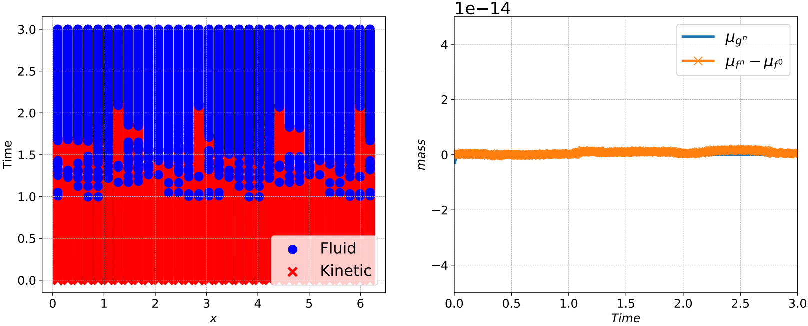
$\rho_i^{\varepsilon,n} \approx \frac{1}{\Delta x} \int_{\mathcal{X}_i} \rho^\varepsilon(t^n, x)\, \mathrm{d}x, \quad g_{i+\frac{1}{2},j}^{\varepsilon,n} \approx \frac{1}{\Delta x \Delta v} \int_{K_{i+\frac{1}{2},j}} g^\varepsilon(t^n, x, v)\, \mathrm{d}x\, \mathrm{d}v$

Hybrid scheme $\varepsilon = 0.1$



$$f_0(x, v) = (2\pi)^{-3/2}(e^{-|v-3|^2/2} + e^{-|v+3|^2})v_x^4(1 + 0.05\cos(2x))$$

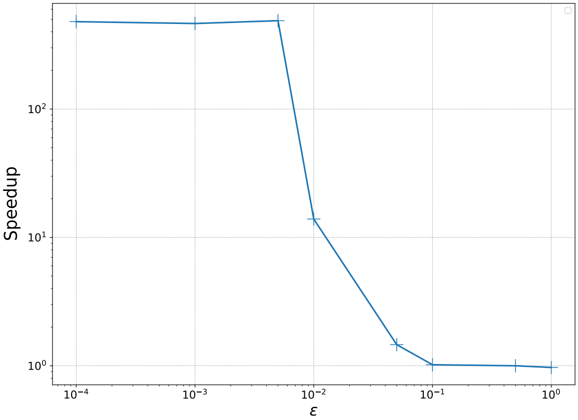
State of cells and conservation property



$$f_0(x,v) = (2\pi)^{-3/2}(e^{-|v-3|^2/2} + e^{-|v+3|^2})v_x^4(1 + 0.05\cos(2x)),\ \varepsilon = 0.1$$

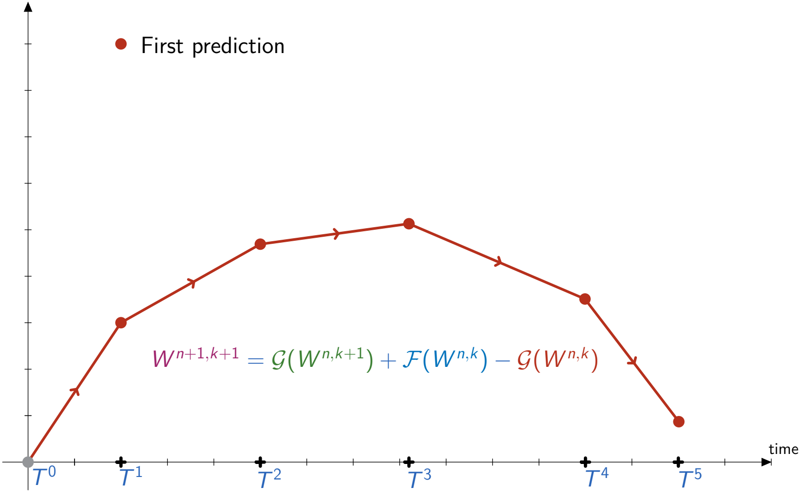
Performances

$$Speedup = \frac{\text{Kinetic CPU time}}{\text{Hybrid CPU time}}$$



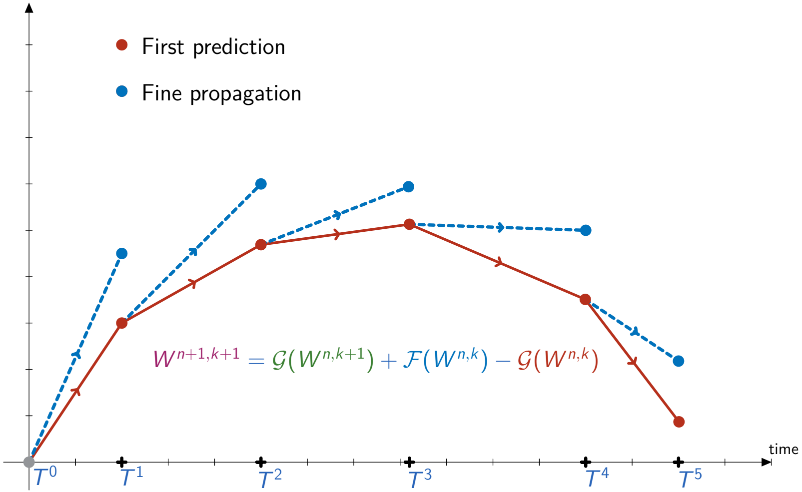
Hybridization in time (hyperbolic)

The parareal method[2]



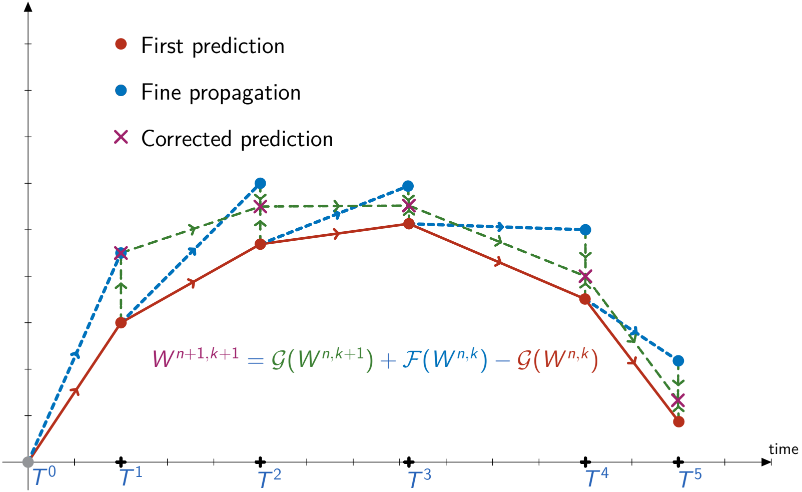
[2] Lions et al. 2001; Maday and Turinici 2002.

The parareal method[3]



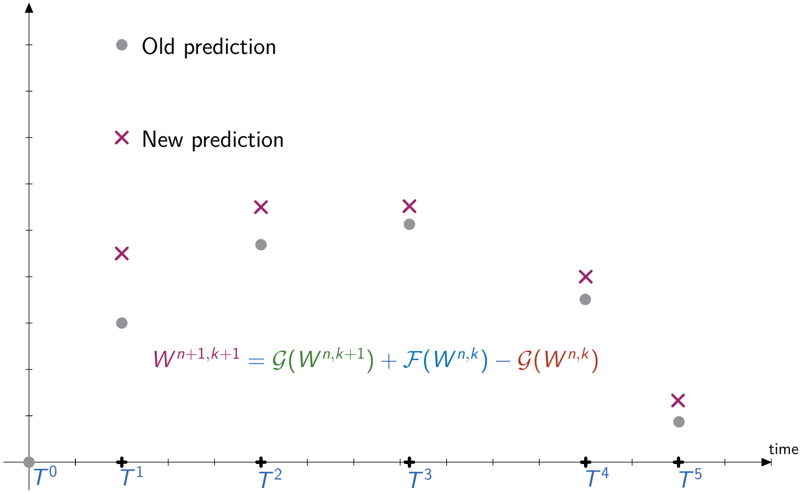
[3] Lions et al. 2001; Maday and Turinici 2002.

The parareal method[4]



[4] Lions et al. 2001; Maday and Turinici 2002.

The parareal method[5]



[5] Lions et al. 2001; Maday and Turinici 2002.

Back to Kinetic: linking the scales

Definition 1 (Projection to fluid)

For a distribution function $f(t,x,v) \in L^1((1+v)^2 \,\mathrm{d}v)$, we define the projection $\mathcal{P}f(t,x) = U(t,x)$ as

$$\mathcal{P}f = U = \begin{pmatrix} \int_{\mathbb{R}^{d_v}} f \,\mathrm{d}v \\ \frac{1}{\rho}\int_{\mathbb{R}^{d_v}} vf \,\mathrm{d}v \\ \frac{1}{d_v \rho}\int_{\mathbb{R}^{d_v}} |v-u|^2 f \,\mathrm{d}v \end{pmatrix} = \begin{pmatrix} \rho \\ u \\ \theta \end{pmatrix}$$

Definition 2 (Lifting to kinetic)

For a macroscopic data $U(t,x) \in \mathbb{R}^5$, we define the lifting of order L, $\mathcal{L}^{(L)}U(t,x,v) = f(t,x,v)$ as

$$f = \mathcal{L}^{(L)}U = \mathcal{M}_{\rho,u,\theta}\left(1 + \sum_{l=1}^{L} \varepsilon^l h^{(l)}\right)$$

The functions h^l are obtained through a Chapman-Enskog expansion.

Kinetic Fluid Parareal Algorithm[6]

Require: $U^{0,0}$

for $n = 1, \dots, N_g$ **do** ▷ First coarse guess
    $U^{n,0} \leftarrow \mathcal{G}\left(U^{n-1,0}\right)$
end for

while $k \leq K$ **or** error $\geq$ tol **do** ▷ Parareal iterations

    for $n = k, \dots, N_g$ **do** ▷ Parallel computation of the jumps
        $\Delta^n = \mathcal{PFL}(U^{n-1,k-1}) - \mathcal{G}\left(U^{n-1,k-1}\right)$
    end for

    for $n = k, \dots, N_g$ **do** ▷ Sequential corrections
        $U^{n,k+1} = \mathcal{G}\left(U^{n-1,k+1}\right) + \Delta^n$
    end for

    Compute the successive error and $k \leftarrow k + 1$
end while

[6] Laidin and Rey 2025.

Exterior force

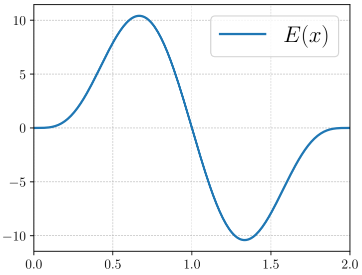
$$f_0(x,v) = \mathcal{M}_{\rho_1(x),u_1(x),\theta_1(x)}(v)+\mathcal{M}_{\rho_2(x),u_2(x),\theta_2(x)}(v),\ x \in [0,2],\ v \in [-8,8]$$

where

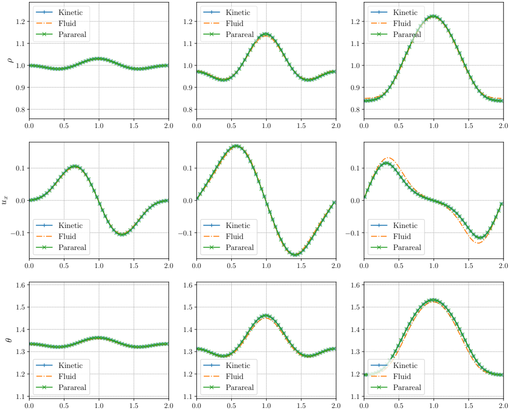
$$(\rho_1(x), u_1(x), \theta_1(x)) = (1, 1, 0, 0, 1)$$

and

$$(\rho_2(x), u_2(x), \theta_2(x)) = (1, -1, 0, 0, 1)$$

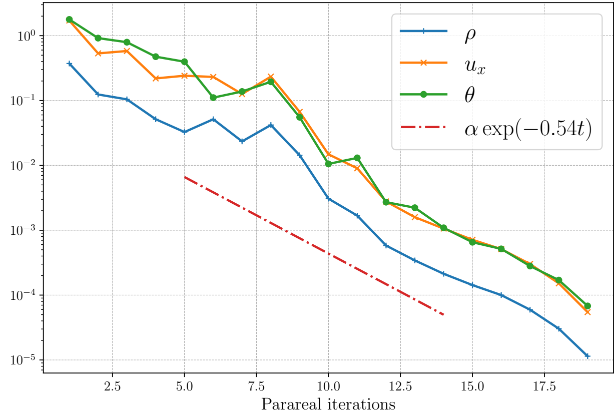


Exterior force, $\varepsilon = 10^{-5}$, $K = 20$



Kinetic (Blue), Parareal (Green), Euler (Orange)

Exterior force, convergence



Convergence of the successive errors

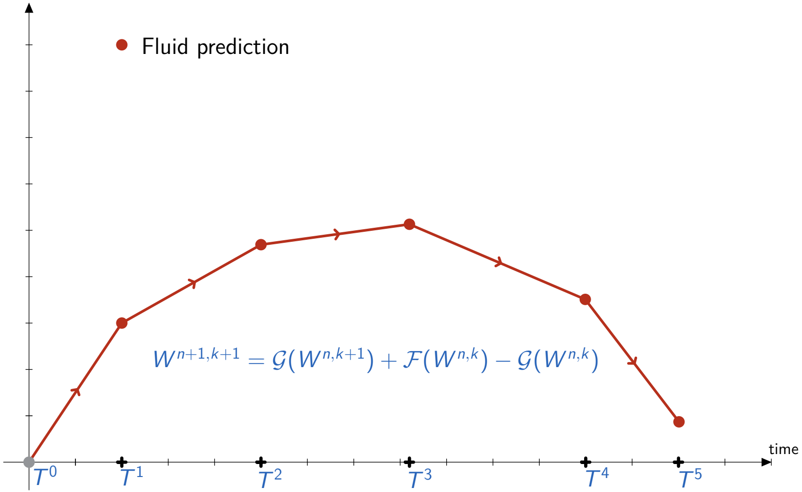
Exterior force, Speedup

Parareal Iterations	Final error	Runtime (s)	Kinetic Runtime	Speedup
3	1.387E-03	71.7	672.8	9.38
6	7.782E-06	153.0	672.8	4.52
9	2.421E-08	218.4	672.8	3.08

Performance on 32 threads, $\varepsilon = 10^{-5}$

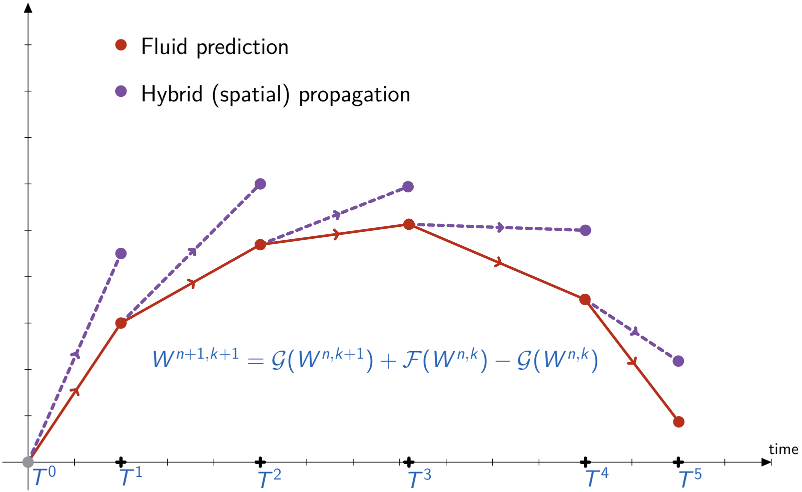
Hybridization in space and time (diffusive)

Space and time hybridization[7]



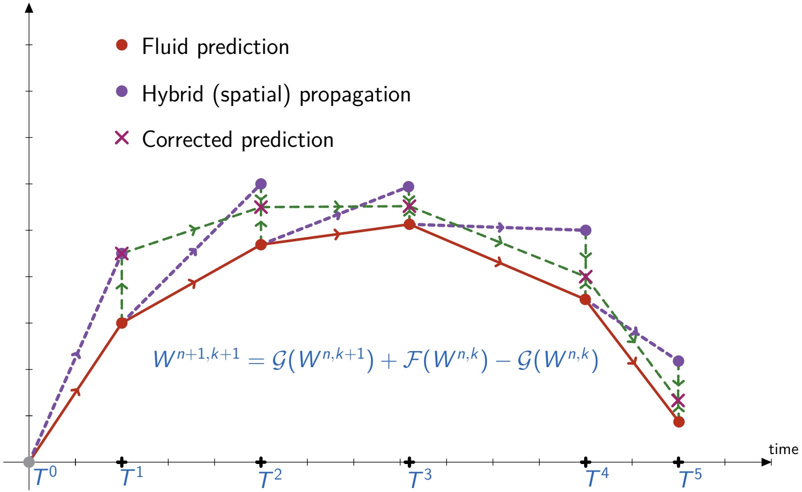
[7] Laidin 2025.

Space and time hybridization[8]



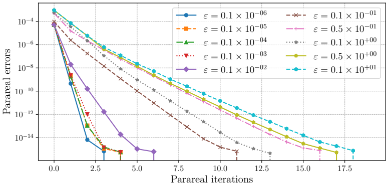
[8] Laidin 2025.

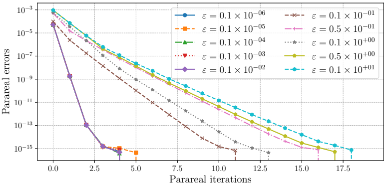
Space and time hybridization[9]



[9] Laidin 2025.

Convergence[10]





Parareal error without (Left) and with (Right) domain adaptation

[10] Laidin 2025.

Speedup

	Para. Off		Para. On		
ε	Adapt. Off	Adapt. On	Adapt. Off	Adapt. On	Fluid
1.0	**1.00**	**0.92**	**2.27**	**2.16**	**384**
10^{-1}	**1.00**	**0.92**	**2.73**	**2.56**	**384**
10^{-2}	**1.00**	**0.97**	**3.09**	**2.96**	**384**
10^{-3}	**1.00**	**4.67**	**5.65**	***24.02***	**384**
10^{-4}	**1.00**	**506**	**7.39**	**72.80**	**384**

Speedups of the space-time hybrid methods

Conclusion and perspectives

Conclusion

→ **Dynamic domain adaptation**

- Very fast in transient / near fluid regimes
- Robust w.r.t. the Knudsen number

→ **Multiscale parareal algorithm**

- Parallel in time
- Independent of phase-space solvers
- Robust w.r.t. the Knudsen number

→ **Space time hybrid method**

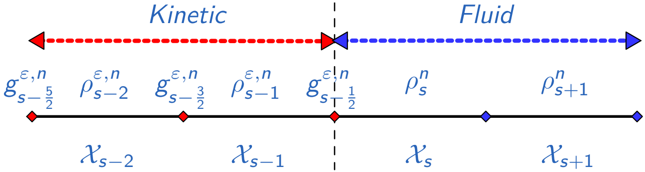
- AP schemes, spatial hybridation, parallel in time
- "Playing on all fields"

Perspectives

- Higher order lifting procedure / learning
- Learn the domain indicators
- Parallelization: shared $\rightarrow$ distributed $+$ hybrid paradigm, accelerators
- Different physics: incompressibility, quasineutrality, gyrokinetic
- More complex models: Boltzmann / Landau, energy transport
- Error estimates

Thank you for your attention !

Interface conditions



Blast wave, pointwise errors