A hyperelastic model for cardiac deformation applied to multimodal registration of MRI and optical mapping data

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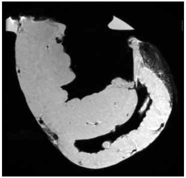
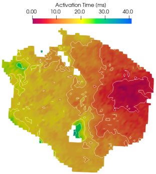
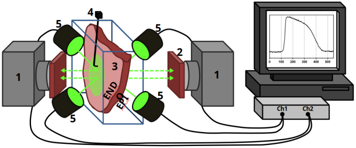
Motivation

MRI



High-resolution MRI (9.4T)





Optical mapping

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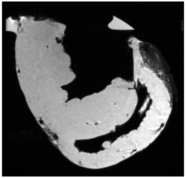

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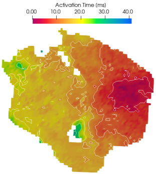
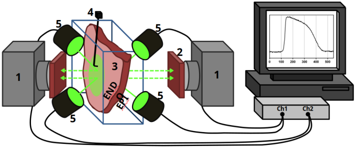
Microstructure - Fibers





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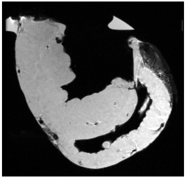

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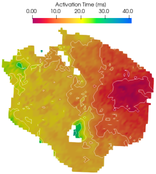

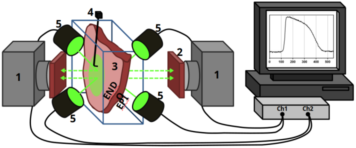

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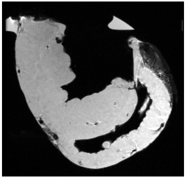
Goal: combine structural and functional information to better understand arrhythmia, responsible for a lot of cardiovascular deaths.



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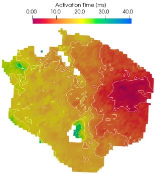

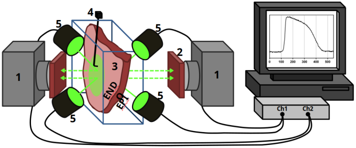

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Optical mapping
Electrical activation map

Key idea

We want to deform the MRI geometry onto the optical mapping geometry.

$$\varphi = \mathrm{id} + \boldsymbol{u} : \Omega_0 \to \Omega$$

- Ω_0: reference domain (resting position);
- Ω: deformed domain (streched position);
- $\boldsymbol{u}$: displacement field.

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- Ω_0: reference domain (resting position);
- Ω: deformed domain (streched position);
- $\boldsymbol{u}$: displacement field.

The displacement $\boldsymbol{u}$ is obtained by solving a nonlinear hyperelastic equilibrium problem:

$$\begin{cases} -\mathrm{div}(\mathrm{P}(\boldsymbol{u})) &= \boldsymbol{f} \quad \text{in } \Omega_0, \\ \mathrm{P}(\boldsymbol{u}) \cdot \mathbf{n} &= \boldsymbol{g} \quad \text{on } \Gamma_N, \\ \boldsymbol{u} &= 0 \quad \text{on } \Gamma_D, \end{cases}$$

where

- $\partial\Omega_0 = \Gamma_D \cup \Gamma_N$, Γ_D is the Dirichlet boundary and Γ_N is the Neumann boundary
- P is the first Piola–Kirchhoff stress tensor,
- $\mathbf{n}$ the outward unit normal vector to $\partial\Omega_0$,
- $\boldsymbol{f}$ denotes the body force in Ω_0 and $\boldsymbol{g}$ denotes the prescribed traction on Γ_N.

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Kinematics and hyperelastic definitions

The deformation is described by the map

$$\varphi : \Omega_0 \to \mathbb{R}^d, \qquad \varphi(X) = X + \boldsymbol{u}(X), \qquad X \in \Omega_0,$$

where $\boldsymbol{u} : \Omega_0 \to \mathbb{R}^d$ is the displacement field.

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The main quantities are

- $\mathrm{F} := \nabla\varphi = \mathrm{Id} + \nabla\boldsymbol{u}$ is the deformation gradient,
- $J := \det \mathrm{F}$ measures the local change of volume,
- $\mathbf{C} := \mathrm{F}^T\mathrm{F}$ is the right Cauchy–Green tensor and measures strain in Ω_0.

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Hyperelastic material

A hyperelastic material is fully described by a **stored energy density** W.
The associated **stress tensor** is defined by

$$\mathrm{P} = \frac{\partial W}{\partial \mathrm{F}}.$$

This framework is well suited to **large deformations** and **soft biological tissues**.

Examples of hyperelastic materials

- **Saint Venant–Kirchhoff** $\rightarrow$ a simple extension of linear elasticity
- **Neo-Hookean** $\rightarrow$ simple isotropic model for rubber-like soft materials
- **Mooney–Rivlin** $\rightarrow$ an enriched Neo-Hookean model

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In this work

We use a nearly-incompressible **Holzapfel-Gasser-Ogden type energy** to model cardiac tissue, with **reduced invariants**.

Invariants

The HGO model is written with respect to scalar invariants of the tensor $\mathbf{C}$.

Principal invariants

$$\det(\mathbf{C} - \lambda \mathrm{Id}) = \sum_{p=0}^{d} (-1)^p I_p \lambda^{d-p}, \qquad I_0 := 1.$$

In particular, in 3D,

- $I_1 := \mathrm{tr}(\mathbf{C})$ measures the amount of stretch;
- $I_2 := \frac{1}{2}\left(I_1^2 - \mathrm{tr}(\mathbf{C}^2)\right)$ measures area changes;
- $I_3 = \det(\mathbf{C}) = J^2$ measures volume change.

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Fiber invariant

To represent the local fiber direction, we denote:

- $\mathbf{a} : \Omega_0 \to \mathbb{R}^d$ the tangential unit vector in the fiber direction;
- the additional invariant that measures the squared stretch in the fiber direction,

$$I_4 := \mathbf{a}^\mathsf{T} \mathbf{C}\, \mathbf{a} = \|\mathrm{F}\mathbf{a}\|^2.$$

The Holzapfel–Gasser–Ogden energy for fiber-reinforced cardiac tissue

Assumptions on the material

- Nearly incompressible: local volume changes are strongly penalized, so $J \simeq 1$;
- Transverse isotropy induced by the fiber field $\mathbf{a} : \Omega_0 \to \mathbb{R}^d$.
- **Isochoric split:** shape changes and volume changes are modeled separately.

The hyperleastic energy is given by,

$$W(\mathbf{C}(\boldsymbol{u})) = \underbrace{\widetilde{W}(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_4)}_{\text{isochoric part}} + \underbrace{W_{\mathrm{vol}}(J)}_{\text{volumetric part}}$$

$$= \underbrace{W_{\mathrm{lin}}(\widetilde{I}_1, \widetilde{I}_2) + W_{\mathrm{iso}}(\widetilde{I}_1) + W_{\mathrm{fib}}(\widetilde{I}_4)} + W_{\mathrm{vol}}(J).$$

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Since near-incompressibility is handled separately by $W_{\mathrm{vol}}(J)$, the isochoric part is written in terms of reduced invariants given by

$$\widetilde{I}_1 = J^{-2/d} I_1, \qquad \widetilde{I}_2 = J^{-4/d} I_2, \qquad \widetilde{I}_4 = J^{-2/d} I_4.$$

Interpretation of the energy terms and material parameters

$$W(\mathbf{C}(\boldsymbol{u})) = W_{\mathrm{lin}}(\widetilde{I}_1, \widetilde{I}_2) + W_{\mathrm{iso}}(\widetilde{I}_1) + W_{\mathrm{fib}}(\widetilde{I}_4) + W_{\mathrm{vol}}(J)$$

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- A volume penalizing term: $W_{\mathrm{vol}}(J) = \kappa(J-1) - \kappa \ln J$,
  - this term enforces near-incompressibility
  - κ: bulk modulus parameter

From stored energy to the equilibrium equations

We seek a displacement $\boldsymbol{u} : \Omega_0 \to \mathbb{R}^d$ such that

$$\boldsymbol{u} \in \operatorname*{arg\,min}_{\boldsymbol{v} \in V} \mathrm{E}(\boldsymbol{v}), \qquad V := \left\{ \boldsymbol{v} \in H^1(\Omega_0; \mathbb{R}^d) \ : \ \boldsymbol{v} = 0 \text{ on } \Gamma_D \right\}.$$

where the total energy is

$$\mathrm{E}(\boldsymbol{u}) = \underbrace{\int_{\Omega_0} W\big(\mathbf{C}(\boldsymbol{u})\big)\,\mathrm{d}x}_{\text{hyperelastic energy}} \underbrace{- \int_{\Omega_0} \boldsymbol{f} \cdot \boldsymbol{u}\,\mathrm{d}x - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}\,\mathrm{d}s}_{\text{external forces}}.$$

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Instead, we seek $\boldsymbol{u}$ such that

$$\mathrm{DE}(\boldsymbol{u};\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in V. \tag{1}$$

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After integration by parts (1), we recover the full hyperelastic equilibrium system.

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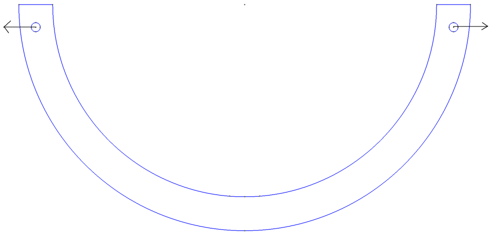
Test case in 2D



	Symbol	Value
Inner radius	r_i	0.85
Outer radius	r_o	1.00
Holes' radius	ε	0.02

- Opposite horizontal tractions on the two holes:

$$\boldsymbol{g}_{left} = -g_0\,\mathbf{e}_x, \qquad \boldsymbol{g}_{right} = +g_0\,\mathbf{e}_x.$$

- No body force is considered, $\boldsymbol{f} = 0$.
- A Dirichlet condition is imposed on 2% of the inner arc to remove rigid-body motions.

Finite element discretization using FreeFem++

The domain Ω is discretized with a triangular mesh in 2D.

- Maillage global : 10614 triangles

The displacement $\boldsymbol{u}$ is approximated in a $\mathbb{P}_1$ Lagrange finite element space:

$$\boldsymbol{u}_h \in V_h.$$

Discrete equilibrium: Find $\boldsymbol{u}_h \in V_h$ such that

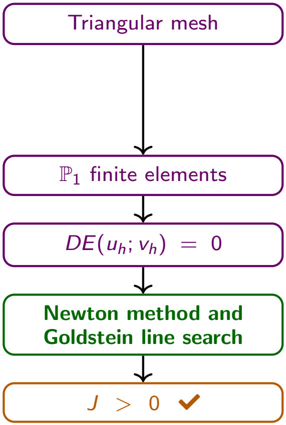
$$\mathrm{DE}(\boldsymbol{u}_h; \boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in V_h.$$

Admissibility constraint: We only consider deformations such that

$$\boxed{J = \det(\nabla\varphi) > 0}$$

to avoid element inversion.

Parameters	ν_1	C_0	C_1	C_2	C_3	κ
Values	1	10	0.5	15	0.9	800



Finite element discretization using FreeFem++

The domain Ω is discretized with a triangular mesh in 2D.

- Maillage global : 10614 triangles
- Maillage local : 1782 triangles

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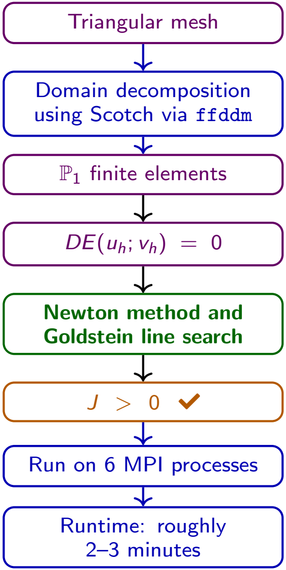
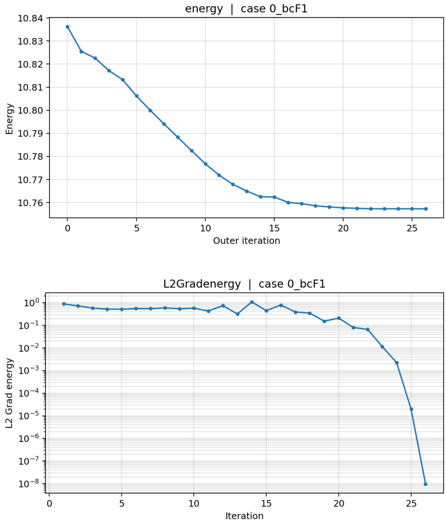


Energy convergence indicators



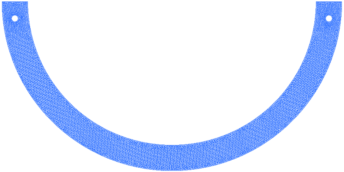
Results

- The energy decreases progressively during the iterations.
- The L^2-norm of the energy gradient tends to zero at convergence.

Quasi-incompressible criteria :

- $J = \det(\nabla\varphi) > 0$ ✔
- $0.985 < J < 1.02$ ✔

Deformed domain



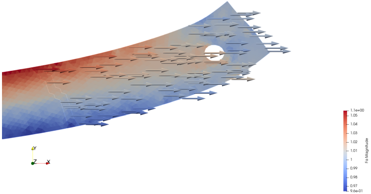
$\varphi = \mathrm{id} + \boldsymbol{u}$

Reference domain

Deformed domain

Post-processing:

- The mesh is progressively deformed from the reference to the deformed configuration.
- No mesh element inversion occurs ✔
- **Observed volume variation:** 0.045%
- The **fiber field** is transported from the reference configuration to the deformed domain.



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Conclusion and next steps

What was established

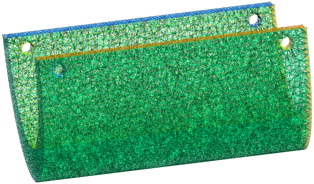
- Modeling the cardiac mechanical behavior using a Holzapfel–Ogden energy.
- Solving the 2D toy model validating the established code.

On going work

- Solve the direct problem on a 3D toy domains.
- Solving the associated inverse problem in 2D.

Next steps

- Solving the associated inverse problem in 3D.
- Move to the 3D case on segmentation of a heart tissue sample.



Thank you for your attention

Research interests

- **Numerical analysis:** high-order boundary conditions, error estimation, spectral problems, linear elasticity models, and nonlinear hyperelasticity with large deformations.
- **Scientific computing:** surface/volume finite elements, curved meshes, shape optimization, and registration methods.

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