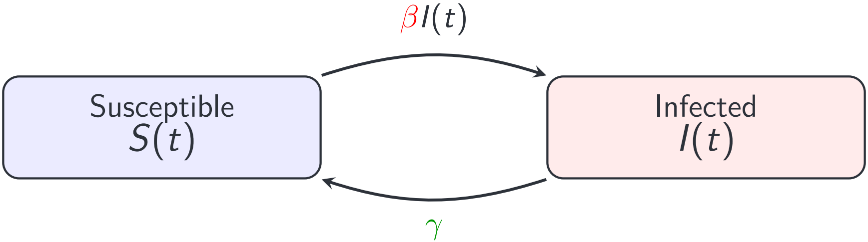
Spectral analysis and global dynamics of a structured SIS model with mutation

Enriqué DEBRABANT, Université de Tours
Supervised by Pierre Gabriel and Sten Madec

CANUM 2026, June 4th 2026

Classical SIS dynamics



The classical SIS model

Classical SIS dynamics

$$\begin{cases} S'(t) = \gamma I(t) - \beta S(t)I(t), \\ I'(t) = \beta S(t)I(t) - \gamma I(t). \end{cases}$$

Interpretation

$S(t)$: proportion of susceptible individuals;

$I(t)$: proportion of infected individuals;

$\beta > 0$: transmission rate;

$\gamma > 0$: recovery rate.

Reduction to one equation

Conservation

$$S'(t) + I'(t) = 0 \qquad \Longrightarrow \qquad S(t) + I(t) = 1.$$

Closed equation

Using $S(t) = 1 - I(t)$, the system becomes

$$\boxed{I'(t) = (\beta - \gamma)I(t) - \beta I(t)^2.}$$

Equilibria and explicit solutions

Explicit solution

For $I(0) = I_0$ and $\beta \neq \gamma$,

$$I(t) = \frac{(\beta - \gamma) I_0}{\beta I_0 + (\beta - \gamma - \beta I_0) e^{-(\beta-\gamma)t}}.$$

If $\beta = \gamma$,

$$I(t) = \frac{I_0}{1 + \beta I_0 t}.$$

Equilibria

$$I_0^* = 0, \qquad I_e^* = 1 - \frac{\gamma}{\beta} \quad (\beta > \gamma).$$

Numerical simulation

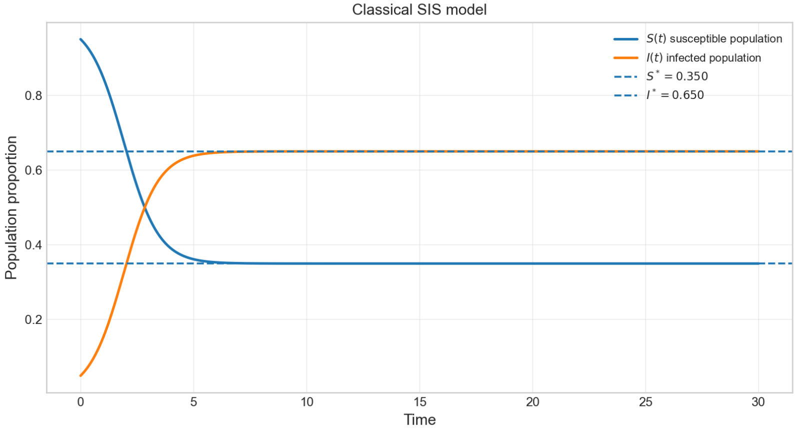


Figure: Convergence to the endemic equilibrium

Why introduce structure?

Classical SIS

$$I(t)$$

β, γ constant

all infected individuals are treated identically

Structured SIS

$$I(t,x), \qquad x \in \Omega$$

$$k(x,y), \qquad \gamma(x)$$

different strains may transmit, mutate and recover differently

Mutation-selection SIS model

Evolution equation

$$\partial_t I(t,x) = S(t) \int_\Omega k(x,y) I(t,y)\, dy - \gamma(x) I(t,x).$$

Interpretation

- $k(x,y)$: transmission from trait y to trait x, including mutation;
- $\gamma(x)$: recovery rate of individuals infected by trait x;
- $S(t)$: proportion of susceptible individuals,
- $I(t,x)$: proportion of infected by trait x individuals.

Since the total population is conserved, we normalize

$$S(t) + \int_\Omega I(t,x)\,dx = 1.$$

Closed equation on I

$$\partial_t I(t,x) = (1 - m(t)) \int_\Omega k(x,y) I(t,y)\,dy - \gamma(x) I(t,x),$$

where

$$m(t) := \int_\Omega I(t,x)\,dx.$$

$$\boxed{S(t) = 1 - m(t).}$$

The spectral family L_m

Definition

For each fixed infected mass $m \in [0,1]$, define

$$L_m\varphi = (1-m)K\varphi - \Gamma\varphi,$$

that is

$$L_m\varphi(x) = (1-m)\int_\Omega k(x,y)\varphi(y)\,dy - \gamma(x)\varphi(x).$$

Interpretation

The operator L_m describes the linear infection dynamics when the total infected mass is fixed at level m.

Endemic equilibria and spectral bound

Endemic equilibrium

An endemic equilibrium $I^* \geq 0$, $I^* \neq 0$ satisfies

$$0 = (1 - m^*)KI^* - \Gamma I^*, \qquad m^* = \int_\Omega I^*(x)\,dx.$$

Equivalently,

$$L_{m^*} I^* = 0.$$

Definition

$$s(L_m) := \sup\{Re(\lambda) \text{ for } \lambda \text{ in the spectrum of } L_m\}.$$

Possible equilibrium masses are selected by $s(L_m) = 0$.

Monotonicity and saturation

The map

$$m \longmapsto s(L_m)$$

is continuous, non-increasing and may saturate at the level $-\underline{\gamma}$.

Regular case

If $\underline{\gamma} > 0$, a plateau may occur, but only below zero:

$$s(L_m) = -\underline{\gamma} < 0.$$

Thus $s(L_m) = 0$ still selects a unique mass.

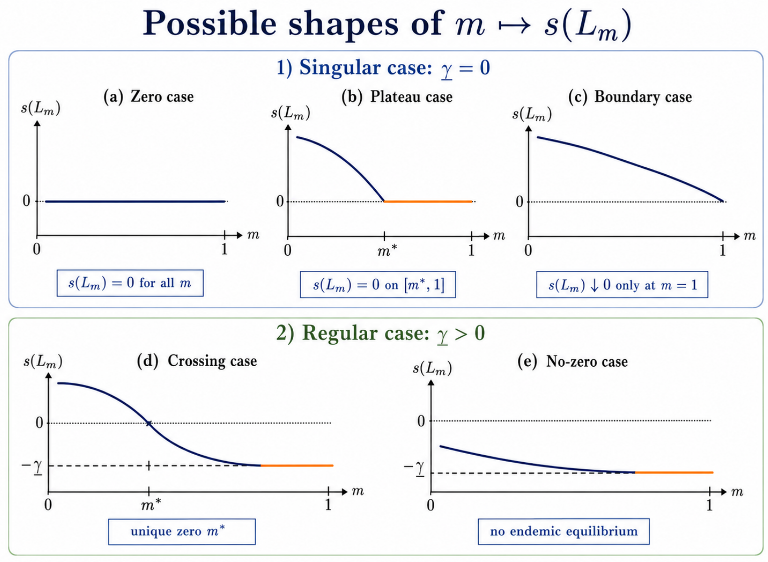
Singular case

If $\underline{\gamma} = 0$, the saturation level is zero:

$$s(L_m) = 0 \quad \text{for } m \in [\underline{m}, 1].$$

Hence several endemic masses may be admissible.

Illustration of the possible map $m \mapsto s(L_m)$



Numerical illustration: spectral plateau

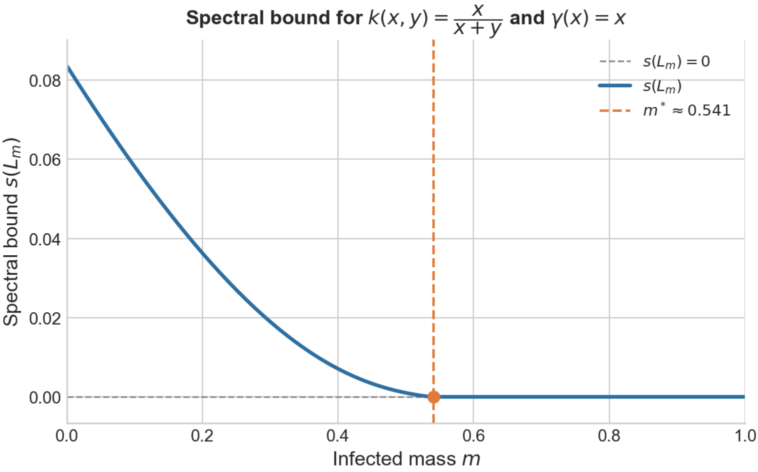


Figure: A spectral plateau may appear in the singular case.

Disease-free equilibrium

Linearization

The disease-free equilibrium is

$$I \equiv 0.$$

The linearized operator is

$$L_0 = K - \Gamma.$$

Threshold principle

$$s(L_0) < 0 \implies \text{extinction},$$
$$s(L_0) > 0 \implies \text{persistence}.$$

Epidemiological meaning of $\mathcal{R}_0$

Basic reproduction number

Average number of secondary infections from one individual in a fully susceptible population.

$\mathcal{R}_0 < 1$ — extinction

Each infection produces *less than one* new case on average.
$\implies$ disease dies out.

$\mathcal{R}_0 > 1$ — invasion

Each infection produces *more than one* new case on average.
$\implies$ epidemic spreads.

SIS classic model

In the first model we define $\mathcal{R}_0$ by the formula $\mathcal{R}_0 := \frac{\beta}{\gamma}$.

Epidemiological meaning of $\mathcal{R}_0$

Structured populations

When infections are distributed over traits $x \in \Omega$, the quantity $\mathcal{R}_0$ is no longer a scalar rate ratio, but the **spectral radius of a positive operator**:

$$\mathcal{R}_0 = r(K\Gamma^{-1}) \quad \text{with} \quad K\Gamma^{-1}[f](x) = \int_\Omega \frac{k(x,y)}{\gamma(y)} f(y)dy.$$

Structured populations and singular case

If $\underline{\gamma} = 0$, the operator $K\Gamma^{-1}$ may not be well defined. We use instead the generalized quantity

$$\mathcal{R}_0 := \lim_{\lambda \downarrow 0} r\left(K(\lambda I + \Gamma)^{-1}\right).$$

Relation with the reproduction number

Regular case (Horst R Thieme, 2009).

If $\underline{\gamma} > 0$, the next-generation operator $\mathcal{R}_0 = r(K\Gamma^{-1})$ is well defined. Then

$$s(L_0) \gtrless 0 \quad \Longleftrightarrow \quad \mathcal{R}_0 \gtrless 1.$$

Singular case

If $\underline{\gamma} = 0$, the generalized reproduction number give us the following result :

$$s(L_0) = 0 \;\Leftrightarrow\; R_0 \leq 1, \quad \text{and} \quad s(L_0) > 0 \;\Leftrightarrow\; R_0 > 1.$$

Long-time dynamics according to $s(L_0)$

$$m(t) := \int_\Omega I(t,x)\,dx.$$

Regime	Assumption	Conclusion
Subcritical	$s(L_0) < 0,\ \underline{\gamma} > 0$	$m(t) \to 0$ exponentially
Critical	$s(L_0) = 0,\ \underline{\gamma} > 0$	$m(t) = O(t^{-1})$
Supercritical	$s(L_0) > 0$	$\liminf_{t\to+\infty} m(t) > 0$

Singular subcritical case

When $\underline{\gamma} = 0$, the extinction result for $s(L_0) = 0$ is not obtained by the present argument.

A spectral Lyapunov functional

Let $\Phi > 0$ satisfy $L_0^*\Phi = s(L_0)\Phi$ and set $Y(t) := \langle \Phi, I(t) \rangle$. Then

$$Y'(t) = s(L_0)Y(t) - m(t)\langle K^*\Phi, I(t)\rangle.$$

Subcritical case

If $s(L_0) < 0$, then

$$Y'(t) \leq s(L_0)Y(t),$$

hence

$$Y(t) \to 0, \qquad m(t) \to 0.$$

Supercritical case

If $s(L_0) > 0$, then

$$Y'(t) \geq \big(s(L_0) - Cm(t)\big)Y(t).$$

Therefore the infection cannot vanish.

Zero-recovery set

In the singular case, some strains may have no recovery:

$$Z := \{x \in \Omega : \gamma(x) = 0\}.$$

Main difficulty

When $\underline{\gamma} = 0$, endemic equilibria may no longer be regular densities in $L^1(\Omega)$. They may appear as measures:

$$\mu^* \in \mathcal{M}_+(\Omega).$$

Vanishing recovery may produce concentration phenomena.

Concentration of equilibria

Let

$$\mu^* = \mu^*_{ac} + \mu^*_{sing}$$

be the Lebesgue decomposition of an equilibrium measure.

Localization of the singular part

Any singular part must be supported on the zero-recovery set:

$$\mathrm{supp}(\mu^*_{sing}) \subset Z.$$

Singularities can only occur on incurable strains.

Zero-recovery set

The theory predicts that singular components can only concentrate on Z.

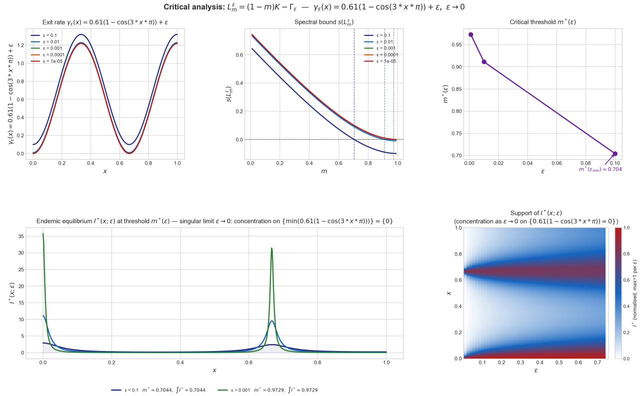


Figure: Concentration of the equilibrium near the zero-recovery set.

Idea of the proof: concentration on Z

Stationary equation

An equilibrium measure μ^* satisfies

$$(1 - m)K\mu^* = \Gamma\mu^*.$$

Regularizing effect

Since $K\mu^*$ is absolutely continuous, the measure $\Gamma\mu^*$ is also absolutely continuous.

Consequence

On $\Omega \setminus Z$, one has $\gamma(x) > 0$, so μ^* is absolutely continuous there. Therefore

$$\mathrm{supp}(\mu^*_{\mathrm{sing}}) \subset Z.$$

Infectious potential

Define the infectious potential

$$B(y) := \int_{\Omega \setminus Z} \frac{k(x,y)}{\gamma(x)}\, dx, \qquad \overline{B} := \sup_{y \in \Omega} B(y).$$

Dichotomy

- $B(y) = +\infty$ for all $y \in Z \implies \int_Z d\mu^* = 0.$
- $0 < \overline{B} < +\infty \implies$ for $m > \frac{\overline{B} - 1}{\overline{B}}$, $\int_Z d\mu^* > 0.$

Regular approximation of the singular regime

$$\gamma_\varepsilon(x) = \gamma(x) + \varepsilon, \qquad \varepsilon \downarrow 0.$$

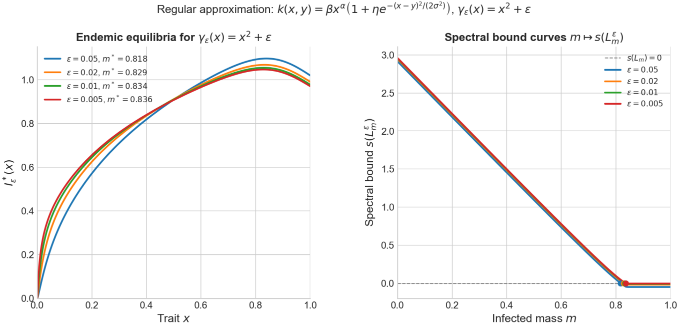


Figure: Regularized equilibria and spectral bounds as $\varepsilon \to 0$.

Conclusion

Main message

$s(L_0)$ is the relevant epidemic threshold.

- The nonlinear SIS dynamics is governed by the spectral bound of L_0.
- In the regular case, this recovers the classical $\mathcal{R}_0$-threshold.
- In the singular case, spectral plateaus and measure-valued equilibria may appear.
- Singular components concentrate on the zero-recovery set

$$Z = \{\gamma = 0\}.$$

Further directions

- Prove convergence of the total infected mass $m(t)$ toward an endemic mass m^*.
- Prove convergence of the infected distribution toward an endemic equilibrium:

$$I(t, \cdot) \longrightarrow I^*.$$

- Perturbations of mutation-selection operators.
- Extension to co-infection system.

Thank you for your attention

Questions?