A Hyper-Reduced Neural-Network-Augmented Semi-Smooth Newton Method for Nonlinear Parametric Variational Inequalities

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We present a model order reduction (MOR) framework for nonlinear parametric variational inequalities [4] arising in computational mechanics. The work is motivated by the long-term challenge of reducing computational costs in the simulation of CO_2 storage in geological formations [3]. While the present contribution is methodological in nature and does not address this application directly, it lays the groundwork for such problems.
Starting from a mixed primal–dual discretization with projection-based complementarity conditions, we solve the resulting nonlinear, nonsmooth problem by a semi-smooth Newton (SSN) method [5] cast in primal–dual active-set form. Proper Orthogonal Decomposition (POD) bases are then built for the primal and dual variables, leading to a projected reduced-order model (ROM) solved by SSN in the reduced space.
When the solution manifold is poorly approximated by low-dimensional linear spaces (e.g., slow Kolmogorov n-width decay), nonlinear MOR strategies can be advantageous. Building on recent approaches, a neural-network-augmented ROM (NN-ROM) [2] is introduced. More precisely, two feedforward networks learn corrections for the truncated POD components of the primal and dual variables ; this nonlinear manifold approximation is then embedded within the SSN iterations. To address the remaining online costs that scale with the high-dimensional model (HDM) size, hyper-reduction is investigated via a sparse cubature scheme based on greedy nonnegative least squares (greedy-NNLS) [1].
The methodology is assessed on two test cases : a $2D$ obstacle problem with a cubic nonlinearity in the state equation, and a $3D$ frictional contact problem with a nonlinearity in the constraint equation arising from the projection operator associated with the Coulomb friction law. Numerical experiments on the $2D$ obstacle case show that the hyper-reduced NN-ROM maintains a relative error comparable to those of the other reduced models, while yielding substantial speedups over the HDM. Extension to the 3D frictional contact setting and full validation are ongoing and may be presented at the talk.

[1] S. S. An, T. Kim, D. L. James. *Optimizing cubature for efficient integration of subspace deformations.* ACM Transactions on Graphics, **27(5)**, 1–10, 2008. doi :10.1145/1409060.1409118.

[2] J. Barnett, C. Farhat, Y. Maday. *Neural-network-augmented projection-based model order reduction for mitigating the Kolmogorov barrier to reducibility.* Journal of Computational Physics, **492**, 112420, 2023. doi :10.1016/j.jcp.2023.112420.

[3] J. Droniou, G. Enchéry, I. Faille, A. Haidar, R. Masson. *A bubble VEM-fully discrete polytopal scheme for mixed-dimensional poromechanics with frictional contact at matrix–fracture interfaces.* Computer Methods in Applied Mechanics and Engineering, **422**, 116838, 2024. doi :10.1016/j.cma.2024.116838.

[4] C. Grossmann. *Numerical methods for variational inequalities and optimal control.* In *Numerical Treatment of Partial Differential Equations : Translated and Revised by Martin Stynes*, pp. 435–498. Springer, Berlin, Heidelberg, 2007. doi :10.1007/978-3-540-71584-9_7.

[5] B. Wohlmuth. *Variationally consistent discretization schemes and numerical algorithms for contact problems.* Acta Numerica, **20**, 569–734, 2011. doi :10.1017/s0962492911000079.

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