Strong stability of characteristic discretizations of IBVP: the case of lattice Boltzmann schemes

Thomas Bellotti

CNRS – EM2C & Fédération de Mathématiques de CentraleSupélec – CentraleSupélec – Université Paris-Saclay – France

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Introduction

Target problem and discretization

Although not utterly important, we are concerned with is the *one-dimensional transport equation* at velocity $V \neq 0$:

$$\begin{cases} \partial_t u(t,x) + V\partial_x u(t,x) = 0, & t > 0, \quad x > 0, \\ u(0,x) = u^{\circ}(x), & \qquad\quad\; x > 0, \\ u(t,0) = g(t), & t > 0, \qquad\qquad\qquad (\text{if} \quad V > 0). \end{cases}$$

For the *numerical schemes*:

- Space discretization: $x_j := j\Delta x$, with $j \in \mathbb{N}$ and $\Delta x > 0$.
- Time discretization: $t^n := n\Delta t$ with $n \in \mathbb{N}$.
- Time-space scaling: $\Delta t := \Delta x/\lambda$ with $\lambda > 0$.
- Courant number $\mathscr{C} := V/\lambda$.

Kinetic schemes and lattice Boltzmann: today's main example

Forget about boundaries and linear problems:

$$\partial_t u + \partial_x(\varphi(u)) = 0 \quad \rightarrow \quad \begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + c^2 \partial_x u = \epsilon^{-1}(\varphi(u) - v) \end{cases}$$

$$\leftrightarrow \qquad \partial_t f_{\pm} \pm c \partial_x f_{\pm} = \epsilon^{-1}\left(\frac{u}{2} \pm \frac{\varphi(u)}{2c} - f_{\pm}\right).$$

Kinetic scheme [Brenier, '83] and [Bouchut, '03]

(rhs – relaxation) $u_j^{n\star} = u_j^n, \quad v_j^{n\star} = (1 - e^{-\epsilon^{-1}\Delta t}) v_j^n + e^{-\epsilon^{-1}\Delta t}\varphi(u_j^n) \xrightarrow{\epsilon \to 0^+} \varphi(u_j^n).$

(lhs – transport) Consistent Finite Volume scheme for transport equations.

D_1Q_2 lattice Boltzmann scheme

Formally replace $e^{-\epsilon^{-1}\Delta t} = s \in (0, 2]$, take $c = \lambda = \Delta x/\Delta t$, and use upwind.

$$u_j^{n\star} = u_j^n, \quad v_j^{n\star} = (1-s)v_j^n + s\varphi(u_j^n).$$

$$f_{\pm,j}^{n+1} = f_{\pm,j\mp 1}^{n\star} \qquad \text{(careful with the boundary).}$$

Issue: characteristic "blending"

The characteristic structure linked to φ is "replaced" by that of $\pm c$ (or $\pm\lambda$).

A scalar problem solved with a “system” scheme

- If $s \neq 1$, both u_j^n and v_j^n “survive” at each iteration (v cannot be computed from u).
- However, we are *only interested* in u_j^n, the approximation to the solution u.

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Solution when $j \in \mathbb{Z}$ (no boundary): from a system to a scalar scheme

$\forall$ lattice Boltzmann scheme: get rid of v (and its generalizations) and *recast* the scheme as multi-step only on u_j^n. If the scheme is linear

$$(u_j^{n+1}, v_j^{n+1}, \dots) = \boldsymbol{E}(u_j^n v_j^n, \dots), \qquad \text{just consider} \qquad \det(z\boldsymbol{I} - \boldsymbol{E}) = 0.$$

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When $j \in \mathbb{N}$ (one boundary): sometimes but not always

One *cannot always eliminate* v. Today's examples:

- *Anti-bounce-back* boundary condition.

$$f_{+,-1}^{n\star} = -f_{-,0}^{n\star} \quad : \quad \text{can be done (fully by hand).}$$

- *Extrapolation* boundary condition of order $\sigma \geq 1$.

$$f_{+,-1}^{n\star} = \sum_{j=0}^{\sigma-1} (-1)^j \binom{\sigma}{j+1} f_{+,j}^{n\star} \quad : \quad \text{can be done (fully by hand).}$$

- *Kinetic Dirichlet* boundary condition.

$$f_{+,-1}^{n\star} = 0 \quad : \quad \text{I was not able to.}$$

We have to work on the "system" scheme . . . but with care

Take $s = 2$, $\mathscr{C} < 0$ (outflow), and $\sigma = 1$ extrapolation ($f_{+,-1}^{n\star} = f_{+,0}^{n\star}$).
"Stability": *absence of shared eigenvalues* between bulk and boundary scheme.

- *Scheme fully rewritten on u*

$$\underbrace{u_0^{n+1} = u_0^n + \mathscr{C}(u_0^n - u_1^n)}_{\text{upwind}} \qquad \text{and} \qquad \underbrace{u_j^{n+1} = u_j^{n-1} + \mathscr{C}(u_{j-1}^n - u_{j+1}^n),}_{\text{leap-frog}} \quad j \geq 1.$$

Stable (example in [Strikwerda, '**04**]). Look for a time-space eigenvalue (z, κ):

$$\begin{cases} z - 1 - \mathscr{C}(1-\kappa) = 0, \\ z - z^{-1} - \mathscr{C}(\kappa^{-1} - \kappa) = 0, \end{cases} \qquad \rightarrow \qquad (z,\kappa) = (1,1), \quad \text{but} \ldots$$

for $\epsilon > 0$, we have $|\kappa(1+\epsilon)| > 1$.

- *Scheme both* on u and v. Naive computation:

$$\begin{cases} z^{-1}\det(z\boldsymbol{I} - \boldsymbol{B}) = z - \kappa z^{-1} - \mathscr{C}(1-\kappa) = 0, \\ z^{-1}\det(z\boldsymbol{I} - \boldsymbol{E}) = z - z^{-1} - \mathscr{C}(\kappa^{-1} - \kappa) = 0, \end{cases} \qquad \rightarrow \qquad (z,\kappa) = (\pm 1, 1).$$

We have $|\kappa(-1-\epsilon)| < 1$: *instability*. How can the two be compatible?

A numerical simulation

We take $\mathscr{C} = -\frac{1}{2}$ and a Dirac delta at zero time on the boundary. Movie.

- Very *different behaviors* between u and v.
- We *recognize unstable modes* (and their *group velocity*).
- Even when the mode $(z, \kappa) = (-1, 1)$ is not clearly present, we "see" its group velocity ([B. & Tenna, *hal-05563076* ('**26**)]). Spoiler: it is a *saddle point* of something.

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Plan of the talk[1]

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Numerical schemes, boundary conditions, and strong stability

Algorithm

- A set of $q = 2, 3, \dots$ discrete velocities $c_1, \dots, c_q \in \mathbb{Z}$, with associated distribution functions $f_1, \dots, f_q$. We define:

$$r := \max_{i \in [\![1, q]\!]} c_i \qquad \text{and} \qquad p := -\min_{i \in [\![1, q]\!]} c_i.$$

- A moment matrix $\boldsymbol{M} \in \mathrm{GL}_q(\mathbb{R})$: from distribution to moments $(m_1, \dots, m_q)^{\mathsf{T}} = \boldsymbol{M}(f_1, \dots, f_q)^{\mathsf{T}}$.
- The equilibria $\boldsymbol{m}^{\mathrm{eq}} : \mathbb{R} \to \mathbb{R}^q$, fulfilling the conservation constraint $m_1^{\mathrm{eq}}(m_1) = m_1$. Hence m_1 approximates u. Generally, $m_1 = \sum_{i=1}^{q} f_i$.
- The relaxation parameters $s_2, \dots, s_q \in [0, 2]$.

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- The relaxation parameters $s_2, \dots, s_q \in [0, 2]$.

1. A *relaxation*

$$\boldsymbol{m}_j^{n\star} := (\boldsymbol{I}_q - \mathbf{diag}(s_1, \dots, s_q))\boldsymbol{m}_j^n + \mathbf{diag}(s_1, \dots, s_q)\boldsymbol{m}^{\mathrm{eq}}(m_{1,j}^n), \qquad j \in \mathbb{N}.$$

Linear framework, we consider $\boldsymbol{m}^{\mathrm{eq}}(m_1) = \boldsymbol{\epsilon} m_1$, so $\boldsymbol{m}_j^{n\star} = \boldsymbol{K}\boldsymbol{m}_j^n$.

2. A *transport* phase, for $i \in [\![1, q]\!]$

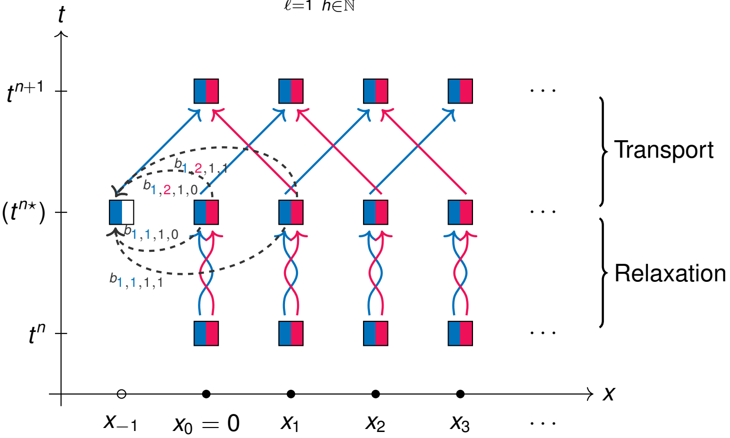
$$f_{i,j}^{n+1} = f_{i,j-c_i}^{n\star}, \qquad j \geq \max(0, c_i).$$

Boundary conditions

Same transport rule: prepare data at the ghost points $x_{-r}, \dots, x_{-1}$ using linear combinations of post-relaxation data.

$$f_{i,j}^{n+1} = f_{i,j-c_i}^{n\star}, \qquad j \in \mathbb{N},$$

$$\text{with} \qquad f_{i,-j}^{n\star} := \sum_{\ell=1}^{q} \sum_{h\in\mathbb{N}} b_{i,\ell,j,h} f_{\ell,h}^{n\star} + \boxed{g_{i,-j}^{n}}, \qquad j \in [\![1, \max(0, c_i)]\!].$$



Different notions of strong stability

Well-established for Finite Differences [Gustafsson, Kreiss & Sundström, **'72**]

Definition (Strong stability)

The lattice Boltzmann scheme is *strongly stable* (SS) if $\exists C > 0$ s.t. $\forall \alpha > 0$, $\forall (g_{i,-j}^n)_{n\in\mathbb{N}}$ with i s.t. $c_i > 0$ and $j \in [\![1, c_i]\!]$, and for all $\Delta x > 0$

$$\frac{\alpha}{1+\alpha\Delta t}\sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}\Delta t\Delta x\, e^{-2\alpha n\Delta t}|\boldsymbol{m}_j^n|^2 + \boxed{\sum_{n\in\mathbb{N}}\sum_{j=0}^{r-1}\Delta t e^{-2\alpha n\Delta t}|\boldsymbol{m}_j^n|^2}$$

$$\leq C\sum_{n\in\mathbb{N}}\sum_{c_i>0}\sum_{j=1}^{c_i}\Delta t\, e^{-2\alpha n\Delta t}(g_{i,-j}^n)^2.$$

The scheme is *strongly stable on the observed output* (SSOO) if

$$\frac{\alpha}{1+\alpha\Delta t}\sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}\Delta t\Delta x\, e^{-2\alpha n\Delta t}(m_{1,j}^n)^2 + \boxed{\sum_{n\in\mathbb{N}}\sum_{j=0}^{r-1}\Delta t e^{-2\alpha n\Delta t}(m_{1,j}^n)^2}$$

$$\leq C\sum_{n\in\mathbb{N}}\sum_{c_i>0}\sum_{j=1}^{c_i}\Delta t\, e^{-2\alpha n\Delta t}(g_{i,-j}^n)^2.$$

In practice…what does this mean? On the numerical example

$$\underbrace{\frac{\alpha}{1+\alpha\Delta t}\sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}}\Delta x\, e^{-2\alpha n\Delta t}(m_{1,j}^{n})^2}_{\text{OUT-bulk}}+\underbrace{\sum_{n\in\mathbb{N}} e^{-2\alpha n\Delta t}(m_{1,0}^{n})^2}_{\text{OUT-boundary}}\leq C\underbrace{\sum_{n\in\mathbb{N}} e^{-2\alpha n\Delta t}(g_{+,-1}^{n})^2}_{\text{IN-boundary}}.$$

- *Anti-bounce-back*. We have $\sum_{j\in\mathbb{N}}(m_{1,j}^{n})^2=O(n)$ and $(m_{1,0}^{n})^2=O(1)$, thus

$$\frac{\text{OUT-bulk}}{\text{IN-boundary}}\approx\frac{\lambda\alpha\Delta t\, e^{-2\alpha\Delta t}}{(1+\alpha\Delta t)(1-e^{-2\alpha\Delta t})^2}\xrightarrow{\alpha\Delta t\to 0^+}+\infty,$$
$$\frac{\text{OUT-boundary}}{\text{IN-boundary}}\approx\frac{1}{1-e^{-2\alpha\Delta t}}\xrightarrow{\alpha\Delta t\to 0^+}+\infty.$$

- *Extrapolation* $\sigma=1$. We have $\sum_{j\in\mathbb{N}}(m_{1,j}^{n})^2=O(1)$ and $(m_{1,0}^{n})^2=O(n^{-3})$

$$\frac{\text{OUT-bulk}}{\text{IN-boundary}}\approx\frac{\lambda\alpha\Delta t}{(1+\alpha\Delta t)(1-e^{-2\alpha\Delta t})}\xrightarrow{\alpha\Delta t\to 0^+}\text{cst},$$
$$\frac{\text{OUT-boundary}}{\text{IN-boundary}}\approx\mathrm{Li}_3(e^{-2\alpha\Delta t})\xrightarrow{\alpha\Delta t\to 0^+}\zeta(3).$$

Practically check stability–instability

Discrete solutions: spectrum

The overall scheme thus reads

$$\begin{cases} \boldsymbol{m}_j^{n+1} = \boldsymbol{E}\boldsymbol{m}_j^{n}, & j \geq r, \\ \boldsymbol{m}_j^{n+1} = \boldsymbol{B}_j\boldsymbol{m}_j^{n} + \boldsymbol{g}_j^{n}, & j \in [\![0, r-1]\!], \\ \boldsymbol{m}_j^{0} = \boldsymbol{0}, & j \in \mathbb{N}, \end{cases}$$

where it is important to note that

$$\boldsymbol{E} = \boldsymbol{E}(\kappa) = \sum_{j=-r}^{p} \boldsymbol{E}_j \kappa^j, \qquad \text{where} \qquad \boldsymbol{E}_j = \sum_{\substack{i \in [\![1, q]\!] \\ \text{s.t. } c_i = -j}} \boldsymbol{M}\boldsymbol{e}_i\boldsymbol{e}_i^{\mathsf{T}}\boldsymbol{M}^{-1}\boldsymbol{K} \in \mathcal{M}_q(\mathbb{R}) \boxed{\not\subset \mathrm{GL}_q(\mathbb{R})},$$

$$\boldsymbol{g}_j^n := \boldsymbol{M}\sum_{c_i > j} \boldsymbol{e}_i g_{i, j-c_i}^n \in \mathbb{R}^q \qquad \text{(the source term has a certain alignment).}$$

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Taking the discrete-time Laplace transform (the z-transform):

Vectorial homogeneous linear difference equation parametrized by z

$$\begin{cases} z\tilde{\boldsymbol{m}}_{j+r}(z) - \sum_{\ell=0}^{r+p} \boldsymbol{E}_{\ell-r}\tilde{\boldsymbol{m}}_{j+\ell}(z) = \boldsymbol{0}_q, & j \in \mathbb{N}, \\ z\tilde{\boldsymbol{m}}_j(z) - \sum_{\ell=-j}^{w} \boldsymbol{B}_{j,\ell}\tilde{\boldsymbol{m}}_{j+\ell}(z) = \tilde{\boldsymbol{g}}_j(z), & j \in [\![0, r-1]\!]. \end{cases}$$

Discrete solutions: spectrum

$$z\tilde{\boldsymbol{m}}_{j+r}(z) - \sum_{\ell=0}^{r+p} \boldsymbol{E}_{\ell-r}\tilde{\boldsymbol{m}}_{j+\ell}(z) = \boldsymbol{0}_q,$$

thus consider $\boldsymbol{L}_z(\kappa) := -\boldsymbol{E}_p\kappa^{r+p} - \boldsymbol{E}_{p-1}\kappa^{r+p-1} - \cdots + (z\boldsymbol{I} - \boldsymbol{E}_r)\kappa^{r} - \cdots - \boldsymbol{E}_{-r}$,

and its $(\kappa)-$*spectrum*, solution to $\det \boldsymbol{L}_z(\kappa) = \kappa^{qr}\det(z\boldsymbol{I} - \boldsymbol{E}(\kappa)) = 0$.

Moreover

$$\det(z\boldsymbol{I}_q - \boldsymbol{E}(\kappa)) = \underbrace{\sum_{\ell=-\overline{r}}^{\overline{p}} d_\ell(z)\kappa^{\ell}}_{(\kappa=\kappa(z))} = \underbrace{z^q - \mathrm{tr}(\boldsymbol{E}(\kappa))z^{q-1} + \cdots + (-1)^q\det(\boldsymbol{E}(\kappa))}_{(z=z(\kappa) \text{ usually with } |\kappa|=1)}.$$

Caveats:

- $\kappa \equiv 0$: boundary-attached terms (absent today).
- Bounds: $\overline{r} \leq \sum_{c_i>0} c_i$ and $\overline{p} \leq -\sum_{c_i<0} c_i$. Inequalities can be strict.
- $d_{\overline{r}}(z)$ (resp, $d_{\overline{p}}(z)$) can vanish for some z: $\kappa(z) = 0$ (resp, $\kappa(z) = \infty$).

Stable discrete solutions

We consider

- Von Neumann stability: for all $|\kappa| = 1$, then $\rho(\boldsymbol{E}(\kappa)) \leq 1$.
- Left stencils: $r = \overline{r} = 1$ (so $d_{\overline{r}}(z) \not\equiv 0$).

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Hyperbolic dichotomy: $|\kappa_{\mathrm{s}}(z)| < 1$ and $|\kappa_{\mathrm{u},1}(z)|, \dots, |\kappa_{\mathrm{u},\overline{p}}(z)| > 1$ when $|z| > 1$.

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- For κ_{s}: select $\boldsymbol{\varphi}_{\mathrm{s}}(z) \in \ker(\boldsymbol{L}_z(\kappa_{\mathrm{s}}(z)))$ (normalization $\varphi_{\mathrm{s},1}(z) \equiv 1$).
- For $\kappa \equiv 0$ (alg. mult. $q-1$): $\ker(\boldsymbol{L}_z(0)) = \ker(\boldsymbol{E}_{-1}) = \mathrm{span}\{\boldsymbol{\varphi}_0^1, \dots, \boldsymbol{\varphi}_0^{q-1}\}$.

Stable (summable at $j \to +\infty$) solutions of the bulk scheme read

$$\tilde{\boldsymbol{m}}_j(z) = C_{\mathrm{s}}(z)\boldsymbol{\varphi}_{\mathrm{s}}(z)\kappa_{\mathrm{s}}(z)^j + \sum_{\ell=1}^{q-1} C_0^{\ell}(z)\boldsymbol{\varphi}_0^{\ell}\delta_{0j}, \qquad j \in \mathbb{N}.$$

Eigenvectors are not “super-safe” for parametric problems (here z)

When the spectrum splits, Riesz projectors (on generalized eigenspaces) are better.

Determining the constants from the boundary scheme

$\exists ! + \in [\![1, q]\!]$ such that $c_+ = 1$ (incoming discrete velocity). Into the boundary scheme:

$$\Delta_{\mathrm{KL}}(z) C_{\mathrm{s}}(z) = \Sigma_{\mathrm{bulk}}(z) \tilde{g}_{+,-1}(z) \qquad \text{and} \qquad \Delta_{\mathrm{KL}}(z) C_0^{\ell}(z) = \boxed{0}, \quad \ell \in [\![1, q-1]\!],$$

where

$$\Delta_{\mathrm{KL}}(z) := \det \left[(z\boldsymbol{I}_q - \textstyle\sum_{\ell \geq 0} \boldsymbol{B}_{0,\ell} \kappa_{\mathrm{s}}(z)^{\ell}) \varphi_{\mathrm{s}}(z) \; (z\boldsymbol{I}_q - \boldsymbol{B}_{0,0}) \varphi_0^1 \; \cdots \; (z\boldsymbol{I}_q - \boldsymbol{B}_{0,0}) \varphi_0^{q-1} \right],$$

$$\text{and} \qquad \Sigma_{\mathrm{bulk}}(z) = \mathrm{cst}(\varphi_0^1, \dots, \varphi_0^{q-1}) \times d_{-1}(z).$$

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$$\text{and} \qquad \Sigma_{\mathrm{bulk}}(z) = \mathrm{cst}(\varphi_0^1, \dots, \varphi_0^{q-1}) \times d_{-1}(z).$$

We can simplify the Kreiss-Lopatinskii determinant:

$$\Delta_{\mathrm{KL}}(z) = \overbrace{\left(\boldsymbol{e}_{+}^{\mathsf{T}} \kappa_{\mathrm{s}}(z)^{-1} - \sum_{\ell=0}^{w} \sum_{h=1}^{q} \boldsymbol{e}_h^{\mathsf{T}} b_{+,h,1,\ell} \kappa_{\mathrm{s}}(z)^{\ell} \right) \boldsymbol{M}^{-1} \boldsymbol{K} \varphi_{\mathrm{s}}(z)}^{=: \langle \mathrm{KL} \rangle (z)} \Sigma_{\mathrm{bulk}}(z),$$

so that—at least formally

$$C_{\mathrm{s}}(z) = \frac{\tilde{g}_{+,-1}(z)}{\langle \mathrm{KL} \rangle (z)} \qquad \text{and} \qquad C_0^{\ell}(z) = 0, \quad \ell \in [\![1, q-1]\!].$$

Singularities in the stable eigenvector

We thus have (recall $\langle \mathrm{KL}\rangle(z) = \boldsymbol{v}^{\mathsf{T}}(z)\varphi_{\mathrm{s}}(z)$)

$$\tilde{\boldsymbol{m}}_j(z) = \tilde{g}_{+,-1}(z)\,\underbrace{\frac{1}{\langle \mathrm{KL}\rangle(z)}}_{\in\mathbb{C}}\,\underbrace{\varphi_{\mathrm{s}}(z)}_{\in\mathbb{C}^q}\,\kappa_{\mathrm{s}}(z)^j, \qquad j\in\mathbb{N}.$$

Stability = continuous dependence of the solution on the datum $g^n_{+,-1}$. Between her and us: inversion of the Laplace transform and possible *poles* regardless of the datum.

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Stability = continuous dependence of the solution on the datum $g^n_{+,-1}$. Between her and us: inversion of the Laplace transform and possible *poles* regardless of the datum.

- $\varphi_{\mathrm{s}}(z)$ has a pole (independent of the boundary condition) at $z = z^{\odot}$ ($|z^{\odot}| \geq 1$). If not compensated, instabilities are *stronger* on some components (e.g., on v).
  - $\langle \mathrm{KL}\rangle(z^{\odot}) = 0$: instability.
  - $\langle \mathrm{KL}\rangle(z^{\odot}) \neq 0$ finite: stable on some components and unstable on others (SSOO).
  - $\langle \mathrm{KL}\rangle(z^{\odot}) = \infty$: stable on every component by (zero–pole) compensation (SS).
- $\varphi_{\mathrm{s}}(z)$ is continuous at $z = z^{\odot}$. Instabilities are *equally strong* on all components and cannot be compensated.
  - $\langle \mathrm{KL}\rangle(z^{\odot}) = 0$: instability.
  - $\langle \mathrm{KL}\rangle(z^{\odot}) \neq 0$ finite: stable (SS).

Renormalization of the eigenvector φ_{s}

Does not change anything on the outcome: as $\langle \mathrm{KL}\rangle(z)$ is linear in $\varphi_{\mathrm{s}}(z)$.

Back to the example and numerical simulation (Play the movie again)

Regardless of the boundary conditions:

$$\varphi_{\mathrm{s},u}(z) \equiv 1 \qquad \text{and} \qquad \varphi_{\mathrm{s},v}(z) = \pm\frac{1}{z \mp 1} + O(1).$$

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Quick reminder of the previous slide

- $\varphi_{\mathrm{s}}(z)$ has a pole (independent of the boundary condition) at $z = z^{\odot}$ ($|z^{\odot}| \geq 1$).
    - $\langle \mathrm{KL} \rangle (z^{\odot}) = 0$: instability.
    - $\langle \mathrm{KL} \rangle (z^{\odot}) \neq 0$ finite: stable on some components and unstable on others (SSOO).
    - $\langle \mathrm{KL} \rangle (z^{\odot}) = \infty$: stable on every component by (zero–pole) compensation (SS).
- ...

Trust me: other modes of modulus ≥ 1 are safe.

- *Anti-bounce-back*: $f^{n\star}_{+,-1} = -f^{n\star}_{-,0} + g^{n}_{+,-1}$.

$$\langle \mathrm{KL} \rangle (1) = \infty \qquad \text{and} \qquad \langle \mathrm{KL} \rangle (-1) = 0 \qquad : \qquad \text{unstable.}$$

- *Extrapolation* $\sigma = 1$: $f^{n\star}_{+,-1} = f^{n\star}_{+,0} + g^{n}_{+,-1}$

$$\langle \mathrm{KL} \rangle (1) = \infty \qquad \text{and} \qquad \langle \mathrm{KL} \rangle (-1) = -1 \neq 0 \qquad : \qquad \text{SSOO.}$$

- *Kinetic Dirichlet*: $f^{n\star}_{+,-1} = g^{n}_{+,-1}$

$$\langle \mathrm{KL} \rangle (1) = \infty \qquad \text{and} \qquad \langle \mathrm{KL} \rangle (-1) = \infty \qquad : \qquad \text{SS.}$$

End of the proof (example)

Consider the extrapolation condition of order $\sigma = 1$ within the previous setting. We have, for $|z| > 1$

$$|\tilde{m}_{1,j}(z)|^2 = \left|\tilde{g}_{+,-1}(z)\frac{1}{\langle \mathrm{KL}\rangle(z)}\kappa_{\mathrm{s}}(z)^j\right|^2 \leq C|\tilde{g}_{+,-1}(z)|^2|\kappa_{\mathrm{s}}(z)|^{2j}.$$

Summing in $j \in \mathbb{N}$:

$$\begin{aligned}\sum_{j\in\mathbb{N}}|\tilde{m}_{1,j}(z)|^2 \leq C|\tilde{g}_{+,-1}(z)|^2\sum_{j\in\mathbb{N}}|\kappa_{\mathrm{s}}(z)|^{2j} &= C|\tilde{g}_{+,-1}(z)|^2\frac{1}{1-|\kappa_{\mathrm{s}}(z)|^2}\\ &\leq C|\tilde{g}_{+,-1}(z)|^2\frac{1}{|z|-1}.\end{aligned}$$

So

$$\frac{|z|-1}{|z|}\sum_{j\in\mathbb{N}}|\tilde{m}_{1,j}(z)|^2 \leq C|\tilde{g}_{+,-1}(z)|^2.$$

By the Parseval identity of z-transform and following [Coulombel, '**13**]:

$$\frac{\alpha}{1+\alpha\Delta t}\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{N}}\Delta x\Delta t\, e^{-2\alpha n\Delta t}|m_{1,j}^n|^2 \leq C\sum_{n\in\mathbb{N}}\Delta t\, e^{-2\alpha n\Delta t}|g_{+,-1}^n|^2.$$

Conclusions and perspectives

Conclusions (take-home message)

Contrarily to the non-characteristic setting, the stable solution (the eigenvector) may *lack a continuous* extension from $|z| > 1$ to $|z| = 1$.
Very different behaviors according to the component (unless compensations).

Conclusions (take-home message)

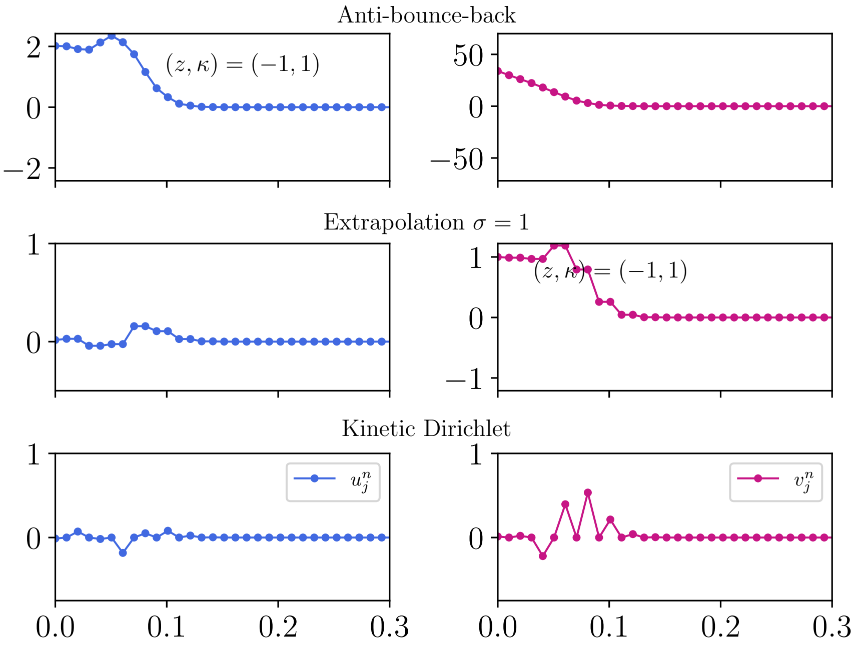
Contrarily to the non-characteristic setting, the stable solution (the eigenvector) may *lack a continuous* extension from $|z| > 1$ to $|z| = 1$.
Very different behaviors according to the component (unless compensations).

Research directions

$$\begin{cases} \boldsymbol{m}_j^{n+1} = \boldsymbol{E}\boldsymbol{m}_j^n + \boldsymbol{F}_j^n, & j \geq r, \\ \boldsymbol{m}_j^{n+1} = \boldsymbol{B}_j\boldsymbol{m}_j^n + \boldsymbol{g}_j^n, & j \in [\![0, r-1]\!], \\ \boldsymbol{m}_j^0 \in \mathbb{R}^q \quad \text{given}, & j \in \mathbb{N}. \end{cases}$$

- *Inner and boundary* sources without specific direction (boundary-attached terms).
  - Duhamel . . . but
  - crossings between $\kappa_{\mathrm{s}}(z)$ and $\kappa \equiv 0$ for $|z| > 1$;
  - crossings between $\kappa_{\mathrm{s}}(z)$ and $\kappa_{\mathrm{u}}(z)$ for $|z| = 1$. Possible solution: Riesz projectors.
- *Non-zero initial data* and *semigroup estimates* ($L_t^{\infty} L_x^2$). Approach: energy methods [Coulombel & Gloria, '**11**].

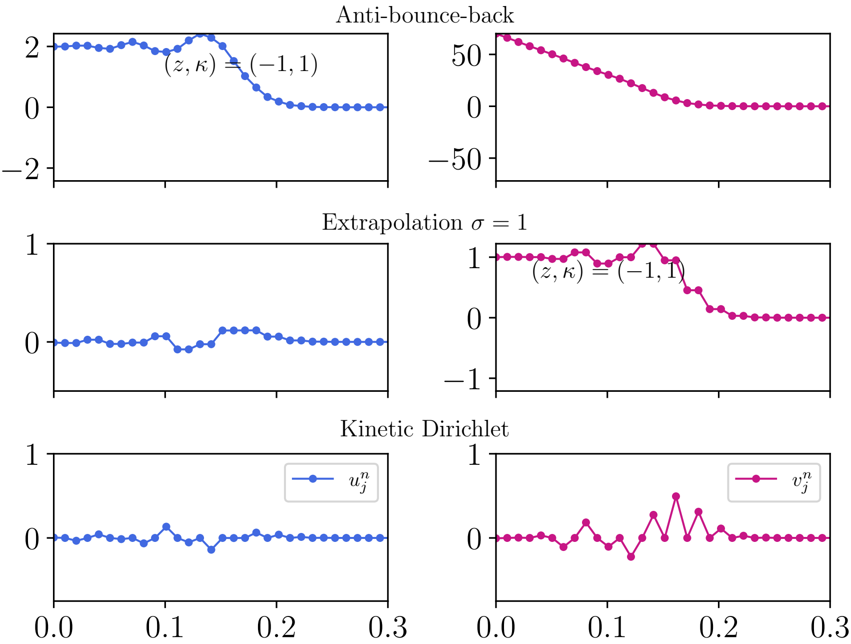
Merci de votre attention !

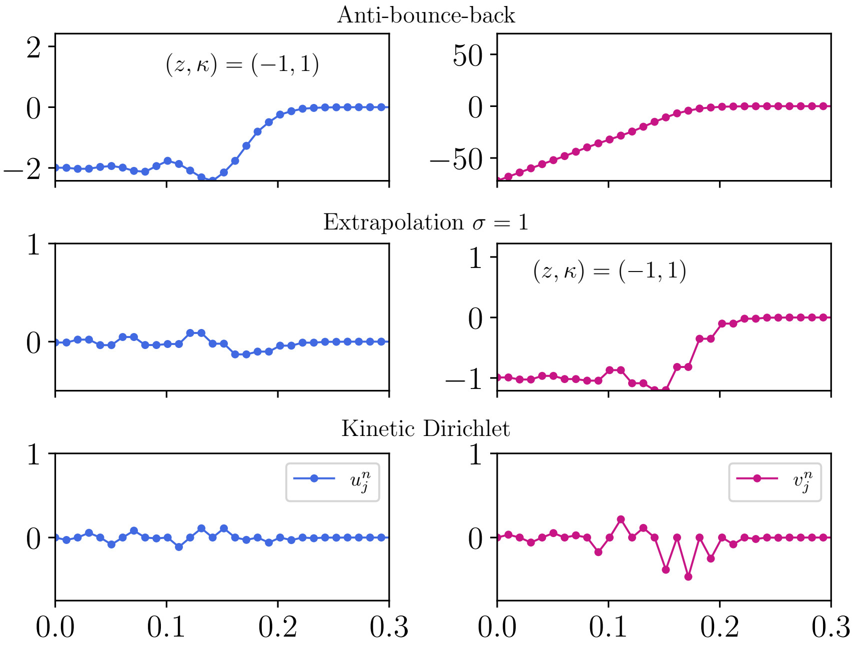


Anti-bounce-back

Extrapolation $\sigma = 1$

Kinetic Dirichlet


Anti-bounce-back
$(z, \kappa) = (-1, 1)$
Extrapolation $\sigma = 1$
$(z, \kappa) = (-1, 1)$
Kinetic Dirichlet
u_j^n
v_j^n
2
0
−2
50
0
−50
1
0
1
0
−1
1
0
1
0
0.0
0.1
0.2
0.3
0.0
0.1
0.2
0.3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$(z, \kappa) = (-1, 1)$
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Kinetic Dirichlet
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0
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1
0
v_j^n
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0.1
0.2
0.3
0.0
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0.2
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