Approximation de mesures multifractales par un système d'Ornstein-Uhlenbeck : convergence faible et application en turbulence lagrangienne.

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CERMICS

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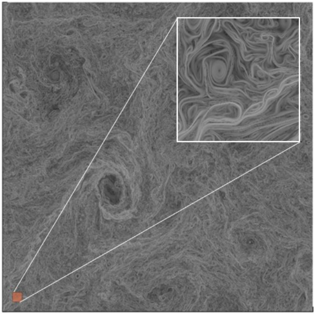
Intro: multifractality in turbulence



Dissipation ε in a direct numerical simulation of the SQG equations, provided by Nicolas Valade (INRIA - Calisto Team)

- Starting point: **Navier-Stokes** equation:
  $\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{1}{\text{Re}} \Delta \vec{u}$
- **Energy dissipation** :
  $\varepsilon(t,x) = \frac{1}{2\text{ Re}} \langle \text{trace} \nabla^T u \nabla u) \rangle (t,x)$
- **Lagrangian** approach:

$$\frac{d}{dt} X_t = u(t, X_t) =: U_t, \qquad \varepsilon_t := \varepsilon(t, X_t),$$

approximated by the **simplified Langevin model**

$$dU_t = -\frac{1}{T_L}(U_t - \langle U \rangle) dt + \sqrt{C_0 \varepsilon_t}\, dB_t.$$

Kolmogorov's refined theory (K62) of turbulence for fluctuations of the Lagrangian **energy dissipation** ε prescribes a multifractal behaviour [Kolmogorov, 1962] [Frisch and Parisi, 1985] [Frisch, 1995] (see also [Barral and Seuret, 2023a] and [Barral and Seuret, 2023b]), that induce

$$\exists \gamma > 0,\ \forall t > 0, \quad \mathbb{E}\left[\left| \frac{1}{\tau} \int_t^{t+\tau} \varepsilon_s ds \right|^p \right] \simeq \tau^{\frac{\gamma^2}{2}(p-p^2)}, \quad \tau \text{ small.}$$

Let $\gamma > 0$. We call a (1D) **Multifractal Random Measure** (MRM) a random measure M_γ on a time interval $[0,T]$ such that there exists p^* for which

$$\forall p \in (0,p^*), \quad \mathbb{E}[M_\gamma([0,\tau])^p] \underset{\tau \to 0}{\sim} C_p \, \tau^{p - \frac{\gamma^2}{2}(p^2 - p)}.$$

- An example of MRM in the litterature (e.g. [Kahane, 1985], [Rhodes and Vargas, 2014], [Shamov, 2016]) arises with the theory of **Gaussian Multiplicative Chaos**:
- consider a centred generalized **Gaussian process** $(Z_t)_{t \in [0,1]}$ with covariance kernel

$$\mathrm{Cov}(Z_s, Z_t) = \gamma^2 \log^+\left(\frac{D}{|t-s|}\right) + g(|t-s|)$$

  for a constant $D > 0$ and a bounded continuous function g.
- For $\gamma \in (0, \sqrt{2})$, the random measure M_γ defined formally by

$$M_\gamma(A) = \int_0^T 1_A(s) \exp\left(Z(s) - \frac{1}{2}\mathbb{E}[Z^2(s)]\right) ds, \quad A \in \mathscr{B}([0,T])$$

  defines a MRM on $[0,T]$ with $p^* = 2/\gamma^2$.

Approximation of MRM by integrated Riemann-Liouville fBm

Main goal: approximate the MRM from Gaussian Multiplicative Chaos by a sequence of **stochastic processes** (useful for both **modelling** and **simulation** perspective). Consider the (rescaled) **Riemann-Liouville fractional Brownian motion** $(B_t^H)_{t\in[0,1]}$ with Hurst parameter $H \in (0, \frac{1}{2})$:

$$B_t^H := \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \quad \text{with } (W_t)_{t\in[0,1]} \text{ a standard Brownian motion.}$$

- For $\gamma \in (0, \sqrt{2})$, it is shown in [Forde et al., 2022] that the sequence $M_\gamma^{(H)}(dt) = \exp(\gamma B_t^H - \frac{\gamma^2}{2}\operatorname{Var}(B_t^H))dt$ satisfies

$$\forall p \in (0, \frac{2}{\gamma^2}),\ \forall \tau \in (0,1), \quad \mathbb{E}[M_\gamma^{(H)}([0,\tau])^p] \xrightarrow[H\to 0]{} \mathbb{E}[M_\gamma([0,\tau])^p],$$

  where M_γ is a MRM. See also [Neuman and Rosenbaum, 2018] for a MRM approximation using renormalized standard fBm.
- We would like to go a step further and approximate MRMs by a broader class of stochastic processes.

Motivation: "Markovian" approximation (from [Carmona et al., 2000])

Let $K_H(r) = r^{H-1/2}$. We start from the formula

$$K_H(r) = c_H \int_0^{+\infty} e^{-rx} x^{-\frac{1}{2}-H} dx, \quad c_H = \frac{1}{\Gamma(1/2-H)}.$$

Introduce a **Gauss quadrature** $(w_i, x_i)_{\{1\le i\le N\}}$ of order N for $\int_0^{+\infty} f(x) x^{-\frac{1}{2}-H} dx$. We approximate K by the sequence $(\overline{K}_H^{(N)})_{N\ge 1}$ where $\overline{K}_H^{(N)}(r) = \sum_{i=1}^N w_i e^{-rx_i}$. This suggests:

$$B_t^H \simeq B_t^{H,N} := c_H \int_0^t \overline{K}_H^{(N)}(t-s) dW_s.$$

Moreover,

$$c_H \int_0^t \overline{K}_H^{(N)}(t-s) dW_s = c_H \sum_{i=1}^N w_i \int_0^t e^{-(t-s)x_i} dW_s = c_H \sum_{i=1}^N w_i Y_t^{x_i}$$

where $(Y_t^{x_i})_{t\in[0,1]}$ is an **Ornstein-Uhlenbeck** process starting from zero :

$$dY_t^{x_i} = -x_i Y_t^{x_i} dt + dW_t, \quad Y_0^{x_i} = 0.$$

Using standard results on the Wasserstein-1 distance between two centred Gaussian laws, we can obtain the upper-bound

$$|\mathbb{E}[\phi(B_t^H))] - \mathbb{E}[\phi(B_t^{H,N}))]| \le C_H \left| \int_0^t ((K_H)^2(s) - (\overline{K}_H^{(N)})^2(s))\, ds \right|$$

for Lipschitz functions $\phi : \mathbb{R} \to \mathbb{R}$.

Convergence at the integrated level: results in the litterature

Question: can we obtain an optimal **rate of convergence** when $N \to +\infty$ for

$$E_{N,H} := \left| \mathbb{E}\left[\phi\left(\int_0^\tau \exp(\gamma B_t^H - \frac{\gamma^2}{2}\mathrm{Var}(B_t^H) dt\right)\right] - \mathbb{E}\left[\phi\left(\int_0^\tau \exp(\gamma B_t^{H,N} - \frac{\gamma^2}{2}\mathrm{Var}(B_t^{H,N}) dt\right)\right] \right| ?$$

- It is possible to use **strong error estimates** for stochastic Volterra equations (see [Alfonsi and Kebaier, 2024]) to upper bound this quantity:

  $$E_{N,H} \leq C_H \, \|\phi\|_{\mathrm{Lip}} \, \|K_H - \overline{K}_H^{(N)}\|_{L^2([0,\tau])}$$

  but this rate might be sub-optimal.

- In parallel, in [Bayer and Breneis, 2023] for the **rough Heston model**,

  $$dS_t = \sqrt{V_t} S_t \left(\rho dW_t + \sqrt{1-\rho^2} d\widehat{W}_t\right), \quad \rho \in [-1,1], \quad W \perp\!\!\!\perp \widehat{W},$$
  $$dV_t = \int_0^t K(t-s)(\theta - \lambda V_s) ds + \int_0^t K(t-s)\gamma\sqrt{V_s} dW_s, \quad \theta, \lambda, \gamma > 0,$$

  the authors obtain the upper-bound:

  $$|\mathbb{E}[\phi(S_t))] - \mathbb{E}[\phi(\overline{S}_t))]| \leq C\|K - \overline{K}\|_{L^1([0,t])}^{\alpha}$$

  with $\alpha \in (0,1)$ ($\alpha = 1$ when ϕ has compact support).

Kernel sensibility of integrated Volterra process

We fix $T > 0$ and $H \in (0, \frac{1}{2})$. We consider possibly **singular kernels** K and $\overline{K}$ with

$$\forall t \in [0,T], \quad \overline{K}(t) \leq C_1 K(t) \leq C_2 t^{H-\frac{1}{2}}; \qquad |K'(t)| \leq C_3 t^{H-\frac{3}{2}}.$$

Our model of interest is the **integrated Volterra process**

$$\begin{aligned} X_t =& X_0 + \int_0^t b(s,V_s)ds + \int_0^t \sigma(s,V_s)dB_s; \qquad B = \rho W + \sqrt{1-\rho^2}\,\widehat{W}, \quad W \perp\!\!\!\perp \widehat{W}, \\ V_t =& \int_0^t K(t-r)dW_r - \alpha \int_0^t K^2(t-r)dr; \qquad t \in [0,T],\ \alpha \in \mathbb{R}, \end{aligned}$$

and its analogue when K is replaced by $\overline{K}$:

$$\begin{aligned} \overline{X}_t =& X_0 + \int_0^t b(s,\overline{V}_s)ds + \int_0^t \sigma(s,\overline{V}_s)dB_s; \\ \overline{V}_t =& \int_0^t \overline{K}(t-r)dW_r - \alpha \int_0^t \overline{K}^2(t-r)dr; \qquad t \in [0,T],\ \alpha \in \mathbb{R}. \end{aligned}$$

- **MRM approximation model** $\Longrightarrow K(t) = \gamma t^{H-\frac{1}{2}}$, $b(s,v) = e^v$, $\sigma(s,v) = 0$ and $\alpha = \frac{1}{2}$.
- **rough Bergomi model** $\Longrightarrow K(t) = \frac{t^{H-\frac{1}{2}}}{\sqrt{2H}}$, $b(s,v) = -\frac{1}{2}e^{\gamma v - \frac{\gamma^2}{2}s^{2H}}$, $\sigma(s,v) = e^{\frac{\gamma v}{2} - \frac{\gamma^2}{4}s^{2H}}$ and $\alpha = 0$.

Main results: weak error for kernel approximation

We consider a test function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi \in C^4$ and $(\phi^{(i)}, 0 \leq i \leq 4)$ have at most **polynomial growth**. We assume that $b(s,\cdot)$ and $\sigma(s,\cdot)$ has at most **exponential growth**.

Theorem 1 (Bossy, Martínez, M., 2025)

For every $\rho \in [-1,1]$ and $\alpha \in \mathbb{R}$, there exists $C_H > 0$ such that:

$$|\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)]| \leq C_H \int_0^T \left\{ \|\overline{K} - K\|_{L^1([0,t])} + \|(\overline{K})^2 - K^2\|_{L^1([0,t])} \right\} dt$$

The situation is in fact **way better** for our specific application case:

Theorem 2 (Bossy, Martínez, M., 2025)

For $\rho = 0$ and $\alpha = \frac{1}{2}$, if $b(s,v) = e^v$ and $\sigma(s,v) = e^{v/2}$ or $\sigma(s,v) = 0$, there exists $C_H > 0$ such that

$$|\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)]| \leq C_H \int_0^T \|\overline{K} - K\|_{L^1([0,t])} dt$$

Proof of Theorem 1 with $\alpha = 0$ available in preprint [Bossy et al., 2025]. General proof with technical adjustments and proof of Theorem 2 will be available in the **new version coming soon** on Arxiv!

Sketch of proof ($\sigma = 0$) - Step 1. The martingale approach

- We consider the filtration $\mathcal{F}_s = \sigma(W_r \; ; \; r \in [0,s])$ for $s \in \mathbb{T}$. The orthogonal decomposition from [Viens and Zhang, 2019] writes:

$$\forall s \geq t \in \mathbb{T}, \qquad V_s = \underbrace{\gamma \int_0^t K(s-r)dW_r - \frac{\gamma^2}{2}\int_0^t K^2(s-r)dr}_{\Theta_t(s) \, \in \, \mathcal{F}_t} \\ + \underbrace{\gamma \int_t^s K(s-r)dW_r - \frac{\gamma^2}{2}\int_t^s K^2(s-r)dr}_{I_t(s) \, \perp\!\!\!\perp \, \mathcal{F}_t}.$$

- We derive that for any test function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\phi(X_T)|\mathcal{F}_t] =& \mathbb{E}\left[\phi\left(X_t + \int_t^T b(V_s^H)ds\right)|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\phi\left(X_t + \int_t^T b(\Theta_t(s) + I_t(s))ds\right)|\mathcal{F}_t\right] \\ &= u(t, X_t, (\Theta_t(s))_{s\in[t,T]}), \end{aligned}$$

where $u(t,x,\omega) = \mathbb{E}_{t,x,\omega}[\phi(X_T)] = \mathbb{E}[\phi(X_T)|X_t = x, \Theta_t = \omega]$ for all $(t,x,\omega) \in \mathbb{T} \times \mathbb{R} \times C([0,T],\mathbb{R})$.

Consider the **flow process**:

$$X_T^{t,x,\omega} = x + \int_t^T b(V_s^{t,\omega})ds;$$
$$V_s^{t,\omega} = \omega_s + \gamma \int_t^s K(s-r)dW_r - \frac{\gamma^2}{2} \int_t^s K^2(s-r)dr, \ s \geq t.$$

We define the function $u : [0,T] \times \mathbb{R} \times C([0,T],\mathbb{R})$ by $u(t,x,\omega) = \mathbb{E}[\phi(X_T^{t,x,\omega})]$ and define the **path derivative** of u in $\omega \in C([0,T],\mathbb{R})$ in the direction $\eta \in C([0,T],\mathbb{R})$ by:

$$\langle \partial_\omega u(t,x,\omega), \eta \rangle = \lim_{\varepsilon \to 0} \frac{u(t,x,\omega + \varepsilon \eta \mathbb{1}_{[t,T]}) - u(t,x,\omega)}{\varepsilon},$$

and we define in the same way the second path derivative $\left\langle \partial_\omega^2 u(t,x,\omega), (\eta, \zeta) \right\rangle$.

Proposition

The function $(t,x,\omega) \mapsto u(t,x,\omega) := \mathbb{E}[\phi(X_T^{t,x,\omega})]$ belongs to $C^{0,2,2}([0,T]\times\mathbb{R}\times C([0,T],\mathbb{R}))$ with

$$\begin{aligned}
\langle \partial_\omega u(t,x,\omega),\eta\rangle =& \mathbb{E}\left[\phi'(X_T^{t,x,\omega})\int_t^T b'(V_s^{t,\omega})\eta(s)ds\right],\\
\langle \partial_\omega^2 u(t,x,\omega),(\eta,\zeta)\rangle =& \mathbb{E}\left[\phi''(X_T^{t,x,\omega})\Big(\int_t^T b'(V_s^{t,\omega})\eta(s)ds\Big)\Big(\int_t^T b'(V_s^{t,\omega})\zeta(s)ds\Big)\right]\\
&+\mathbb{E}\left[\phi'(X_T^{t,x,\omega})\int_t^T b''(V_s^{t,\omega})\eta(s)\zeta(s)ds\right], \quad (\eta,\zeta)\in C([0,T],\mathbb{R}).
\end{aligned}$$

The differentiability in ω is interpreted as **Fréchet differentiability**. In particular the maps $\eta \mapsto \langle \partial_\omega u(t,x,\omega),\eta\rangle$ and $(\eta,\zeta) \mapsto \langle \partial_\omega^2 u(t,x,\omega),(\eta,\zeta)\rangle$ are (bi)-linear continuous.

Sketch of proof - Step 4. Growth control and singular derivatives

Definition

We say that $u \in C^{0,2,2}([0,T] \times \mathbb{R} \times C([0,T],\mathbb{R}))$ has controlled path derivatives, if

$$|\langle \partial_\omega u(t,x,\omega), \eta \rangle| \lesssim G(t,x,\omega) \int_t^T L(s,\omega)|\eta_s| ds$$

$$|\left\langle \partial_\omega^2 u(t,x,\omega), (\eta,\zeta) \right\rangle| \lesssim G(t,x,\omega) \left\{ \left(\int_t^T L(s,\omega)|\eta_s| ds \right) \left(\int_t^T L(s,\omega)|\zeta_s| ds \right) + \int_t^T L(s,\omega)|\eta_s||\zeta_s| ds \right\}$$

for every $\eta, \zeta \in C([0,T],\mathbb{R})$ and appropriate growth functions and $G : [0,T] \times \mathbb{R} \times C([0,T] \times \mathbb{R}) \to \mathbb{R}_+$ and $L : [t,T] \times C([0,T],\mathbb{R}) \to \mathbb{R}_+$.

Proposition

If $u \in C^{0,2,2}([0,T] \times \mathbb{R} \times C([0,T],\mathbb{R}))$ has controlled path derivatives, then we may define the path derivatives in the singular direction $\eta(s) = K(s-t)$:

$$\langle \partial_\omega u(t,x,\omega), K(\cdot - t) \rangle = \lim_{\delta \to 0} \langle \partial_\omega u(t,x,\omega), K((\cdot - t) \vee \delta) \rangle$$

$$\langle \partial_\omega^2 u(t,x,\omega), (K(\cdot - t), K(\cdot - t)) \rangle = \lim_{\delta \to 0} \langle \partial_\omega^2 u(t,x,\omega), (K((\cdot - t) \vee \delta), K((\cdot - t) \vee \delta)) \rangle$$

Sketch of proof - Step 5. Functional Itô formula and PPDE

Let $u \in C^{0,2,2}(\mathbb{T} \times \mathbb{R} \times C([t,T],\mathbb{R}))$ with controlled path derivatives, and suppose that u is continuously differentiable in time. Then we have the following (singular) **functional Itô formula**:

$$
\begin{aligned}
u(t,X_t,\Theta_t) - u(0,x,0) = &\int_0^t \Big(\partial_t u(s,X_s,\Theta_s) + b(V_s)\partial_x u(s,X_s,\Theta_s)\Big) ds \\
&+ \int_0^t \Bigg\{ \frac{1}{2} \Big\langle \partial^2_\omega u(s,X_s,\Theta_s), (\gamma K(\cdot - s), \gamma K(\cdot - s)) \Big\rangle \\
&- \frac{1}{2} \Big\langle \partial_\omega u(s,X_s,\Theta_s), \gamma^2 (K(\cdot - s))^2 \Big\rangle \Bigg\} ds + \int_0^t \langle \partial_\omega u(s,X_s,\Theta_s), \gamma K(\cdot - s) \rangle \, dW_s.
\end{aligned}
$$

We also have the following **path-dependent PDE** for u:

Theorem (adapted from [Bonesini et al., 2023])

Under sufficient regularity hypothesis on u and its path derivatives we have

$$
\begin{aligned}
&\partial_t u(t,x,\omega) + b(\omega_t)\partial_x u(t,x,\omega) \\
&\quad + \frac{1}{2} \Big\langle \partial^2_\omega u(t,x,\omega), (\gamma K(\cdot - t), \gamma K(\cdot - t)) \Big\rangle - \frac{1}{2} \Big\langle \partial_\omega u(t,x,\omega), \gamma^2 (K(\cdot - t))^2 \Big\rangle = 0
\end{aligned}
$$

with terminal condition $\quad u(T,x,\omega) = \phi(x).$

Sketch of proof - Step 6. - The weak error expansion

Let $\overline{\Theta} = (\overline{\Theta}_t(s))_{0 \le t \le s \le T}$ defined by

$$\overline{\Theta}_t(s) = \gamma \int_0^t \overline{K}(s-r)dW_r - \frac{\gamma^2}{2} \int_0^t (\overline{K})^2(s-r)dr.$$

We apply the **functional Itô formula** to $u(t,x,\omega) = \mathbb{E}[\phi(X_T^{t,x})]$ and $(t, \overline{X}_t, \overline{\Theta}_t)$. We use the **PPDE** satisfied by u, the bilinearity and symmetry of the map $\langle \partial^2_\omega u(t,x,\omega), (\cdot,\cdot) \rangle$:

$$\mathbb{E}[\phi(\overline{X}_t)] - \mathbb{E}[\phi(X_T)] = \mathbb{E}[u(T,\overline{X}_t,\overline{\Theta}_t)] - \mathbb{E}[u(0,x,0)]$$

$$= \mathbb{E}\int_0^T \Big\{ \partial_t u(t,\overline{X}_t,\overline{\Theta}_t) + \partial_x u(t,\overline{X}_t,\overline{\Theta}_t) b(X_t) \Big\} dt$$
$$+ \frac{\gamma^2}{2}\mathbb{E}\int_0^T \Big\{ \Big\langle \partial^2_\omega u(t,\overline{X}_t,\overline{\Theta}_t), (\overline{K}(\cdot - t), \overline{K}(\cdot - t)) \Big\rangle - \Big\langle \partial_\omega u(t,\overline{X}_t,\overline{\Theta}_t),\ \overline{K}(\cdot - t)^2 \Big\rangle \Big\} dt$$

$$= \frac{\gamma^2}{2}\mathbb{E}\int_0^T \Big\langle \partial^2_\omega u(t,\overline{X}_t,\overline{\Theta}_t), ((\overline{K}-K)(\cdot - t), (\overline{K}+K)(\cdot - t)) \Big\rangle dt$$
$$- \frac{\gamma^2}{2}\mathbb{E}\int_0^t \langle \partial_\omega u(t,\overline{X}_t,\overline{\Theta}_t), ((\overline{K}-K)(\cdot - t)) \times ((\overline{K}+K)(\cdot - t)) \rangle dt$$

Using the **probabilistic representations** of the path-dependent derivatives we obtain that

$$\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)] = \frac{\gamma^2}{2}(I_1 + I_2 - I_3)$$

with

$$I_1 = \int_0^T \mathbb{E}\left[\phi''(X_T^{t,\overline{X}_t,\overline{\Theta}_t})\left(\int_t^T b'(V_s^{t,X_t})(\overline{K}-K)(s-t)ds\right)\left(\int_t^T b'(V_s^{t,\overline{X}_t})(\overline{K}+K)(s-t)ds\right)\right]dt,$$

$$I_2 = \int_0^T \mathbb{E}\left[\phi'(X_T^{t,\overline{X}_t,\overline{\Theta}_t})\left(\int_t^T b''(V_s^{t,\overline{X}_t})(\overline{K}^2-K^2)(s-t)ds\right)\right]dt$$

$$I_3 = \int_0^T \mathbb{E}\left[\phi'(X_T^{t,\overline{X}_t,\overline{\Theta}_t})\left(\int_t^T b'(V_s^{t,\overline{X}_t})(\overline{K}^2-K^2)(s-t)ds\right)\right]dt$$

Observe that when $b(x) = \exp(x)$, one has $I_2 = I_3$!

Coming back to the problem presented in introduction, we apply **Theorem 2** with $\alpha = \frac{1}{2}$,

$$K(t) = K_H(t) = \gamma t^{H-\frac{1}{1}} \implies V_t = \gamma B_t^H - \frac{\gamma^2}{2}\mathbb{V}\mathrm{ar}(B_t^H)$$

$$\overline{K}(t) = \overline{K}_H^{(N)}(t) = \sum_{i=1}^N w_i e^{-x_i t} \implies \overline{V}_t = \gamma B_t^{H,N} - \frac{\gamma^2}{2}\mathbb{V}\mathrm{ar}(B_t^{H,N})$$

where (x_i, w_i) is a **Gaussian quadrature** of $\gamma c_H \int_0^\infty e^{-xs} x^{-1/2-H} dx$; $b(s,v) = e^v$ and $\sigma(s,v) = 0$.
We obtain

$$\left|\mathbb{E}\left[\phi\left(\int_0^T e^{\gamma B_t^H - \frac{\gamma^2}{2}\mathbb{V}\mathrm{ar}(B_t^H)} dt\right)\right] - \mathbb{E}\left[\phi\left(\int_0^T e^{\gamma B_t^{H,N} - \frac{\gamma^2}{2}\mathbb{V}\mathrm{ar}(B_t^{H,N})} dt\right)\right]\right| \leq C_H \int_0^T |K_H(t) - \overline{K}_H^{(N)}(t)| dt$$

Proposition

Suppose $|\phi(x)| \leq C_\phi(1+|x|^p)$ with $p > 1$ integer. If $p = 2, 3$ and $\gamma^2 < \frac{1}{2}$ or if $p > 3$ and $p < 1 + \frac{\gamma}{2}$, then

$$\sup_{H \in (0,1/2)} C_H < +\infty \quad \text{(!!)}$$

Minimizing $\|\overline{K}_H^{(N)} - K_H\|_{L^1([0,T])}$: seeking for a quadrature method

What to do now ? **Optimizing** the Gaussian quadrature $(x_i, w_i)_{0\leq i\leq N}$. To do so, we cut the integral $\int_0^\infty e^{-xs}x^{-\frac{1}{2}-H}dx$ in **three parts**:

$$\int_0^\infty e^{-xs}x^{-\frac{1}{2}-H}dx = \underbrace{\int_0^a e^{-xs}x^{-\frac{1}{2}-H}dx}_{\text{isolated integrable singularity}} + \underbrace{\int_a^b e^{-xs}x^{-\frac{1}{2}-H}dx}_{\text{bulk}} + \underbrace{\int_b^{+\infty} e^{-xs}x^{-\frac{1}{2}-H}dx}_{\text{residual part}}.$$

- The integral with isolated integrable singularity is approximated with a **Gauss-Jacobi** quadrature with weight function $w(x) = x^{-1/2-H}$ on $[a,b]$.
- The bulk integral is approximated with a **Gauss-Legendre** quadrature with weight function $w(x) = 1$ on $[-1,1]$, then transformed to a quadrature on $[a,b]$ using an **optimal change of variables**.
- Finally, the residual part is simply **neglected**, which requires to take b large enough.

Theorem 3 (Bayer-Breneis 2023, Koyama 2023)

We can construct an approximating kernel $\overline{K}_H^{(N)}$ such that

$$\|\overline{K}_H^{(N)} - K_H\|_{L^1([0,T])} \leq C\, e^{-\alpha_\Phi\sqrt{N}},$$

where $\alpha = \sqrt{\pi/2}(1+2H)$ and the constant C does not depend on H.

Conclusion on MRM approximation

Combining the results of **Theorem 2** and **Theorem 3**, we have obtained that

$$\forall \gamma^2 \in \left(0, \frac{1}{2} \wedge \frac{1}{p-1}\right) \; \forall H \in \left(0, \frac{1}{2}\right), \quad |\mathbb{E}[\phi(\overline{X}_T^{H,N})] - \mathbb{E}[\phi(X_T^H)]| \leq C\, e^{-\alpha_\Phi \sqrt{N}}$$

when $\overline{K}_H^{(N)}(r) = \sum_{i=1}^N w_i e^{-x_i r}$ with the appropriate Gauss quadrature.

- Using that for this range of γ, we know from Forde and co-authors that the convergence $\mathbb{E}[\phi(X_T^H)] = \mathbb{E}[\phi(M_\gamma^{(H)}([0,T]))] \xrightarrow[H\to 0]{} \mathbb{E}[\phi(M_\gamma([0,T]))]$ holds, we deduce that

$$|\mathbb{E}[\phi(\overline{X}_T^{0,N})] - \mathbb{E}[\phi(M_\gamma([0,T]))]]| \leq C\, e^{-\alpha_\Phi \sqrt{N}}.$$

- This provides a **weak approximation** of the 1D MRM M_γ by a sequence of sums of **Ornstein-Uhlenbeck** processes that converges **exponentially fast**.
- Our result only cover the range $\gamma^2 \in (0, \frac{1}{2} \wedge \frac{1}{p-1})$. For $p = 2$, this restricts $\gamma^2 < \frac{1}{2}$ but M_γ is defined for $\gamma^2 \in (0,1)$. In fact, we can show that for $|\phi(x)| \lesssim 1 + x^2$, then

$$\forall \gamma^2 \in (0,1),\; |\mathbb{E}[\phi(\overline{X}_T^{H,N})] - \mathbb{E}[\phi(X_T^H)]| \leq C \left\{ \|K - \overline{K}\|_{L^1([0,T])} \right\}^{2(1-\gamma^2)^-}$$

Numerical simulations – trajectories

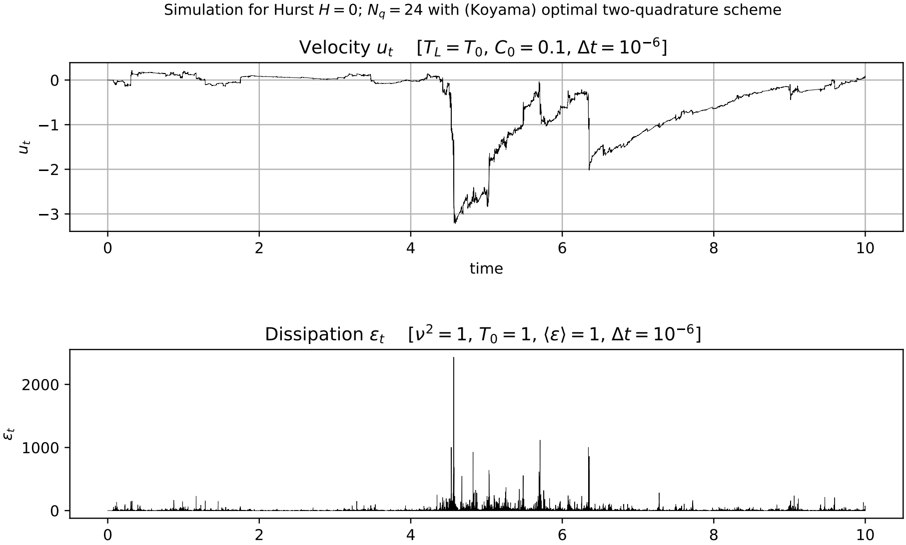


Figure 1: Approximated sample path for $\varepsilon_t = e^{V_t}$ and $dU_t = -\frac{1}{T_L}(U_t - \langle U \rangle)dt + \sqrt{C_0 \varepsilon_t} dB_t$

Numerical simulations – structure fonction for the time-averaged dissipation $M_\gamma[0,\tau] = \int_0^\tau \varepsilon_t dt$

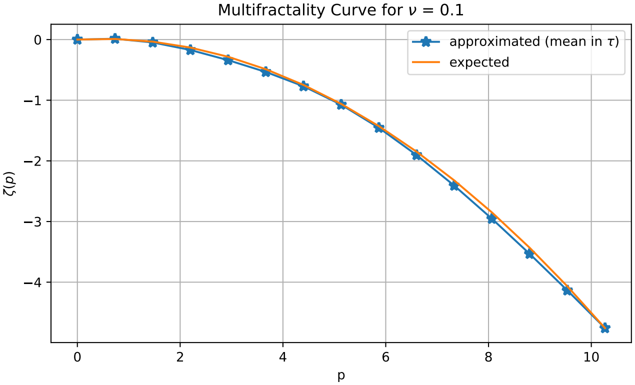


Figure 2: Estimation of the structure fonction $\zeta(p) = \lim_{\tau\to 0} \frac{\log(\mathbb{E}[M_\gamma[0,\tau]^p])}{\log(\tau)}$ for $\gamma^2 = 0.1$, $N = 14$, $N_{MC} = 1,6 \times 10^5$, $\Delta t = 5 \times 10^{-8}$, $H = 0.005$.

Numerical simulations – structure fonction for the velocity increments

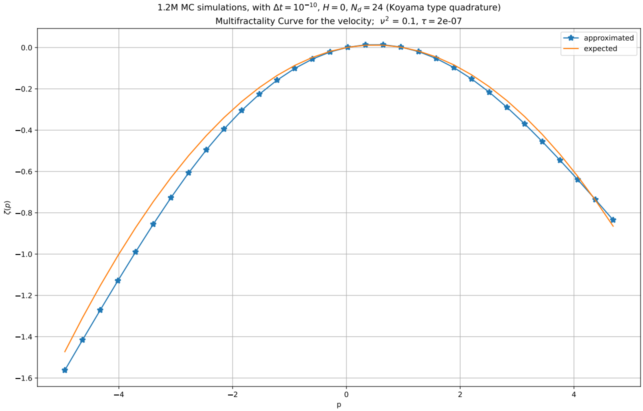


Figure 3: Estimation of $\zeta(p) = \lim_{\tau \to 0} \frac{\log(\mathbb{E}[(U(t+\tau)-U(t))^p])}{\log(\tau)}$ for $\gamma^2 = 0.1$, $N = 24$, $N_{MC} = 1,2 \times 10^6$, $\Delta t = 10^{-10}$, $H = 0$.

Thanks for your attention!

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