The cascade method for the Vlasov equation

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Joint work with **Chunyang Xu** and **Chang Yang**
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2D transport equation

We look for a numerical solver for a **general 2D transport equation**

$$\frac{\partial f}{\partial t}(t,x,y) + A_1(x,y)\frac{\partial f}{\partial x}(t,x,y) + A_2(x,y)\frac{\partial f}{\partial x}(t,x,y) = 0$$

with **divergence free** property

$$\partial_x A_1(x,y) + \partial_y A_2(x,y) = 0$$

without CFL condition, without system to invert.

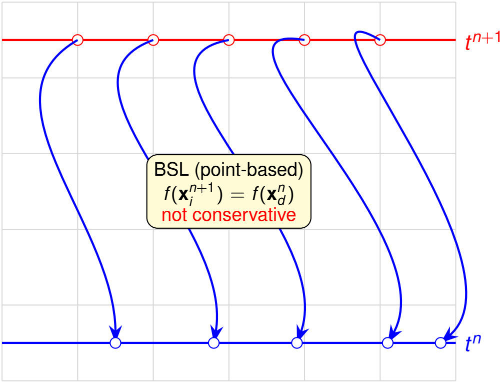
- Backward Semi-Lagrangian (BSL2D) method is **not conservative**
- Splitting breaks the divergence free property

☺ Conservative Semi-Lagrangian (CSL2D)
☺ **Cascade method**

Prototype for gyrokinetic simulation / diocotron instability / limit of Vlasov equation with strong magnetic field / poloidal plane

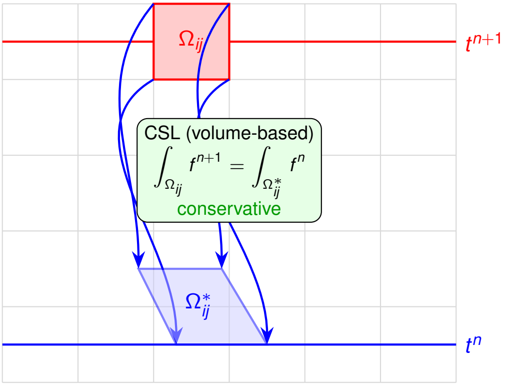
BSL2D

Sonnendrücker, Roche, Bertrand, Ghizzo, JCP 1999



CSL2D - Conservative Semi-Lagrangian

Lauritzen, Nair, Ullrich, JCP 2010 (CSLAM)
Crouseilles, Mehrenberger, Sonnendrücker, JCP 2010 (1D)
Crouseilles, Glanc, Hirstoaga, Madaule, Mehrenberger, EPJD 2014

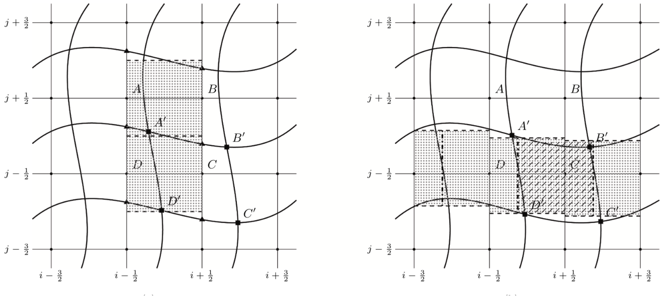


- Tracks the **whole cell** backward in time
- Requires computing mass of irregular region Ω_{ij}^*
- Mass conservative but complex in 2D

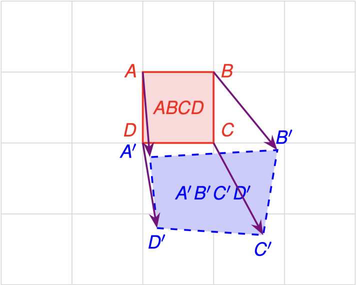
CCSL2D - Why a (conservative) cascade method?

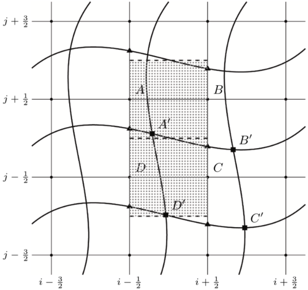
Main idea: Replace one difficult 2D problem by two easy 1D problems

Nair, Scroggs, Semazzi, Mon. Wea. Rev. 2002



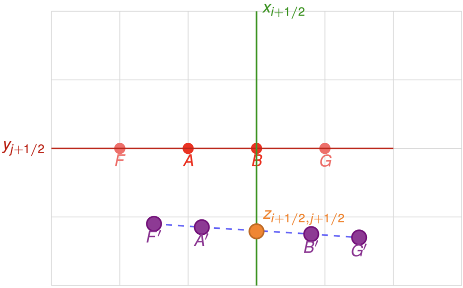
CCSL Step 1: Backtrack cell corners

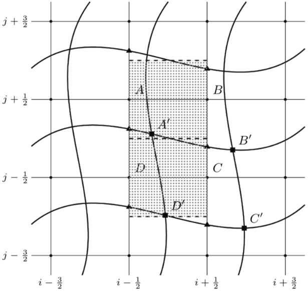




- Trace the four corners of cell *ABCD* backward in time
- Obtain irregular quadrilateral $A'B'C'D'$ at t^n
- Mass is conserved: $\iint_{ABCD} f^{n+1} = \iint_{A'B'C'D'} f^n$

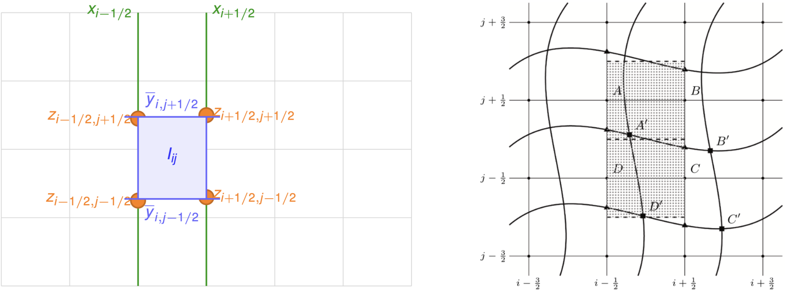
CCSL Step 2: Build intermediate grid points





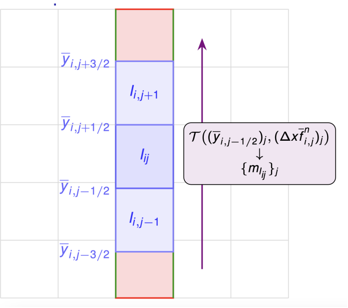
- Points at t^{n+1} on $y_{j+1/2}$: F, A, B, G
- Backtrack to F', A', B', G'
- Third-order Lagrange interpolation at $x = x_{i+1/2}$
- Gives $z_{i+1/2,j+1/2}$

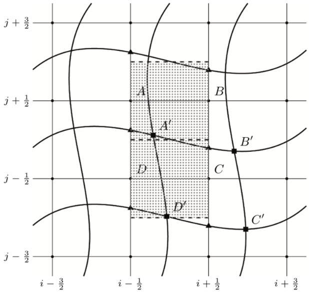
CCSL Step 3: Construct intermediate cells I_{ij}



- Left/right boundaries = vertical grid lines $x_{i-1/2}$, $x_{i+1/2}$
- Bottom boundary = average of $z_{i-1/2,j-1/2}$ and $z_{i+1/2,j-1/2}$
- Top boundary = average of $z_{i-1/2,j+1/2}$ and $z_{i+1/2,j+1/2}$
- Forms a Cartesian cell I_{ij}

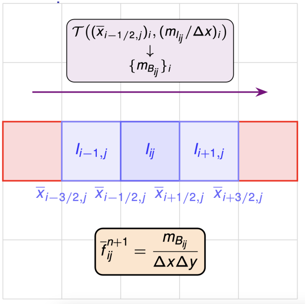
CCSL Step 4: Column-wise 1D conservative remapping

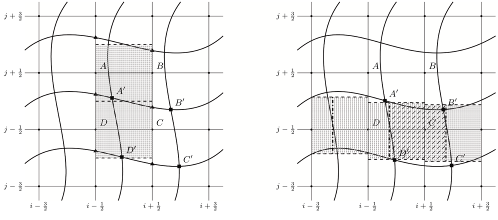




- For each column i, apply 1D CSL reconstruction **vertically**
- Input: horizontal boundaries $\overline{y}_{i,j-1/2}$ and cell averages $\overline{f}^n_{i,j}$
- Output: mass of each intermediate cell $m_{I_{ij}}$
- Uses the same 1D conservative reconstruction as CSL1D

CCSL Step 5: Column-wise 1D conservative remapping





- For each row j, apply 1D CSL reconstruction **horizontally**
- Input: vertical boundaries $\overline{x}_{i-1/2,j}$ and intermediate cell masses $m_{I_{ij}}/\Delta x$
- Output: mass of each backtracked cell $m_{B_{ij}}$
- Finally, divide by $\Delta x \Delta y$ to get $\overline{f}_{ij}^{n+1}$

What can we do?

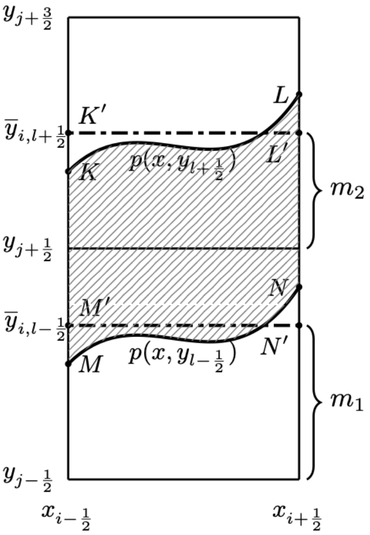
- Theoretical analysis of consistency error

  $\Rightarrow$ study of the dominant error arising from geometric approximation of the backtracked region.
- some modifications are proposed
  - freestream-preserving correction that ensures exact volume conservation
  - maximum-principle limiter
- numerical results on classical benchmarks
  - linear advection
    - rotation
    - swirling deformation flow
  - guiding center model
    - classical diocotron instability
    - "drift kinetic like" model with modified Poisson equation
  - relativistic Vlasov-Maxwell model

All tests are in $2D_x \times 0D_v$ *or* $1D_x \times 1D_v$

Chunyang Xu, Michel Mehrenberger, Chang Yang,
A conservative cascade semi-Lagrangian method for solving the Vlasov equation,
Journal of Computational Physics 558 (2026) 114847

Geometric error on the intermediate cell: $O(h^4)$



$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{p(x,y_{l+\frac{1}{2}})}^{\overline{y}_{i,l+\frac{1}{2}}} f(x,y)\,dy\,dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{p(x,y_{l-\frac{1}{2}})}^{\overline{y}_{i,l-\frac{1}{2}}} f(x,y)\,dy\,dx$$

The cascade method: lemma for geometric error

Lemma 1. *Let $p \in C^2([x_1, x_2])$ and $f \in C^2$ in a neighborhood of the curve $(u, p(u))$. The integral error*

$$\epsilon = \int_{x_1}^{x_2} \int_{\frac{1}{2}(p(x_1)+p(x_2))}^{p(u)} f(u,v)\,dv\,du$$

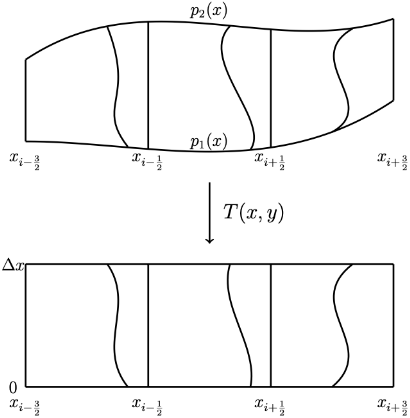
admits the asymptotic expansion:

$$\epsilon = \left(\frac{1}{12} \frac{\partial f}{\partial u} p' + \frac{1}{24} \frac{\partial f}{\partial v} (p')^2 - \frac{1}{12} f\, p'' \right)\bigg|_{(x_1, p(x_1))} h^3 + \mathcal{O}(h^4),$$

where $h = x_2 - x_1$.

- The curve has to be defined and smooth; it is not the case, if time step is too big
- Leads to $O(h^2)$ error for average mass

Mapping for analysing error in the second direction



Some issues

- Freestream preserving is not ensured: if $f = 1$, f does not remain $= 1$ after the operations
- Freestream can be critical for long term accuracy and stability
- High order interpolation in semi-Lagrangian schemes can make violate the maximum principle, in particular the positivity
- not always critical, but sometimes huge values can appear, especially when using discontinuous data

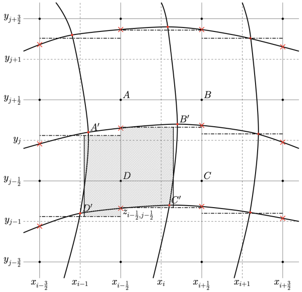
Modifications of the initial method of Nair, Scroggs, Semazzi, 2002

- Conservation of backtracked volume

  $\Rightarrow$ adjust vertical edges

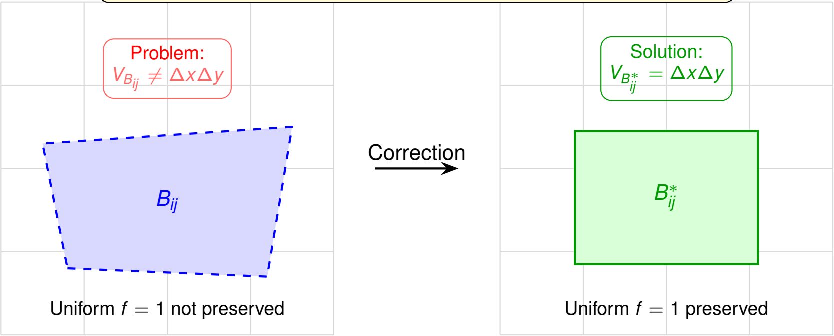
- Conservation of maximum principle

  $\Rightarrow$ linear combination with linear reconstruction



Freestream preservation correction

Freestream preservation: Adjust boundaries to conserve volume



- Original CCSL: backtracked cell B_{ij} does not preserve volume
- Even with $\nabla \cdot A = 0$, $V_{B_{ij}} \neq \Delta x \Delta y$
- Freestream preservation correction: adjust cell boundaries
- Ensures $V_{B_{ij}^*} = \Delta x \Delta y$
- Uniform state $f = 1$ is exactly preserved

Rotation $\Delta t = 0.25$

mesh	CFL	$t = 1$		$t = 4$	
		L^2 error	order	L^2 error	order
40×40	11.10	5.39e-03	–	1.58e-02	–
80×80	22.20	5.83e-04	3.21	4.13e-04	5.26
160×160	44.40	1.45e-04	2.01	7.11e-06	5.86
320×320	88.80	3.63e-05	2.00	3.19e-07	4.48
640×640	177.59	9.06e-06	2.00	2.51e-08	3.67
1280×1280	355.19	2.29e-06	1.98	8.62e-10	4.86

Swirling Deformation Flow test

For the swirling deformation flow test, we adopt the same boundary conditions, initial conditions, and computational domain as those used in Section 4.1.1. The velocity field is defined as follows:

$$\mathbf{a}(x,y) = \Big(-2\pi \cos^2(\frac{x}{2})\sin(y)g(t), 2\pi \sin(x)\cos^2(\frac{y}{2})g(t)\Big), \tag{19}$$

where $g(t) = \cos(\frac{\pi t}{T})$ and $T = 2$.

Swirling Deformation Flow $\Delta t = 0.125$

mesh	CFL	$t=1$		$t=4$	
		L^2 error	order	L^2 error	order
40×40	5	3.55e-02	–	5.48e-02	–
80×80	10	6.07e-03	2.55	4.13e-03	3.73
160×160	20	1.40e-03	2.12	3.33e-04	3.63
320×320	40	3.45e-04	2.02	9.65e-05	1.79
640×640	80	8.36e-05	2.05	3.25e-05	1.57
1280×1280	160	1.95e-05	2.10	1.32e-05	1.3

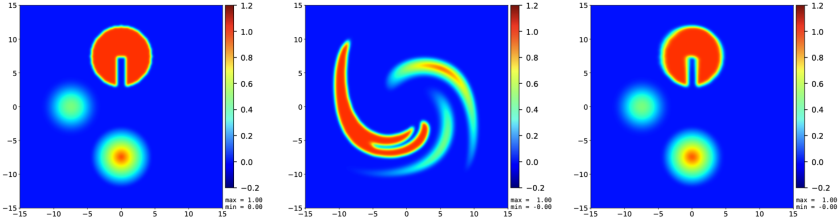
Case of discontinuous data test

We modify the initial condition to use a discontinuous one:

$$f(x,y,0)=\begin{cases}1, & \sqrt{x^2+(y-0.5\pi)^2}\le r_0 \text{ and } (|x|\ge 0.05\pi \text{ or } y\ge 0.5\pi),\\ 1-\dfrac{1}{r_0}\sqrt{x^2+(y+0.5\pi)^2}, & \sqrt{x^2+(y+0.5\pi)^2}\le r_0,\\ \dfrac{1+\cos\left(\dfrac{\pi}{r_0}\sqrt{(x+0.5\pi)^2+y^2}\right)}{4}, & \sqrt{(x+0.5\pi)^2+y^2}\le r_0,\\ 0, & \text{otherwise,}\end{cases} \tag{20}$$

where $r_0 = 0.3\pi$. In Fig. 5, we plot the contour maps of the solution under the CCSL method at times $t = 0$, $t = 1$, and $t = 2$. It can be observed that the CCSL method can well capture complex structures and maintain the maximum principle.

Case of discontinuous data $\Delta t = 0.03125$, $CFL = 5$, $N_x = N_y = 160$



(a) t=0.0 (b) t=1.0 (c) t=2.0

Guiding center model

$$\begin{cases} \dfrac{\partial f}{\partial t} + \nabla \cdot (\mathbf{E}^{\perp} f) = 0, \\ -\Delta \phi = f, \ \mathbf{E}^{\perp} = \left(-\dfrac{\partial \phi}{\partial y}, \dfrac{\partial \phi}{\partial x} \right) \end{cases}$$

1. Solve the Poisson equation using $f_{i,j}^n$ to obtain $\mathbf{E}_{i,j}^n$;
2. Utilize $\mathbf{E}_{i,j}^n$ for backtracking over half a time step $\frac{\Delta t}{2}$, converting $f_{i,j}^n$ to $f_{i,j}^*$;
3. Solve the Poisson equation using $f_{i,j}^*$ to obtain $\mathbf{E}_{i,j}^*$;
4. Use $\mathbf{E}_{i,j}^*$ for backtracking over a full time step Δt, converting $f_{i,j}^n$ to $f_{i,j}^{n+1}$;

Guiding center model

We consider the diocotron test case [41]. The initial condition is given by

$$f(x,y,0)=\begin{cases}(1+\varepsilon\cos(l\theta))\exp\left(-4(r-6.5)^2\right), & r^-\le\sqrt{x^2+y^2}\le r^+\\ 0, & \text{otherwise,}\end{cases}$$

where $r=\sqrt{x^2+y^2}$ and $\theta=\text{atan2}(y,x)$. We set the parameters as $\varepsilon=0.1$, $r^-=5$, $r^+=8$, $l=6$. A square computational domain $[-15,15]^2$ is adopted, with a uniform grid resolution of 1024×1024 in the x- and y-directions. Zero boundary conditions are imposed for both f and the electric potential ϕ; accordingly, the Poisson equation is solved using the finite difference method. We fix the time step size as $\Delta t=1$.

Guiding center model

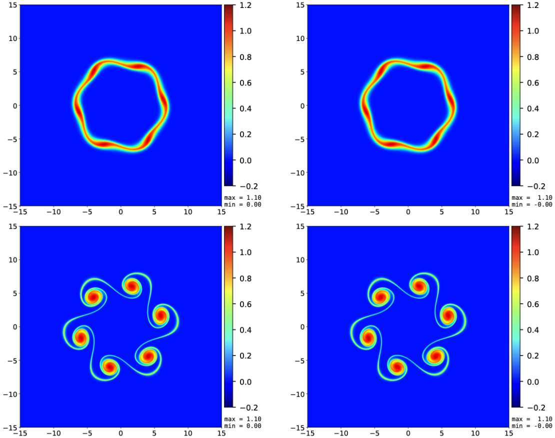


Figure: Cascade (left), BSL (right)

$t = 10, 30$

Guiding center model

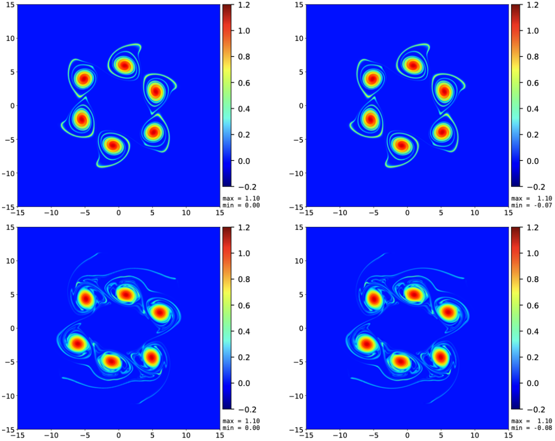
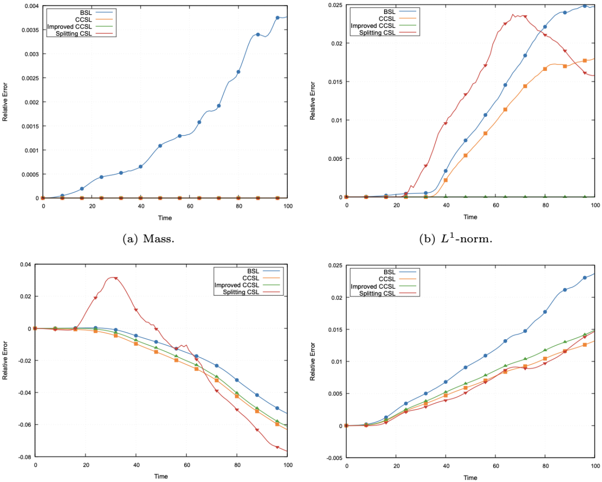


Figure: Cascade (left), BSL (right)

$t = 50, 100$

Guiding center model: conservation of invariants



(a) Mass. (b) L^1-norm.

(c) L^2-norm. (d) Total energy.

Guiding center model, with modified Poisson equation

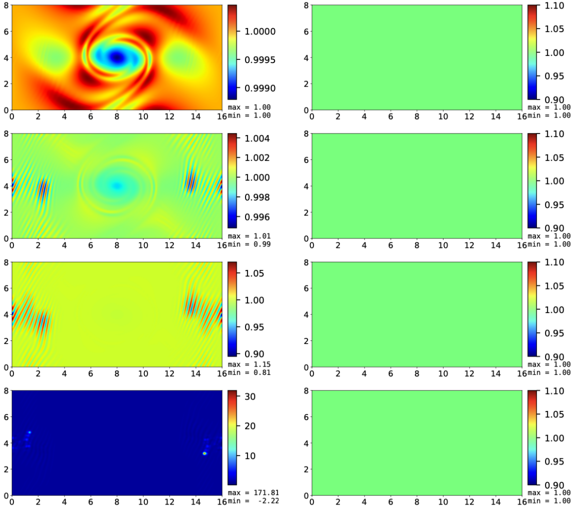
In this test, the formulation of the guiding center model remains unchanged except for the Poisson equation, which is modified as follows:

$$-\Delta\phi = kf + \varepsilon \exp\left[-\frac{(x/l_x - 0.5)^2 + (y/l_y - 0.5)^2}{2\sigma^2}\right], \qquad (21)$$

where k, ε, and σ are constants, and l_x and l_y denote the domain lengths in the x- and y-directions, respectively. To emphasize the influence of the perturbation, we set $k = 10$, $\varepsilon = 0.8$, and $\sigma = 0.1$ in this test.

The computational domain is $[0, 16] \times [0, 8]$, discretized by a uniform grid of 128×64 cells. Periodic boundary conditions are applied in both directions. The initial condition is defined as $f(x, y, 0) = 1$, corresponding to a uniform distribution.

Guiding center model, with modified Poisson equation



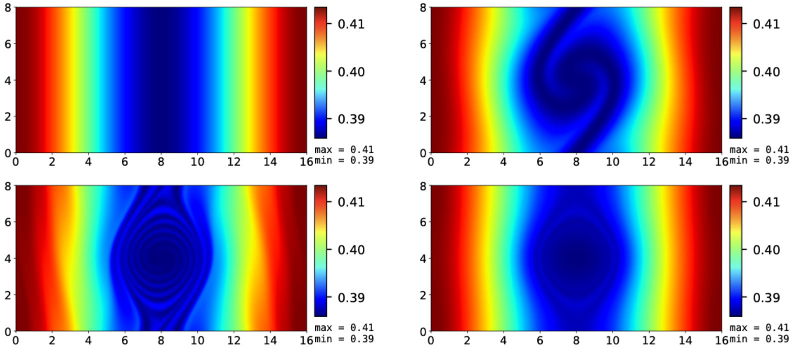
Guiding center model, with modified Poisson equation

The computational domain is $[0, 16] \times [0, 8]$, discretized by a uniform grid of 256×128 cells. The initial condition is defined as

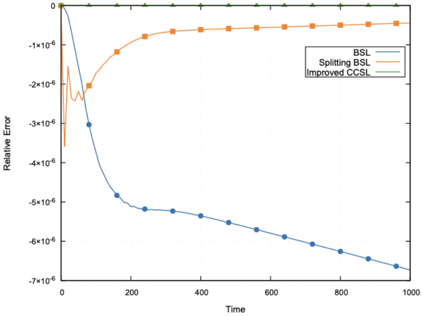
$$\begin{cases} f(x,y,0) = \dfrac{1}{\sqrt{2\pi\, m_i\, T(x)}}, \\ T(x) = 1 - \dfrac{L_x}{74\pi} \cos\left(\dfrac{2\pi x}{L_x}\right), \end{cases} \tag{22}$$

which develops vortex structures during the evolution. Unless otherwise specified, the time step is $\Delta t = 1$, and periodic boundary conditions are applied in both directions.

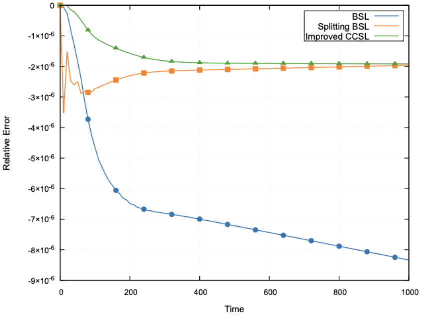
Guiding center model, with modified Poisson equation



Guiding center model, with modified Poisson equation

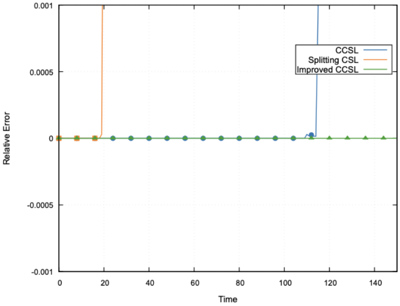


(a) L^1-norm.

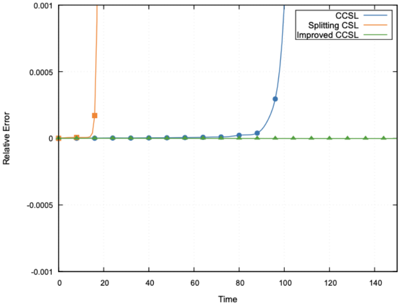


(b) L^2-norm.

Guiding center model, with modified Poisson equation



(a) L^1-norm.



(b) L^2-norm.

Vlasov-Maxwell 1dx1d

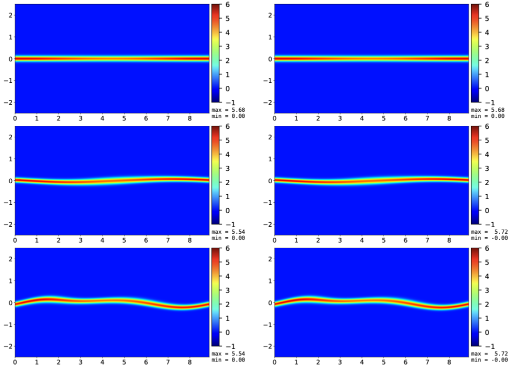
The governing equations of the 1D relativistic Vlasov–Maxwell system are written as

$$\frac{\partial f}{\partial t} + \frac{p}{m\gamma}\frac{\partial f}{\partial x} + \left(eE_x - \frac{mc^2}{2\gamma}\frac{\partial a^2}{\partial x} \right) \frac{\partial f}{\partial p} = 0, \tag{23}$$

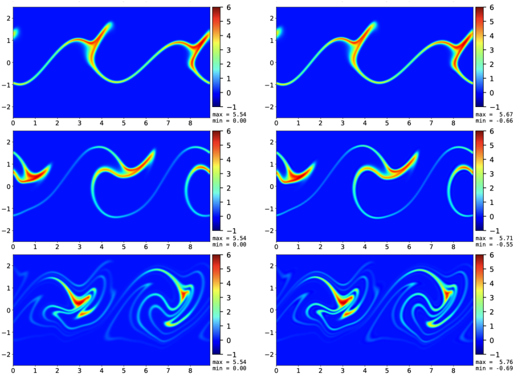
where $\gamma = \sqrt{1 + p^2/(m^2c^2) + a^2}$ is the relativistic factor, and $a(x,t) = eA(x,t)/mc$ is the normalized amplitude of the potential vector $A = (0, A_y, A_z)$. The electromagnetic fields evolve according to

$$\begin{cases} \dfrac{\partial E_y}{\partial t} = -c^2\dfrac{\partial B_z}{\partial t} + \omega_p^2 A_y \rho_\gamma, \; \dfrac{\partial E_z}{\partial t} = c^2\dfrac{\partial B_y}{\partial t} + \omega_p^2 A_z \rho_\gamma, \\ \dfrac{\partial B_y}{\partial t} = \dfrac{\partial E_z}{\partial x}, \dfrac{\partial B_z}{\partial t} = -\dfrac{\partial E_y}{\partial x}, \end{cases} \tag{24}$$

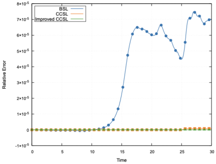
Vlasov-Maxwell 1dx1d



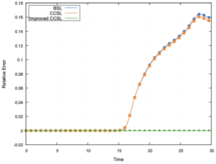
Vlasov-Maxwell 1dx1d



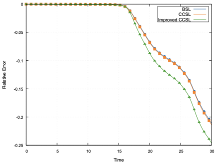
Vlasov-Maxwell 1dx1d



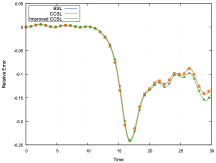
(a) Mass.



(b) L^1-norm.



(c) L^2-norm.



(d) Total energy.

Conclusion

- Conclusion
  - Study of geometrical error, numerical results
  - Improvement for being both conservative and for freestream preserving
  - Improvement for maximum principle
  - Numerical results in plasma physics classical benchmarks
  - Special test for freestream
- Perspectives
  - parallelization
  - higher dimensional simulations
  - other geometries