

Enriching continuous Lagrange finite element approximation spaces using neural networks

Michel Duprez¹, Emmanuel Franck², **Frédérique Lecourtier**¹ and Vanessa Lleras³

¹Project-Team MIMESIS, Inria, Strasbourg

²Project-Team MACARON, Inria, Strasbourg

³IMAG, University of Montpellier, Montpellier

Joint work with:

H. Barucq, F. Faucher, N. Victorion and V. Michel-Dansac.

MS : "Perspectives et avancées récentes en apprentissage pour la résolution numérique d'équations aux dérivées partielles"

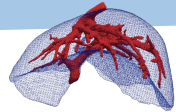


CANUM 2026



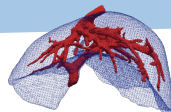
Scientific context

Context: Create real-time digital twins of an organ (e.g. liver).



Objective: Develop an hybrid finite element / neural network method.
accurate quick + parameterized

Scientific context



Context: Create real-time digital twins of an organ (e.g. liver).

Objective: Develop an hybrid finite element / neural network method.
accurate quick + parameterized

Parametric Poisson problem (with homogeneous Dirichlet BC):

For one or several $\mu \in \mathcal{M}$, find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\Delta u(\mathbf{x}; \mu) = f(\mathbf{x}; \mu), & (\mathbf{x}, \mu) \in \Omega \times \mathcal{M}, \\ u(\mathbf{x}; \mu) = 0, & (\mathbf{x}, \mu) \in \partial\Omega \times \mathcal{M}, \end{cases} \quad (\mathcal{P})$$

with $\Omega \subset \mathbb{R}^d$ a domain ($d = 1, 2, 3$) and $\mathcal{M} \subset \mathbb{R}^p$ the parameter space (p the number of parameters).

Classical approaches

Physics-Informed Neural Networks¹

Standard PINNs: Find the optimal weights θ^* , such that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(\omega_r J_r(\theta) + \omega_b J_b(\theta) \right), \quad (\mathcal{P}_\theta)$$

with

residual loss

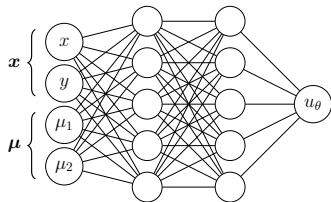
$$J_r(\theta) = \int_{\mathcal{M}} \int_{\Omega} |\Delta u_\theta(\mathbf{x}, \boldsymbol{\mu}) + f(\mathbf{x}, \boldsymbol{\mu})|^2 dx d\boldsymbol{\mu},$$

boundary loss

$$J_b(\theta) = \int_{\mathcal{M}} \int_{\partial\Omega} |u_\theta(\mathbf{x}, \boldsymbol{\mu})|^2 dx d\boldsymbol{\mu},$$

with ω_r and ω_b are some weights.

u_θ is a neural network, e.g. fully-connected NN.
(see example in 2D with 2 parameters)



Monte-Carlo method: Discretize the cost functions by random process.

¹[Raissi, Perdikaris, and Karniadakis, 2019]

Finite Element Method¹

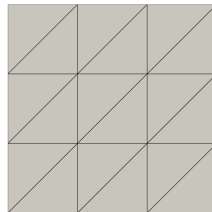
Variational Problem:

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

with h the mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h^0 a continuous, piecewise polynomial space of degree k .



Cartesian mesh

¹[Ern and Guermond, 2004]

Finite Element Method¹

Variational Problem:

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

with h the mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h^0 a continuous, piecewise polynomial space of degree k .

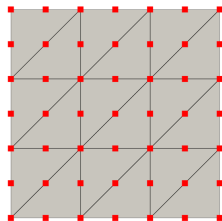
Linear system: Let $(\phi_1, \dots, \phi_{N_{\text{dofs}}})$ a basis of V_h^0 .

Find $U \in \mathbb{R}^{N_{\text{dofs}}}$ such that

$$AU = b,$$

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}} \times N_{\text{dofs}}},$$

$$b = (l(\phi_j))_{1 \leq j \leq N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}}}.$$



■ degrees of freedom ($k = 2$)

¹[Ern and Guermond, 2004]

Finite Element Method¹

Variational Problem:

$$\text{Find } u_h \in V_h^0 \text{ such that, } \forall v_h \in V_h^0, a(u_h, v_h) = l(v_h), \quad (\mathcal{P}_h)$$

with h the mesh size, a and l the bilinear and linear forms given by

$$a(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h, \quad l(v_h) = \int_{\Omega} f v_h,$$

and V_h^0 a continuous, piecewise polynomial space of degree k .

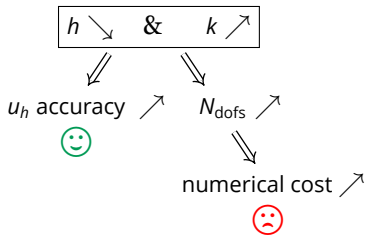
Linear system: Let $(\phi_1, \dots, \phi_{N_{\text{dofs}}})$ a basis of V_h^0 .

Find $U \in \mathbb{R}^{N_{\text{dofs}}}$ such that

$$AU = b,$$

$$A = (a(\phi_i, \phi_j))_{1 \leq i, j \leq N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}} \times N_{\text{dofs}}},$$

$$b = (l(\phi_j))_{1 \leq j \leq N_{\text{dofs}}} \in \mathbb{R}^{N_{\text{dofs}}}.$$



¹[Ern and Guermond, 2004]

Enriched finite element method using PINNs

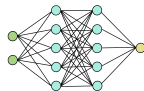
Pipeline of the Enriched FEM

OFFLINE : PINN training

Inputs

$\mathbf{x} \in \Omega$: space coordinates

$\boldsymbol{\mu} \in \mathcal{M}$: parameter



Parametric PINN



Output

$u_{\theta}(\mathbf{x}, \boldsymbol{\mu})$: prediction of the PDE solution $u(\mathbf{x}, \boldsymbol{\mu})$

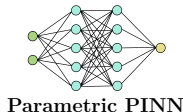
Pipeline of the Enriched FEM

OFFLINE : PINN training

Inputs

$\mathbf{x} \in \Omega$: space coordinates

$\boldsymbol{\mu} \in \mathcal{M}$: parameter



Output

$u_\theta(\mathbf{x}, \boldsymbol{\mu})$: prediction of the PDE solution $u(\mathbf{x}, \boldsymbol{\mu})$

ONLINE (fixed $\boldsymbol{\mu}$) : PINN evaluation + Enriched FEM resolution

PINN evaluation

$u_\theta(\cdot, \boldsymbol{\mu})$: prediction for the given parameter $\boldsymbol{\mu}$



FEM solver
enriched by u_θ



Output

enriched FEM solution
(depending on the mesh size h)

Complete ONLINE process : quick + accurate

Additive approach

Variational Problem: Let $u_\theta \in H^{k+1}(\Omega)$ be a PINN prior.

$$\text{Find } C_h^+ \in V_h \text{ such that, } \forall v_h \in V_h^0, a(C_h^+, v_h) = l(v_h) - a(u_\theta, v_h), \quad (\mathcal{P}_h^+)$$

with the **enriched trial space** V_h^+ defined by

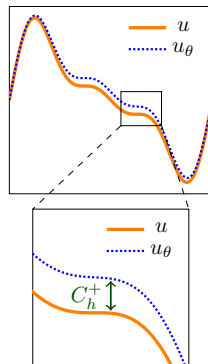
$$V_h^+ = \{u_h^+ = u_\theta + C_h^+, \quad C_h^+ \in V_h^0\}.$$

General Dirichlet BC: If $u = g$ on $\partial\Omega$, then

$$C_h^+ = g - u_\theta \quad \text{on } \partial\Omega,$$

with u_θ the PINN prior.

We expect that the modified problem will give the same results as the standard one on coarser meshes.



Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

Let $u_h \in V_h$ be the solution of (\mathcal{P}_h) with V_h the standard trial space. Then,

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Convergence analysis

Theorem 1: Convergence analysis of the standard FEM [Ern and Guermond, 2004]

Let $u_h \in V_h$ be the solution of (\mathcal{P}_h) with V_h the standard trial space. Then,

$$\|u - u_h\|_{L^2} \leq C_{L^2} h^{k+1} |u|_{H^{k+1}}.$$

Theorem 2: Convergence analysis of the enriched FEM [Lecourtier et al., 2026]

Let $u_h^+ \in V_h^+$ be the solution of (\mathcal{P}_h^+) with V_h^+ the enriched trial space. Then,

$$\|u - u_h^+\|_{L^2} \leq \frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} (C_{L^2} h^{k+1} |u|_{H^{k+1}}).$$

$$\frac{|u - u_\theta|_{H^{k+1}}}{|u|_{H^{k+1}}} < 1 \Rightarrow \text{Gains of the additive approach.}$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Consider the Poisson problem in 2D with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (-0.5\pi, 0.5\pi)^2$ and $\mathcal{M} = [-0.5, 0.5]^2$ ($p = 2$ parameters).

Analytical solution: $\mu = (\mu_1, \mu_2) \in \mathcal{M}$

$$u(\mathbf{x}, \mu) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{2}\right) \sin(2x) \sin(2y).$$

Parametric PINN training:

MLP 5 layers (sine - 40, 60, 60, 60, 40); LBFGs optimizer (5000 epochs - 6000 col. pts).

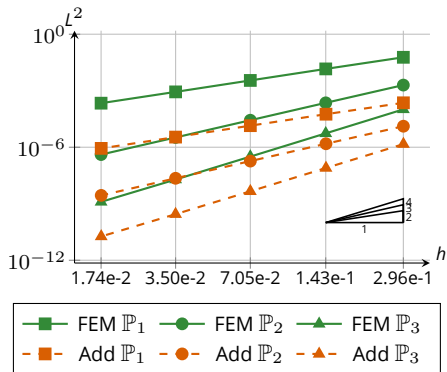
Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

$$\varphi(\mathbf{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi).$$

Online phase - 1 parameters instance

$$\mu^{(1)} = (0.05, 0.22)$$

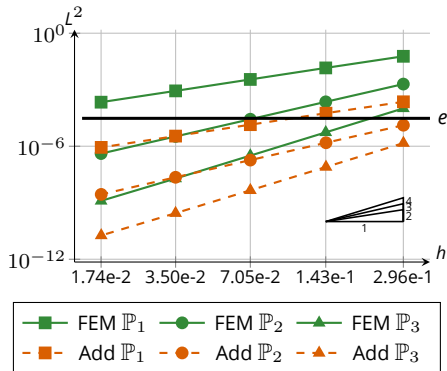
Error estimates in L^2 norm:



Online phase - 1 parameters instance

$$\mu^{(1)} = (0.05, 0.22)$$

Error estimates in L^2 norm:



N_{dofs} required to reach the same error e :

k	e	N_{dofs}	
		FEM	Add
1	$1 \cdot 10^{-3}$	14 161	64
	$1 \cdot 10^{-4}$	143 641	576
2	$1 \cdot 10^{-4}$	6 889	225
	$1 \cdot 10^{-5}$	31 329	1 089
3	$1 \cdot 10^{-5}$	6 724	784
	$1 \cdot 10^{-6}$	20 164	2 704

$$h \approx N_{\text{dofs}}^{-1/2}$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Consider the Poisson problem in 3D with Dirichlet BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (-0.5\pi, 0.5\pi)^3$ and $\mathcal{M} = [-0.5, 0.5]^3$ ($p = 3$ parameters).

Analytical solution: $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathcal{M}$

$$u(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2 + (z - \mu_3)^2}{2}\right) \sin(2x) \sin(2y) \sin(2z).$$

Parametric PINN training:

MLP 5 layers (tanh - 40, 60, 60, 60, 40); LBFGs optimizer (5000 epochs - 40000 col. pts).

Imposing the Dirichlet BC exactly in the PINN with the levelset φ defined by

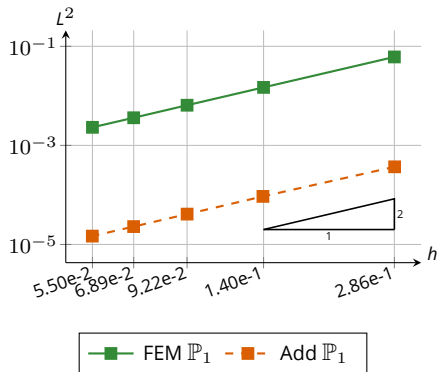
$$\varphi(\mathbf{x}) = (x + 0.5\pi)(x - 0.5\pi)(y + 0.5\pi)(y - 0.5\pi)(z + 0.5\pi)(z - 0.5\pi).$$

Training time: 12 minutes on NVIDIA RTX 2000 (8GB VRAM).

Online phase - 1 parameters instance I

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

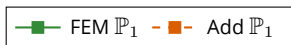
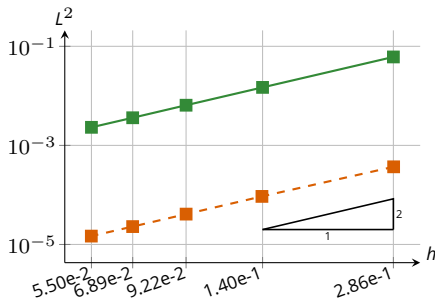
Error estimates in L^2 norm:



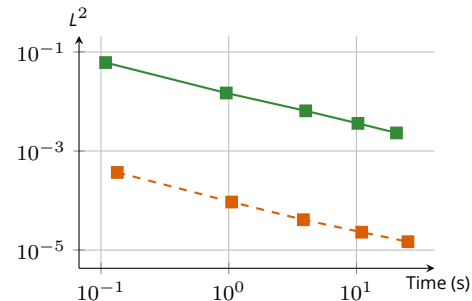
Online phase - 1 parameters instance I

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:



L^2 error w.r.t. online time*:

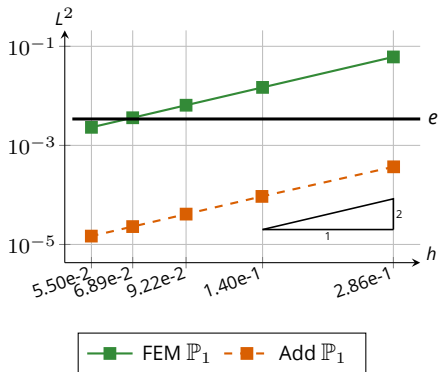


* mesh + assembly + solve
 + prior derivatives evaluation
 + training time

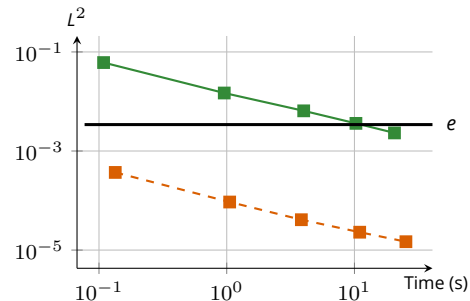
Online phase - 1 parameters instance I

$$\mu^{(1)} = (0.05, 0.22, 0.1)$$

Error estimates in L^2 norm:



L^2 error w.r.t. online time*:



* mesh + assembly + solve
+ prior derivatives evaluation
+ training time

Online phase - 1 parameters instance II

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

N_{dofs} and online time required to reach the same error e :

e	N		N_{dofs}		Online time*	
	FEM	Add	FEM	Add	FEM	Add
$1 \cdot 10^{-3}$	152	12	$3.51 \cdot 10^6$	$1.73 \cdot 10^3$	$(7.65 \cdot 10^1)$	$5.2 \cdot 10^{-2}$
$1 \cdot 10^{-4}$	484	39	$1.13 \cdot 10^8$	$5.93 \cdot 10^4$	$(2.8 \cdot 10^3)$	$9.26 \cdot 10^{-1}$
$1 \cdot 10^{-5}$	1539	122	$3.65 \cdot 10^9$	$1.82 \cdot 10^6$	$(1.02 \cdot 10^5)$	$(5.26 \cdot 10^1)$

$\div 1471 \rightarrow$

Cartesian mesh: $N_{\text{dofs}} = N^3$ nodes (\mathbb{P}_1 elements).

* The results in brackets are extrapolations.

Online phase - 1 parameters instance II

$$\boldsymbol{\mu}^{(1)} = (0.05, 0.22, 0.1)$$

N_{dofs} and online time required to reach the same error e :

e	N		N_{dofs}		Online time*	
	FEM	Add	FEM	Add	FEM	Add
$1 \cdot 10^{-3}$	152	12	$3.51 \cdot 10^6$	$1.73 \cdot 10^3$	$(7.65 \cdot 10^1)$	$5.2 \cdot 10^{-2}$
$1 \cdot 10^{-4}$	484	39	$1.13 \cdot 10^8$	$5.93 \cdot 10^4$	$(2.8 \cdot 10^3)$	$9.26 \cdot 10^{-1}$
$1 \cdot 10^{-5}$	1539	122	$3.65 \cdot 10^9$	$1.82 \cdot 10^6$	$(1.02 \cdot 10^5)$	$(5.26 \cdot 10^1)$

$\div 1471 \rightarrow$

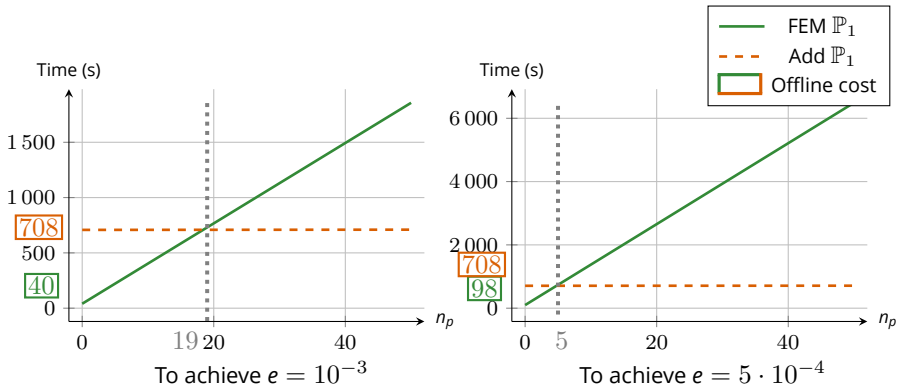
Cartesian mesh: $N_{\text{dofs}} = N^3$ nodes (\mathbb{P}_1 elements).

* The results in brackets are extrapolations.

! Training time !

Parametric framework

How many parameters instance n_p are needed to balance training?



Online: assembly + solve + prior derivatives evaluation

Offline: mesh + training time

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering an Anisotropic Elliptic problem with Dirichlet BC:

$$\begin{cases} -\operatorname{div}(D\nabla u) = f, & \text{in } \Omega \times \mathcal{M}, \\ u = 0, & \text{on } \partial\Omega \times \mathcal{M}, \end{cases}$$

with $\Omega = (0, 1)^2$ and $\mathcal{M} = [0.4, 0.6]^2 \times [0.01, 1] \times [0.1, 0.8]$ ($p = 4$ parameters).

Right-hand side and diffusion matrix: $\boldsymbol{\mu} = (\mu_1, \mu_2, \sigma, \epsilon) \in \mathcal{M}$

$$f(\mathbf{x}, \boldsymbol{\mu}) = \exp\left(-\frac{(x - \mu_1)^2 + (y - \mu_2)^2}{0.025\sigma^2}\right).$$

$$D(\mathbf{x}, \boldsymbol{\mu}) = \begin{pmatrix} \epsilon x^2 + y^2 & (\epsilon - 1)xy \\ (\epsilon - 1)xy & x^2 + \epsilon y^2 \end{pmatrix}.$$

Parametric PINN training:

MLP¹ 5 layers (tanh - 40, 60, 60, 60, 40). Adam optimizer (15000 epochs - 8000 col. pts).

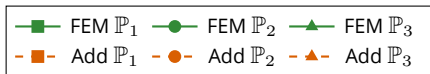
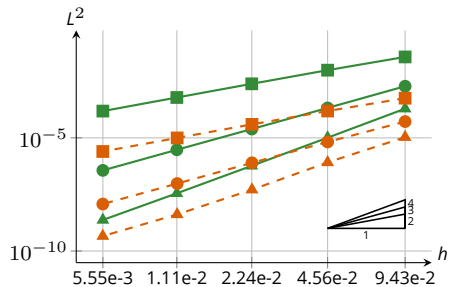
Imposing the Dirichlet BC exactly in the PINN with the a level-set function.

¹with Fourier Features [Tancik, Srinivasan, and al, 2020]

Online phase

Error estimates: 1 parameter instance

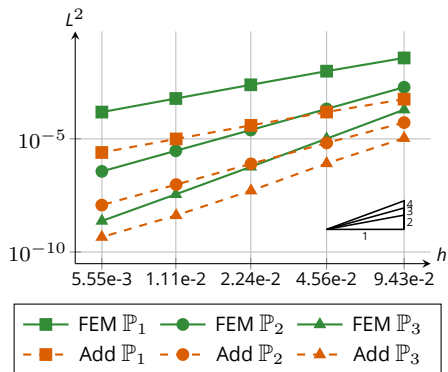
$$\mu^{(1)} = (0.51, 0.54, 0.52, 0.55)$$



Online phase

Error estimates: 1 parameter instance

$$\boldsymbol{\mu}^{(1)} = (0.51, 0.54, 0.52, 0.55)$$



Gains achieved: $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Considering $n_p = 50$ parameters instances

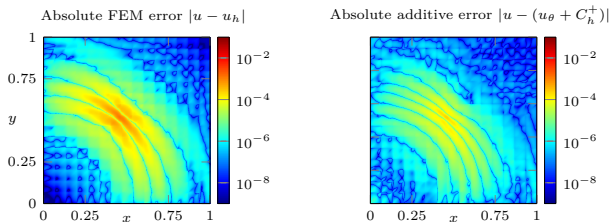
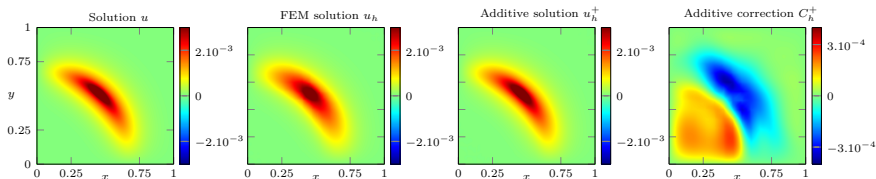
$$\mathcal{S} = \left\{ \boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(n_p)} \right\}.$$

**Gains in L^2 rel error
of our method w.r.t. FEM**

k	min	max	mean
1	7.12	82.57	35.67
2	3.54	35.88	18.32
3	1.33	26.51	8.32

Cartesian mesh: N^2 nodes with $N = 20$.

Numerical solutions and errors



$$\boldsymbol{\mu}^{(2)} = (0.46, 0.52, 0.05, 0.12) \quad (k = 2, N = 16)$$

Numerical results

2D Poisson problem on Square - Dirichlet BC

3D Poisson problem on Cube - Dirichlet BC

2D Anisotropic Elliptic problem on Square - Dirichlet BC

2D Poisson problem on Annulus - Mixed BC

Problem considered

Problem statement: Considering the Poisson problem with mixed BC:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \times \mathcal{M}, \\ u = g, & \text{on } \Gamma_E \times \mathcal{M}, \\ \frac{\partial u}{\partial n} + u = g_R, & \text{on } \Gamma_I \times \mathcal{M}, \end{cases}$$

with $\Omega = \{(x, y) \in \mathbb{R}^2, 0.25 < x^2 + y^2 < 1\}$ and $\mathcal{M} = [2.4, 2.6]$ ($p = 1$ parameter).

Analytical solution and boundary conditions: $\mu = \mu_1 \in \mathcal{M}$

$$u(\mathbf{x}; \mu) = 1 - \frac{\ln(\mu_1 \sqrt{x^2 + y^2})}{\ln(4)},$$

$$g(\mathbf{x}; \mu) = 1 - \frac{\ln(\mu_1)}{\ln(4)} \quad \text{and} \quad g_R(\mathbf{x}; \mu) = 2 + \frac{4 - \ln(\mu_1)}{\ln(4)}.$$

Parametric PINN training:

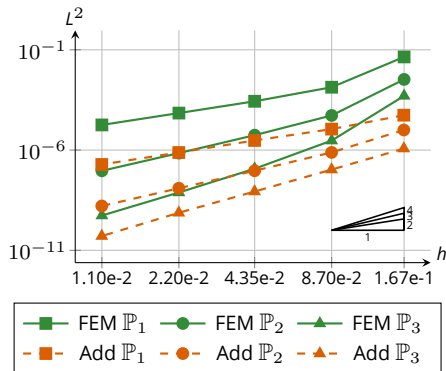
MLP 5 layers (tanh - 5×40); LBFGs optimizer (4000 epochs - 6000 col. pts).

Imposing the mixed BC exactly in the PINN¹ (+ Sobolev training).

Online phase

Error estimates: 1 parameters instance

$$\mu^{(1)} = 2.51$$



Gains achieved: $\|u - u_h\|_{L^2} / \|u - u_h^+\|_{L^2}$

Considering $n_p = 50$ parameters instances

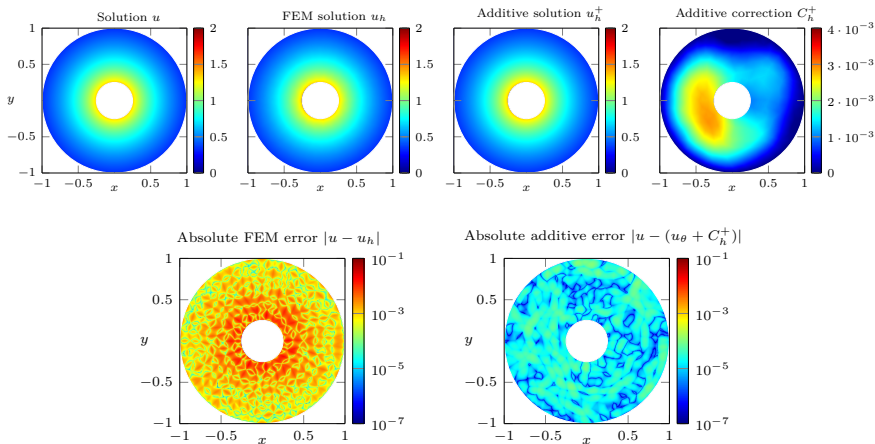
$$\mathcal{S} = \left\{ \mu^{(1)}, \dots, \mu^{(n_p)} \right\}.$$

**Gains in L^2 rel error
of our method w.r.t. FEM**

k	min	max	mean
1	15.12	137.72	55.5
2	31	77.46	58.41
3	18.72	21.49	20.6

Mesh size: $h = 1.33 \cdot 10^{-1}$

Numerical solutions and errors



$$\mu^{(1)} = 2.51 \quad (k = 1, h = 1.67 \cdot 10^{-1})$$

Conclusion

- **Offline/online framework:**
 - **Offline phase:** Training a PINN over a parametric space.
 - **Online phase:** (fixed parameter instance)
Resolution of the FEM correction problem using the PINN prior.
- **Numerical results:** Quick and accurate on several parametric PDEs.
- **Theoretical results:** Convergence analysis showing the PINN prior can improve the FEM error.

Ongoing works:

- Extension to non-linear problems : First promising results for the incompressible Navier-Stokes equations. [Appendix 1](#)
- Extensions to DG methods and non-conforming FEM, currently under development in ScimBa (SciML's Python library) using auto-differentiation



See the paper

References

- Coulaud, G., M. Le, and R. Duvigneau [2024]. "Investigations on Physics-Informed Neural Networks for Aerodynamics". In.
- Ern, A. and J.-L. Guermond [2004]. "Theory and Practice of Finite Elements". In.
- Lecourtier, F., H. Barucq, M. Duprez, F. Faucher, E. Franck, V. Lleras, V. Michel-Dansac, and N. Victorion [2026]. "Enriching continuous lagrange finite element approximation spaces using neural networks". In: ESAIM: M2AN.**
- Raissi, M., P. Perdikaris, and G. E. Karniadakis [2019]. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations". In: J. Comput. Phys.
- Sukumar, N. and A. Srivastava [2022]. "Exact imposition of boundary conditions with distance functions in physics-informed deep neural networks". In: Comput. Method. Appl. M.
- Tancik, M., P. Srinivasan, and al [2020]. "Fourier Features Let Networks Learn High Frequency Functions in Low Dimensional Domains". In.



See Scimba

Thank you for your
attention!



See the paper

Appendix 1 : Non-linear test case

PINN approach

Enriched FEM using PINN

Numerical results

A1 – Heated cavity test case

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity)¹ :

We consider $\Omega = [-1, 1]^2$ a squared domain and $\mathbf{e}_y = (0, 1)$.

Find the velocity $\mathbf{u} = (u_1, u_2)$, the pressure p and the temperature T such that

$$\left\{ \begin{array}{ll} (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \Delta \mathbf{u} - g(\beta T + 1)\mathbf{e}_y = 0 & \text{in } \Omega \quad (\text{momentum}) \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \quad (\text{incompressibility}) \\ \mathbf{u} \cdot \nabla T - k_f \Delta T = 0 & \text{in } \Omega \quad (\text{energy}) \\ + \text{suitable BC} & \end{array} \right. \quad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, ν the viscosity and k_f the thermal conductivity. [Coulaud, Le, and Duvigneau, 2024]

¹The approach can be extended to other test cases.

A1 – Heated cavity test case

Objective: Simulation on a range of parameters $\boldsymbol{\mu} = (\nu, k_f) \in \mathcal{M} = [0.01, 0.1]^2$.

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :

We consider $\mathbf{x} = (x, y) \in \Omega$ and $\mathbf{e}_y = (0, 1)$.

Find $\mathbf{U} = (\mathbf{u}, p, T) = (u_1, u_2, p, T)$ such that

$$\left\{ \begin{array}{ll} R_{mom}(\mathbf{U}; \mathbf{x}, \boldsymbol{\mu}) = 0 & \text{in } \Omega \quad (\text{momentum}) \\ R_{inc}(\mathbf{U}; \mathbf{x}, \boldsymbol{\mu}) = 0 & \text{in } \Omega \quad (\text{incompressibility}) \\ R_{ener}(\mathbf{U}; \mathbf{x}, \boldsymbol{\mu}) = 0 & \text{in } \Omega \quad (\text{energy}) \\ + \text{ suitable BC} & \end{array} \right. \quad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, ν the viscosity and k_f the thermal conductivity. [Coulaud, Le, and Duvigneau, 2024]

A1 – Heated cavity test case

Objective: Simulation on a range of parameters $\boldsymbol{\mu} = (\nu, k_f) \in \mathcal{M} = [0.01, 0.1]^2$.

Stationary incompressible Navier-Stokes equations (with buoyancy and gravity) :

We consider $\mathbf{x} = (x, y) \in \Omega$ and $\mathbf{e}_y = (0, 1)$.

Find $\mathbf{U} = (\mathbf{u}, p, T) = (u_1, u_2, p, T)$ such that

$$\begin{cases} R_{mom}(\mathbf{U}; \mathbf{x}, \boldsymbol{\mu}) = 0 & \text{in } \Omega & \text{(momentum)} \\ R_{inc}(\mathbf{U}; \mathbf{x}, \boldsymbol{\mu}) = 0 & \text{in } \Omega & \text{(incompressibility)} \\ R_{ener}(\mathbf{U}; \mathbf{x}, \boldsymbol{\mu}) = 0 & \text{in } \Omega & \text{(energy)} \end{cases} \quad (\mathcal{P})$$

with $g = 9.81$ the gravity, $\beta = 0.1$ the expansion coefficient, ν the viscosity and k_f the thermal conductivity. [Coulaud, Le, and Duvigneau, 2024]

Boundary Conditions:

No-slip BC : $\mathbf{u} = 0$ on $\partial\Omega$ **Isothermal BC :** $T = 1$ on the left wall ($x = -1$)
 $T = -1$ on the right wall ($x = 1$)

Adiabatic BC : $\frac{\partial T}{\partial n} = 0$ on the top and bottom walls ($y = \pm 1$, denoted by Γ_{ad})

Appendix 1 : Non-linear test case

PINN approach

Enriched FEM using PINN

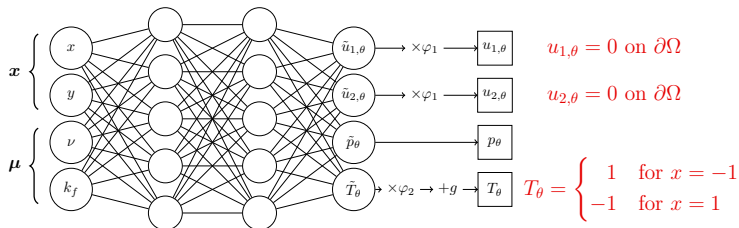
Numerical results

A1.1 – Neural Network considered

We consider a parametric NN with 4 inputs and 4 outputs, defined by

$$U_{\theta}(\mathbf{x}, \boldsymbol{\mu}) = (u_{1,\theta}, u_{2,\theta}, p_{\theta}, T_{\theta})(\mathbf{x}, \boldsymbol{\mu}).$$

The Dirichlet boundary conditions are imposed on the outputs of the MLP by a **post-processing** step. [Sukumar and Srivastava, 2022]



We consider two levelsets functions φ_1 and φ_2 , and the linear function g defined by

$$\varphi_1(x, y) = (x - 1)(x + 1)(y - 1)(y + 1),$$

$$\varphi_2(x, y) = (x - 1)(x + 1) \quad \text{and} \quad g(x, y) = 1 - (x + 1).$$

A1.1 – PINN training

Approximate the solution of (\mathcal{P}) by a PINN : Find the optimal weights θ^* , such that

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \left(J_{inc}(\theta) + J_{mom}(\theta) + J_{ener}(\theta) + J_{ad}(\theta) \right), \quad (\mathcal{P}_\theta)$$

where the different cost functions¹ are defined by

adiabatic condition

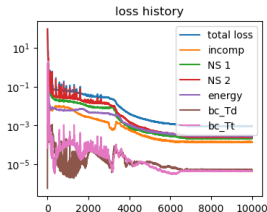
$$J_{ad}(\theta) = \int_{\mathcal{M}} \int_{\Gamma_{ad}} \left| \frac{\partial T_\theta(\mathbf{x}, \boldsymbol{\mu})}{\partial n} \right|^2 d\mathbf{x} d\boldsymbol{\mu},$$

3 residual losses

$$J_\bullet(\theta) = \int_{\mathcal{M}} \int_{\Omega} |R_\bullet(U_\theta(\mathbf{x}, \boldsymbol{\mu}); \mathbf{x}, \boldsymbol{\mu})|^2 d\mathbf{x} d\boldsymbol{\mu},$$

with U_θ the parametric NN and \bullet the PDE considered (i.e. *inc*, *mom* or *ener*).

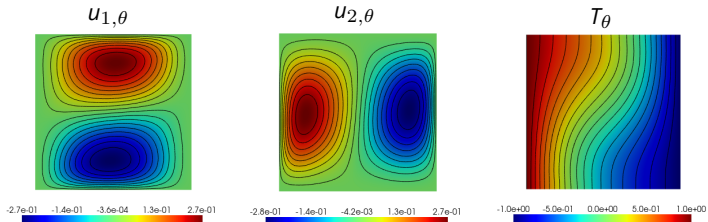
Network - MLP		Training (ADAM / LBFGs)			
<i>layers</i>	40, 60, 60, 60, 40	<i>lr</i>	7e-3	<i>N_{col}</i>	40000
<i>σ</i>	sine	<i>n_{epochs}</i>	10000	<i>N_{bc}</i>	30000



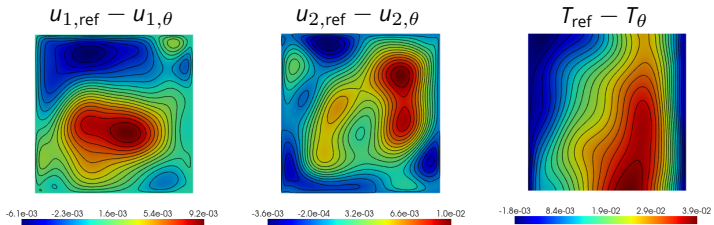
¹ Discretized by a random process using Monte-Carlo method.

A1.1 – Prediction on $\mu^{(1)} = (0.1, 0.1)$

Prediction :



Error map :



**L^2 error :
(relative)**

$$2.98 \times 10^{-2}$$

$$3.17 \times 10^{-2}$$

$$3.90 \times 10^{-2}$$

Appendix 1 : Non-linear test case

PINN approach

Enriched FEM using PINN

Numerical results

A1.2 – Enriched space using PINN

Considering the PINN prior $U_\theta = (\mathbf{u}_\theta, p_\theta, T_\theta)$, we define the **mixed finite element space additively enriched** by the PINN as follows:

$$M_h^+ = \{U_h^+ = U_\theta + C_h^+, \quad C_h^+ \in M_h^0\}$$

with $M_h^0 = [V_h^0]^2 \times Q_h \times W_h^0$, $U_h^+ = (\mathbf{u}_h^+, p_h^+, T_h^+) \in M_h^+$ and $C_h^+ = (\mathbf{c}_{h,\mathbf{u}}^+, c_{h,p}^+, c_{h,T}^+)$.

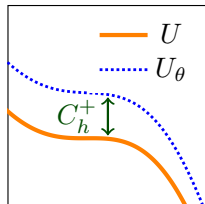
We can then define the three finite element subspaces of M_h^+ as follows:

$$V_h^+ = \{\mathbf{u}_h^+ = \mathbf{u}_\theta + \mathbf{c}_{h,\mathbf{u}}^+, \quad \mathbf{c}_{h,\mathbf{u}}^+ \in [V_h^0]^2\},$$

$$Q_h^+ = \{p_h^+ = p_\theta + c_{h,p}^+, \quad c_{h,p}^+ \in Q_h\},$$

$$W_h^+ = \{T_h^+ = T_\theta + c_{h,T}^+, \quad c_{h,T}^+ \in W_h^0\},$$

where $\mathbf{c}_{h,\mathbf{u}}^+$, $c_{h,p}^+$ and $c_{h,T}^+$ becomes the unknowns.



A1.2 – Weak form - Additive approach

Weak problem¹ : Find $C_h^+ = (C_{h,u}^+, C_{h,p}^+, C_{h,T}^+) \in M_h^0$ s.t., $\forall (\mathbf{v}_h, q_h, w_h) \in M_h^0$,

$$\begin{aligned} & \int_{\Omega} [(\mathbf{u}_{\theta} \cdot \nabla) \mathbf{u}_{\theta} + (\mathbf{u}_{\theta} \cdot \nabla) C_{h,u}^+ + (C_{h,u}^+ \cdot \nabla) \mathbf{u}_{\theta} + (C_{h,u}^+ \cdot \nabla) C_{h,u}^+] \cdot \mathbf{v}_h \, dx \\ & + \mu \left(\int_{\Omega} \nabla \mathbf{u}_{\theta} : \nabla \mathbf{v}_h \, dx + \int_{\Omega} \nabla C_{h,u}^+ : \nabla \mathbf{v}_h \, dx \right) + \left(\int_{\Omega} \nabla p_{\theta} \cdot \mathbf{v}_h \, dx - \int_{\Omega} C_{h,p}^+ \nabla \cdot \mathbf{v}_h \, dx \right) \\ & - g \int_{\Omega} (1 + \beta(T_{\theta} + C_{h,T}^+)) \mathbf{e}_y \cdot \mathbf{v}_h \, dx = 0, \text{ (momentum)} \\ & \int_{\Omega} q_h [\nabla \cdot \mathbf{u}_{\theta} + \nabla \cdot C_{h,u}^+] \, dx + 10^{-4} \int_{\Omega} q_h (p_{\theta} + C_{h,p}^+) \, dx = 0, \text{ (incompressibility + penal)} \\ & \int_{\Omega} [\mathbf{u}_{\theta} \cdot \nabla T_{\theta} + \mathbf{u}_{\theta} \cdot \nabla C_{h,T}^+ + C_{h,u}^+ \cdot \nabla T_{\theta} + C_{h,u}^+ \cdot \nabla C_{h,T}^+] w_h \, dx \\ & + k_f \left(\int_{\Omega} \nabla T_{\theta} \cdot \nabla w_h \, dx + \int_{\Omega} \nabla C_{h,T}^+ \cdot \nabla w_h \, dx \right) = 0, \text{ (energy)} \end{aligned} \tag{\mathcal{P}_h^+}$$

with $\mathbf{U}_{\theta} = (\mathbf{u}_{\theta}, p_{\theta}, T_{\theta})$ the PINN prior and some modified boundary conditions.

¹ Problem solved using Newton's algorithm with the correction vector initialized to 0.

Appendix 1 : Non-linear test case

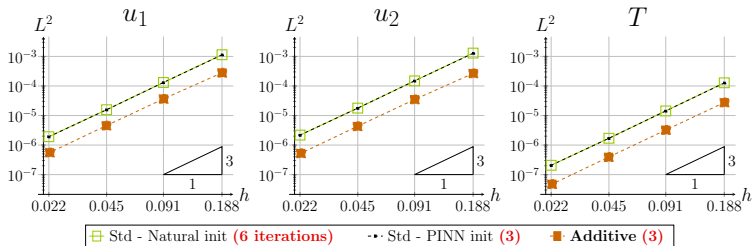
PINN approach

Enriched FEM using PINN

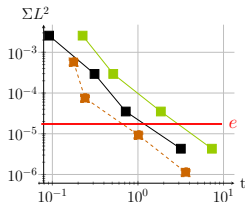
Numerical results

A1.3 – Numerical costs

$\mu^{(1)}$:



N_{dofs} and execution time required to reach the same global L^2 relative error e ¹



Std vs Add	Number of DoFs		Execution times			
	e	Std	Add	(nat)	(PINN)	Add
	$1 \cdot 10^{-3}$	6,031	2,044	0.32	0.16	0.16
	$1 \cdot 10^{-4}$	26,959	10,588	0.99	0.48	0.23
	$1 \cdot 10^{-5}$	121,156	49,231	4.21	1.75	0.96

→ ÷ 2.5 →
→ ÷ 2 →
← ÷ 4 ←

¹Error defined as the sum of the L^2 relatives errors on \mathbf{u} and T .