Numerical simulation of the three-dimensional Vlasov-Poisson system in a torus

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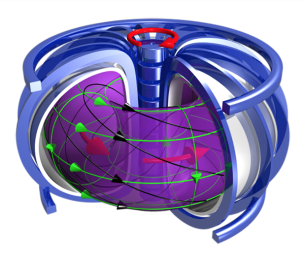
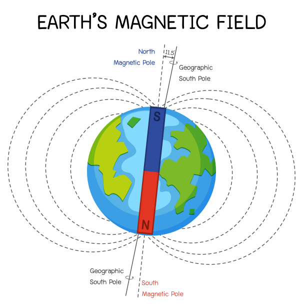
Two examples:

Magnetic fusion confinement applies to tokamak devices to confine particles in a torus.





Earth's magnetic field

- particles move around the planet following the magnetic field lines;
- protects the ozone layer from the solar wind (a stream charge particles from the Sun).

Vlasov-Poisson system with an external magnetic field

We consider the Vlasov-Poisson system with an external electromagnetic field

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\alpha + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}_{\text{ext}}) \cdot \nabla_{\mathbf{v}} f_\alpha = 0, \tag{1}$$

where the phase space $(\mathbf{x}, \mathbf{v}) \in \Omega \times \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^3$ and $t \in \mathbb{R}^+$ is the time variable. The electric field is such that $\mathbf{E}(\mathbf{x}, t) = -\nabla_{\mathbf{x}} \Phi(\mathbf{x}, t)$

$$-4\,\pi\,\varepsilon_0\,\Delta\Phi = \rho(\mathbf{x}, t) = \sum_\alpha q_\alpha \int_{\mathbb{R}^3} f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \tag{2}$$

and $\mathbf{B}_{\text{ext}}$ is the external magnetic field with $\text{div}_{\mathbf{x}} \mathbf{B}_{\text{ext}} = 0$. We define two frequencies:

- the plasma frequency

$$\omega_p^\alpha = \left(\frac{q_\alpha^2\, n_\alpha}{4\,\pi\,\varepsilon_0\, m_\alpha} \right)^{1/2};$$

- the cyclotron frequency

$$\omega_c^\alpha = \frac{q_\alpha\, \|\mathbf{B}_{ext}\|}{m_\alpha}.$$

We set $\varepsilon = \omega_p^\alpha / \omega_c^\alpha \ll 1$.

The Vlasov-Poisson system with an external magnetic field

We consider the system with $\varepsilon \ll 1$ and phase-space $(\mathbf{x}, \mathbf{v}, t) \in \Omega \times \mathbb{R}^3 \times \mathbb{R}^+$

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_\varepsilon + \left(\mathbf{E}_\varepsilon(t, \mathbf{x}) + \dfrac{B(\mathbf{x})}{\varepsilon} \mathbf{v} \wedge \mathbf{e}_\parallel(\mathbf{x}) \right) \cdot \nabla_{\mathbf{v}} f_\varepsilon \ = 0, \\ \mathbf{E}_\varepsilon = -\nabla_{\mathbf{x}} \Phi_\varepsilon, \quad -\Delta_{\mathbf{x}} \Phi_\varepsilon = \rho_\varepsilon, \quad \rho_\varepsilon = \displaystyle\int_{\mathbb{R}^2} f_\varepsilon d\mathbf{v}, \end{cases} \tag{3}$$

with $\mathbf{B}_{\text{ext}}$ is split into the intensity $\frac{1}{\varepsilon} B$ and the parallel direction $\mathbf{e}_\parallel$ such that

$$B \in W^{1,\infty}, \quad B > B_0 > 0, \quad \forall\, \mathbf{x} \in \Omega.$$

Consider the characteristic curves of the Vlasov equation (3)

$$\begin{cases} \dfrac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,t} = \mathbf{v}, \\ \dfrac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} = \mathbf{E}_\varepsilon(t, \mathbf{x}) + \dfrac{B(\mathbf{x})}{\varepsilon} \mathbf{v} \wedge \mathbf{e}_\parallel(\mathbf{x}), \end{cases} \tag{4}$$

GOAL: Our aim is to provide **accurate** and **stable** numerical schemes:

- for any $\Delta t > 0$ which are not constrained by the magnitude of ε.
- preserve structure of asymptotic regime $\varepsilon \ll 1$.

Particle-In-Cell method

- Solve the system of trajectory (4) for N particles.
- Consider a set of particles characterized by a weight $(\omega_k)_{k\in N}$ and their position in phase space $(\mathbf{x}_k, \mathbf{v}_k)_{k\in\mathbb{N}}$ computed by discretizing the characteristic curves at time $t^n = n\Delta t$.
- The solution f is approximated by sum of Dirac mass

$$f_N^n(t, \mathbf{x}, \mathbf{v}) = \sum_{1\leq k\leq N} \omega_k \delta(\mathbf{x} - \mathbf{x}_k^n)\delta(\mathbf{v} - \mathbf{v}_k^n),$$

  and

$$\rho_N^n(t, \mathbf{x}) = \sum_{1\leq k\leq N} \omega_k \delta(\mathbf{x} - \mathbf{x}_k^n).$$

  For the regularization of f_N, we use a smooth function $\varphi_\alpha = \alpha^{-3}\varphi(\cdot/\alpha)$ to replace with $\delta(\cdot)$.
- Project the density ρ_N on a mesh of physical space to solve a discrete Poisson equation.

Table of contents

Study on a simple model 2D $\times$ 2D

Example on $(\mathbf{x},\mathbf{v}) \in$ 2D $\times$ 2D

Consider $\mathbf{e}_\parallel = (0,0,1)$. The characteristic curves of the Vlasov equation on the orthogonal plane $(\mathbf{x},\mathbf{v}) \in \Omega \times \mathbb{R}^2$, in which $\Omega \subset \mathbb{R}^2$, is written as

$$\begin{cases} \dfrac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,t} = \mathbf{v}, \\ \dfrac{\mathrm{d}\,\mathbf{v}}{\mathrm{d}\,t} = \mathbf{E}_\varepsilon(t,\mathbf{x}) - B(\mathbf{x})\dfrac{\mathbf{v}^\perp}{\varepsilon}. \end{cases} \tag{5}$$

Assume that $\mathbf{E}(\mathbf{x}) = -\nabla_{\mathbf{x}}\Phi(\mathbf{x}), \forall \mathbf{x} \in \Omega$, we

- decompose into slow variables $(\mathbf{x}, e)$ with $e = \frac{1}{2}\|\mathbf{v}\|^2$ and fast variable $\mathbf{v}$;
- pass to the regime $\varepsilon \ll 1$ on the slow variables $(\mathbf{x}, e)$, which lead to the new system [1] of

$$\begin{cases} \mathbf{y} = \mathbf{x} - \varepsilon\,\dfrac{\mathbf{v}^\perp}{B(\mathbf{x})} + \mathcal{O}(\varepsilon^2), \\ g = e - \varepsilon\,\mathbf{E}\cdot\dfrac{\mathbf{v}^\perp}{B(\mathbf{x})} + \mathcal{O}(\varepsilon^2), \end{cases} \quad \textit{solving} \quad \begin{cases} \dfrac{\mathrm{d}\,\mathbf{y}}{\mathrm{d}\,t} = -\,\varepsilon\left(\dfrac{\mathbf{E}^\perp}{B}(\mathbf{y}) - g\,\dfrac{\nabla_{\mathbf{x}}^\perp B}{B^2}(\mathbf{y})\right), \\ \dfrac{\mathrm{d}\,g}{\mathrm{d}\,t} = \varepsilon\,g\,\mathbf{E}\cdot\dfrac{\nabla_{\mathbf{x}}^\perp B}{B^2}(\mathbf{y}). \end{cases}$$

[1] F. Filbet, L. M. Rodrigues, 2017, and F. Filbet, L. M. Rodrigues, K. H. Trinh, 2025, 2026

Hamiltonian structure of Vlasov-Poisson

The Vlasov - Poisson system has an Hamiltonian structure [2] that conserves the invariant for all $t \geq 0$

$$\mathcal{E}(t, \mathbf{x}, \mathbf{v}) \;:=\; e + \Phi(\mathbf{x}).$$

Adiabatic invariant when $\varepsilon \to 0$

In the asymptotic limit $\varepsilon \to 0$, the adiabatic invariant [3] $\mu(t)$ is preserved with time $t \geq 0$.

$$\mu(t) \;=\; \frac{g}{B(\mathbf{x})},$$

[2] F. Casas, N. Crouseilles, E. Faou, M. Mehrenberger, (2015)

[3] F. Filbet, L. M. Rodrigues, (2020)

Some modified Crank-Nicolson scheme to solve the characteristic curves

- J. U. Brackbill, D. W. Forslund, H. X. Vu : Accurate numerical solution of charged particle motion in a Magnetic field (1995).
- T.C. Genoni, R.E. Clark, D.R. Welch , A Fast Implicit Algorithm for Highly Magnetized Charged Particle Motion (2010).
- L.F. Ricketson, L. Chacón , An energy-conserving and asymptotic-preserving charged-particle orbit implicit time integrator for arbitrary electromagnetic fields (2020)

A modified Crank-Nicolson scheme is proposed to solve the motion of one particle

- F. Filbet, L. M. Rodrigues, K. H. Trinh, A modified Crank-Nicolson scheme for the Vlasov-Poisson system with a strong external magnetic field, (2026),

New modified Crank-Nicolson schemes (1)

Approach: We use a strategy of Filbet-Rodrigues [4] by using a new variable $(\mathbf{x}, \mathbf{v}) \Rightarrow (\mathbf{x}, \mathbf{w}, e)$ since $e^n \neq \frac{1}{2}\|\mathbf{w}^n\|^2$

$$\begin{cases} \dfrac{\mathbf{x}^{n+1} - \mathbf{x}^n}{\Delta t} = \mathbf{w}^{n+1/2}, \\ \dfrac{e^{n+1} - e^n}{\Delta t} + \dfrac{\Phi(\mathbf{x}^{n+1}) - \Phi(\mathbf{x}^n)}{\Delta t} = 0, \\ \dfrac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} = \mathbf{E}^{n+1/2} - \chi^{n+1/2} \dfrac{\nabla_{\mathbf{x}} B^{n+1/2}}{B^{n+1/2}} - B^{n+1/2} \dfrac{(\mathbf{w}^{n+1/2})^\perp}{\varepsilon}, \\ \mathbf{x}^0 = \mathbf{x}(0), \quad \mathbf{w}^0 = \mathbf{v}(0), \quad e^0 = \dfrac{1}{2}\|\mathbf{v}^0\|^2. \end{cases} \tag{6}$$

Based on the idea that [5] [6]

$$\chi^{n+1/2} \xrightarrow[\varepsilon \ll 1]{fixed\,\Delta t} g^{n+1/2} \text{ and } \chi^{n+1/2} \xrightarrow[\Delta t \to 0]{fixed\,\varepsilon} 0,$$

And the function χ

$$\chi^{n+1/2} = max\left(e^{n+1/2} - \frac{1}{2}\|\mathbf{w}^{n+1/2}\|^2, 0\right).$$

[4] F. Filbet, L.M.Rodrigues, (2017)

[5] FJ. U. Brackbill, D. W. Forslund, H. X. Vu (1995)

[6] L.F. Ricketson, L. Chacón (2020, 2023)

New modified Crank-Nicolson scheme (2)

Theorem: (Consistency in the regime $\varepsilon \ll 1$ for fixed Δt)

We assume that $(\mathbf{x}^n, \mathbf{w}^n, e^n)_{n\in\mathbb{N}}$ be the solution of the CN scheme. Then for any $1 \leq n \leq N_T$

$$\|(\mathbf{x}^n, e^n) - (\mathbf{y}^n, g^n)\| \leq C(\varepsilon + \Delta t^2)$$

where $(\mathbf{y}^n, g^n)$ solves

$$\begin{cases} \dfrac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = -\varepsilon \left(\dfrac{\mathbf{E}^\perp}{B}(\mathbf{y}^{n+1/2}) - g^{n+1/2} \dfrac{\nabla_{\mathbf{y}}^\perp B}{B^2}(\mathbf{y}^{n+1/2}) \right), \\ \dfrac{g^{n+1} - g^n}{\Delta t} = \varepsilon\, g^{n+1/2} \mathbf{E} \cdot \dfrac{\nabla_{\mathbf{y}}^\perp B}{B^2}(\mathbf{y}^{n+1/2}). \end{cases}$$

Furthermore, the total energy $\mathcal{E}^n := g^n + \phi(\mathbf{y}^n)$ and the adiabatic invariant $\mu^n := g^n / b(\mathbf{y}^n)$ satisfy

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} = 0, \qquad \frac{\mu^{n+1} - \mu^n}{\Delta t} = \mathcal{O}\left(\Delta t^2\right).$$

Conclusion: The modified CN scheme preserves the structure of asymptotic regime $\varepsilon \ll 1$.

Study on the full model 3D $\times$ 3D

Slow variables and invariants

We decompose $(\mathbf{x}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ into slow variables $(\mathbf{x}, v_\parallel, e)$ with $e = \frac{1}{2}\|\mathbf{v}\|^2$ and fast variables $\mathbf{v}_\perp$; where

$$v_\parallel := \langle \mathbf{v}, \mathbf{e}_\parallel(\mathbf{x}) \rangle, \quad \mathbf{v}_\perp := \mathbf{v} - v_\parallel \mathbf{e}_\parallel(\mathbf{x}),$$

Invariants

- The Vlasov - Poisson system has an Hamiltonian structure [7] that conserves the invariant for all $t \geq 0$

$$\mathcal{E}(t, \mathbf{x}, \mathbf{v}) := e + \Phi(\mathbf{x}).$$

- In the asymptotic limit $\varepsilon \to 0$, the magnetic moment [8] $\mu(t)$ is preserved with time $t \geq 0$.

$$\mu(t) = \frac{e_\perp}{B(\mathbf{x})}, \quad \textit{where} \quad e_\perp := e - \frac{1}{2}|v_\parallel|^2.$$

[7] F. Casas, N. Crouseilles, E. Faou, M. Mehrenberger, (2015)
[8] F. Filbet, L. M. Rodrigues, (2020)

A modified Crank-Nicolson schemes (1)

Following the strategy [9], we use a new variable $(\mathbf{x}, \mathbf{v}) \Rightarrow (\mathbf{x}, \mathbf{w}, e)$

$$\begin{cases} \dfrac{\mathbf{x}^{n+1} - \mathbf{x}^n}{\Delta t} = \mathbf{w}^{n+1/2}, \\ \dfrac{e^{n+1} - e^n}{\Delta t} + \dfrac{\Phi(\mathbf{x}^{n+1}) - \Phi(\mathbf{x}^n)}{\Delta t} = 0, \\ \dfrac{\mathbf{w}^{n+1} - \mathbf{w}^n}{\Delta t} = \mathbf{E}(\mathbf{x}^{n+1/2}) - \chi^{n+1/2} \dfrac{\nabla_{\mathbf{x}} B}{B}(\mathbf{x}^{n+1/2}) + \dfrac{B(\mathbf{x}^{n+1/2})}{\varepsilon} \mathbf{w}^{n+1/2} \wedge \mathbf{e}_{\parallel}(\mathbf{x}^{n+1/2}), \\ \mathbf{x}^0 = \mathbf{x}(0), \quad \mathbf{w}^0 = \mathbf{v}(0), \quad e^0 = \dfrac{1}{2}\|\mathbf{w}^0\|^2. \end{cases} \tag{7}$$

where the function $\chi^{n+1/2}$

$$\chi^{n+1/2} = \max\left(e^{n+1/2} - \frac{1}{2}\|\mathbf{w}^{n+1/2}\|^2, 0\right).$$

[9] F. Filbet, L.M.Rodrigues, (2017), F. Filbet, L.M.Rodrigues, KH Trinh (2025)

A modified Crank-Nicolson scheme (2)

Proposition: (Consistency in the regime $\varepsilon \ll 1$ for fixed Δt)

We assume that $(\mathbf{x}^n, \mathbf{w}^n, e^n)_{n\in\mathbb{N}}$ be the solution of the CN scheme. Then for any $1 \leq n \leq N_T$,

- the slow variable solution $(\mathbf{x}^n, w_\parallel^n, e^n)_{n\in\mathbb{N}}$ is consistent with the solution $(\mathbf{y}^n, u_\parallel^n, g^n)_{n\in\mathbb{N}}$ of asymptotic model with an error of $\varepsilon + \Delta t^2$ [10],
- the total energy $\mathcal{E}^n := g^n + \Phi(\mathbf{y}^n)$ is conserved for all $\varepsilon > 0$, i.e.

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} = 0,$$

- the magnetic moment $\mu^n := \dfrac{g_\perp^n}{B(\mathbf{y}^n)}$, where $g_\perp^n = g^n - \frac{1}{2}|u_\parallel^n|^2$ satisfies

$$\frac{\mu^{n+1} - \mu^n}{\Delta t} = \mathcal{O}\left(\Delta t^2\right).$$

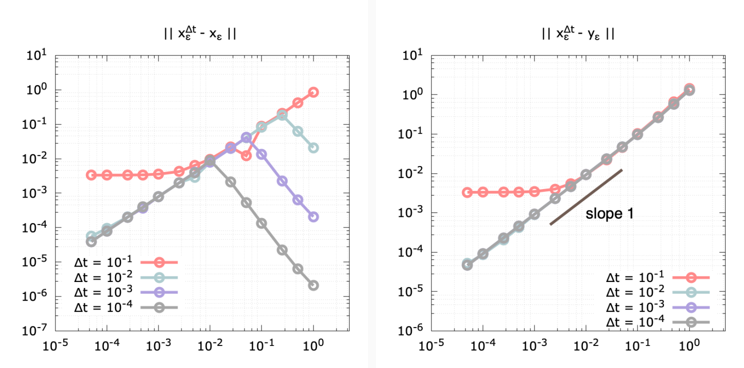
[10] F. Filbet, L.M.Rodrigues and KH Trinh (2026)

Numerical simulation

Numerical error

Tokamak-like equilibrium [11] with various $\varepsilon > 0$ and various $\Delta t > 0$.

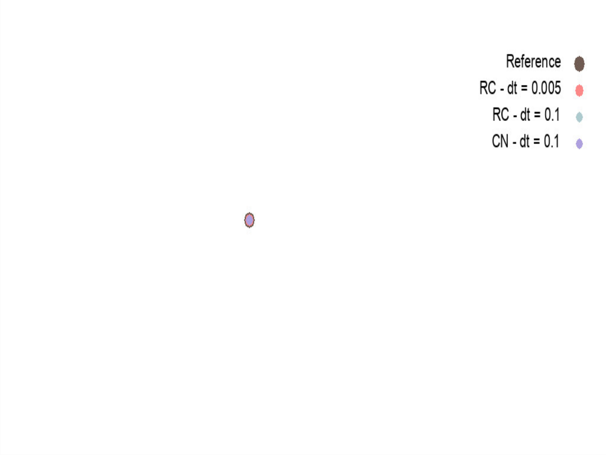
Numerical errors $\|\mathbf{x}^n - \mathbf{x}(t^n)\|$ (left) and $\|\mathbf{x}^n - \mathbf{y}(t^n)\|$ (right).



[11] F. Filbet, L. M. Rodrigues, 2023.

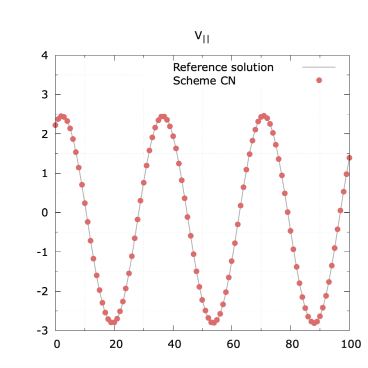
Trajectory of particle

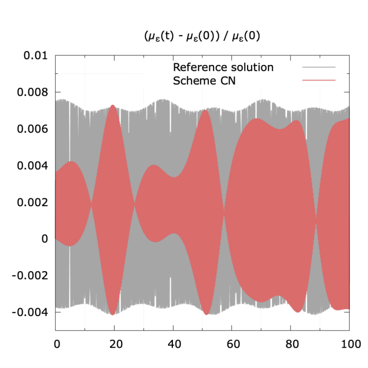
Tokamak-like equilibrium [12] with $\varepsilon = 0.005$ and the final time $T = 650s$.



[12] F. Filbet, L. M. Rodrigues, 2023.

Numerical simulation of $v_{||}$ (left) and μ (right) by the new scheme CN.

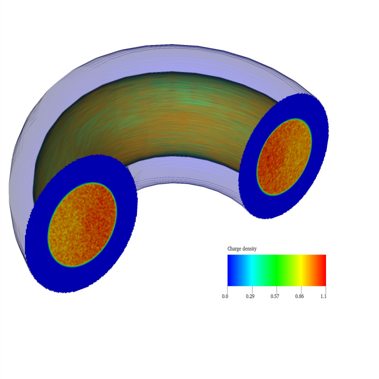


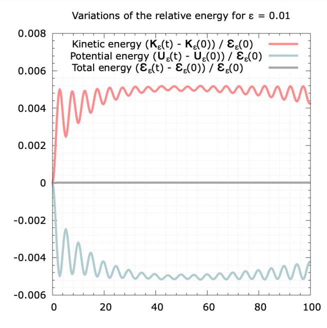


Numerical simulations of Vlasov-Poisson system in a torus

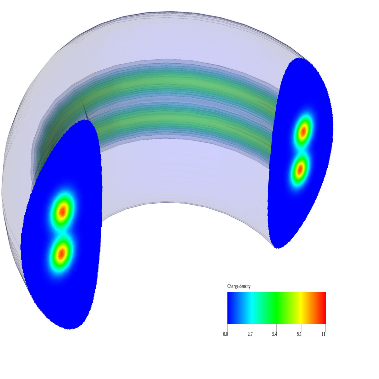
Vlasov-Poisson system

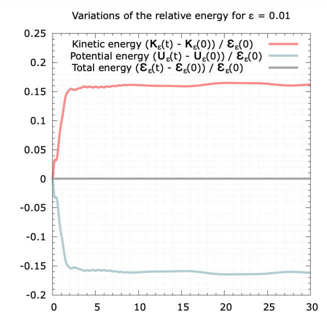
Tokamak-like equilibrium with $\varepsilon = 0.01$ and the time step $\Delta t = 0.1$.





Tokamak-like equilibrium with $\varepsilon = 0.01$ and the time step $\Delta t = 0.1$..





Conclusion

Conclusion

Single particle:

- The new modified CN scheme provides second order approximation of the asymptotic limit of slow variables, adiabatic invariant, and conservation of the energy.
- The scheme gives better approximation than the previous schemes for long time simulation with coarse time step Δt.

Vlasov-Poisson system:

- The proposed scheme provides a long stable simulation for nonhomogeneous magnetic field and the energy conservation.