Flow rate measurement in a heterogeneous fluid with acoustic waves

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This work studies the propogation of acoustic waves in a waveguide with a non-uniform flow and impenetrable obstacles in the context of modelling of non-intrusive ultrasonic flowmeters. We consider a two-dimensional waveguide containing N particles which we denote by $\mathcal{O}$ and $\Omega = ([-L, L]\times[0, h])\backslash\mathcal{O}$ the fluid part of the domain. We denote by $\partial\Theta$ the exterior wall of the waveguide and $\Sigma_{\pm L}$ the outflow/inflow interfaces. The relative fluid velocity $\mathbf{V}_f$ is assumed to be given by an incompressible potential flow $\nabla u_0 + \nabla u$, where $u_0 = (V_f - V_p)x_1$ is the relative free velocity potential, and u the perturbation by the obstacle. We arrive at a model with two sequential problems, a Laplace problem for the flow potential (Equation (1)) and a Convected Helmholtz problem for the propagation of the acoustic waves in the fluid (Equation (2)). First we solve Equation (1) from which we obtain the fluid Mach velocity $\mathbf{M} = c^{-1}(\nabla u+\nabla u_0)$. This is substituted into Equation (2) in which we denote by D_k the convective derivative $D_k = -ik+\mathbf{M}\cdot\nabla$, where k is the wave number. The Dirichlet-to-Neumann (DtN)

$$\begin{cases} \Delta u &= 0, \text{ in } \Omega \\ \partial_{\mathbf{n}} u &= -\partial_{\mathbf{n}} u_0, \text{ on } \partial\Theta\cup\partial\mathcal{O} \\ \partial_{\mathbf{n}} u &= -S_{\pm} u, \text{ on } \Sigma_{\pm L} \end{cases} \quad (1)$$

$$\begin{cases} \nabla\cdot(\nabla\varphi - D_k\varphi\mathbf{M}) + ikD_k\varphi &= 0 \text{ in } \Omega \\ \partial_{\mathbf{n}}\varphi &= 0 \text{ on } \partial\mathcal{O} \\ \partial_{\mathbf{n}}\varphi &= f \text{ on } \partial\Theta \\ (\nabla\varphi - D_k\varphi\mathbf{M})\cdot\mathbf{n} &= -T_{\pm}\varphi \text{ on } \Sigma_{\pm L} \end{cases} \quad (2)$$

operators $S_{\pm}, T_{\pm}$ are constructed via a modal decomposition, and are exact for the Laplace problem, while for the Convected Helmholtz problem they approximate the flow as being uniform outside the bounded domain. First we prove that Problem (2) is well-posed in $H^1(\Omega)$, with the exception of a countable collection of wavenumber, under the assumption that the flow $\mathbf{M}$ is subsonic and tangential to $\partial\mathcal{O}\cup\partial\Theta$. This reduces the problem to proving that the flow model (1) yields a subsonic flow, which we prove under suitable geometrical assumptions. In addition to these theoretical results we present numerical simulations using FreeFEM.
We also explore asymptotic models in the limit of a large number of particles. We consider various homogenisation regimes in the periodic setting, and apply the theory of two-scale convergence to study the convergence to the effective equations. In this case formal calculations yield the following homogenised field equation :

$$\nabla\cdot(\mathbf{A}^*\nabla\varphi_0) + 2ik\mathbf{M}^*\cdot\nabla\varphi_0 + k^2\rho^*\varphi_0 = 0 \quad (3)$$

where $\mathbf{A}^*, \mathbf{M}^*, \rho^*$ are effective tensors independent of ε computed via the periodic cell problems. An interesting scaling is where the size of the particles r and the difference in the free fluid and particle velocities $V_f - V_p$ scale linearly in ε, the size of the periodic cell. While such homogenisation problems have been well-studied for the Helmholtz equation, to our knowledge this has not been presented for the Convected Helmholtz equation with non-uniform flow.

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