Spectral approach for a homogenization problem using boundary integral operators

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The following multi-phase homogenization problem in conductivity is considered : find a periodic function $u \in H^1(\Omega)$ satisfying

$$\begin{cases} -\nabla \cdot (C(\nabla u + \overline{\varepsilon})) = 0 \text{ on } \Omega \\ \langle u \rangle_\Omega = 0 \\ \partial_n u \text{ is antiperiodic on } \partial\Omega, \end{cases}$$

where $\Omega :=]0,1[^d$, $\overline{\varepsilon}$ is a given constant vector. We assume that each phase is made of a material with constant conductivity, i.e. $C := \sum_{k=1}^{p} C_k \mathbb{1}_{\omega_k} > 0$. It is known [1] that u is the unique solution of an equation involving a volume integral operator known as Lippmann-Schwinger and denoted by S. This implies, with additional assumptions, that u may be expressed as

$$u = \sum_{k=1}^{+\infty} \frac{\overline{\varepsilon} \cdot \langle C \nabla \phi_k \rangle_\Omega}{1 - \lambda_k} \phi_k,$$

where (λ_k, ϕ_k) are eigenpairs of S.
In that presentation, we will show that an alternative approach based on boundary integral operators associated to the interface $\gamma := \partial\omega_1 \cup \ldots \cup \partial\omega_p$ can be used. An introduction to such operators in the *not so common* context of periodic boundary conditions will be presented, with a focus on the single layer operator denoted by $\mathscr{S}_\gamma$. This will lead to a representation formula of the form

$$u = \mathscr{S}_\gamma q,$$

with $q \in H^{-\frac{1}{2}}(\gamma)$. We will show that the problem is equivalent to a boundary integral equation involving the so-called Neumann-Poincaré operator, denoted by K'_γ. As a consequence, when there are only two phases, i.e. $p = 2$, the spectral properties of K'_γ can be used to derive the following explicit expansion formula

$$u = \sum_{k=0}^{+\infty} \frac{\langle \overline{\varepsilon} \cdot n, e_k \rangle}{\alpha - \mu_k} \mathscr{S}_\gamma e_k,$$

(μ_k, e_k) being eigenpairs of K'_γ. This result is significantly stronger than the previous one, which is one of the benefits of this approach. The eigenpairs (μ_k, ϕ_k) can be observed in plasmonic resonance phenomena too, which highlights an interesting connection between the two fields. In addition, they will be explored numerically and their links with those of S will be presented.

[1] C. Bellis, H. Moulinec. *Lippmann-Schwinger spectrum, composite materials eigenstates and their role in computational homogenization.* International Journal for Numerical Methods in Engineering, **126(19)**, 38, 2025. doi:10.1002/nme.70130.

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