# Analysis of a two-level **domain decomposition** preconditioner for the time-harmonic Maxwell equations in **anisotropic media**

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CANUM, June 2026

# Overview

# Overview

1. Maxwell setting and additive Schwarz
2. Theoretical results
3. Numerical experiments

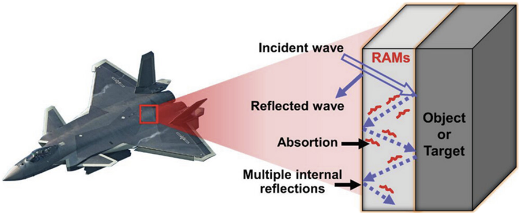
# Electromagnetic waves in complex media

## Time-harmonic Maxwell equations

$$\begin{cases} \operatorname{curl}\left(\mu^{-1} \operatorname{curl} \boldsymbol{E}\right) - \omega^2 \varepsilon_c \boldsymbol{E} = \boldsymbol{F} & \text{in } \Omega \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases} \xrightarrow{\text{after discretization}} \boxed{A_\omega \mathbf{x} = \mathbf{b}},$$

**Notation:**

- $\boldsymbol{E} : \Omega \to \mathbb{C}^3$ electric field
- $\mu : \Omega \to \mathbb{R}^{3\times 3}$ magnetic permeability
- $\varepsilon_c : \Omega \to \mathbb{C}^{3\times 3}$ electric permittivity
- $\omega > 0$ temporal frequency




Motivating example: radar-absorbing materials [Kim et. al 2023]

**How does tensor anisotropy affect iterative solvers?**

# Challenges for efficient solvers

**Maxwell-specific analytical issues:**

- curl operator has a large structured kernel.
- Regularity theory and tensor-weighted spaces.

**Time harmonic waves:**

- Mesh and coarse scales must resolve waves.
- Indefinite operator and strongly non-normal

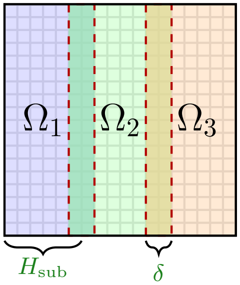
**Solver implication:**

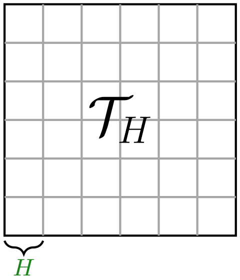
- Krylov solvers can stagnate
- Preconditioners crucial for scalability

**Context:** preconditioned GMRES convergence analysis for scalar absorptive media
Bonazzoli–Dolean–Graham–Spence–Tournier, 2019

**Goal:** extend this theory to anisotropic tensor coefficients.

# Domain decomposition: two-level additive Schwarz (AS)




- Solve local problems on subdomains in parallel
- Combine back summing in the overlap

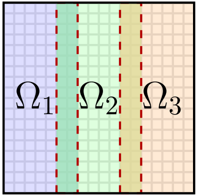
- Discretize and solve on coarse mesh
- Map result back to fine mesh

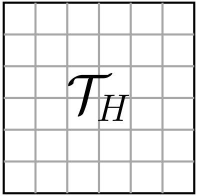
Widlund–Toselli, 2005
Dolean–Nataf–Jolivet, 2015

$$A_\omega \mathbf{x} = \mathbf{b} \Longrightarrow B_{\text{AS}}^{-1} A_\omega \mathbf{u} = B_{\text{AS}}^{-1}\mathbf{b}.$$

**Discrete spaces and restrictions:**

- Fine Nédélec space: $\boldsymbol{Q}_h \subset \boldsymbol{H}_0(\text{curl};\Omega)$.
- Local spaces: $\boldsymbol{Q}_h^\ell = \boldsymbol{Q}_h \cap \boldsymbol{H}_0(\text{curl};\Omega_\ell)$.
- Coarse space: $\boldsymbol{Q}_H \subset \boldsymbol{H}_0(\text{curl};\Omega)$ on $\mathcal{T}_H$.
- Restriction matrices:
  $R^\ell$ : fine DOFs $\to$ local/coarse DOFs.




**Two-level preconditioner**

For $A_\omega^\ell = R^\ell A_\omega (R^\ell)^T$,

$$B_{\text{AS}}^{-1} = \underbrace{(R^0)^T (A_\omega^0)^{-1} R^0}_{\text{coarse correction}} + \underbrace{\sum_{\ell=1}^{N_{\text{sub}}} (R^\ell)^T (A_\omega^\ell)^{-1} R^\ell}_{\text{subdomain corrections}}$$

# Overview

# Variational formulation and material parameters

**Variational problem**

Find $\boldsymbol{E} \in \boldsymbol{H}_0(\mathrm{curl};\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl};\Omega)$

$$\left(\mu^{-1}\,\mathrm{curl}\,\boldsymbol{E},\,\mathrm{curl}\,\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2\,(\epsilon_c \boldsymbol{E},\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{F},\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\,, \qquad \epsilon_c = \epsilon + \mathrm{i}\omega^{-1}\sigma.$$

$\epsilon, \mu, \sigma : \Omega \to \mathbb{R}^{3\times 3}$ are *uniformly SPD*, i.e. for almost every $\boldsymbol{x} \in \Omega$,

$$\xi_-|\boldsymbol{z}|^2 \le (\xi(\boldsymbol{x})\boldsymbol{z}) \cdot \boldsymbol{z} \le \xi_+|\boldsymbol{z}|^2, \qquad \xi = \epsilon, \mu, \sigma.$$

**Coefficient contrasts:** $\kappa_\mu := \dfrac{\mu_+}{\mu_-}, \qquad \kappa_\epsilon := \dfrac{\epsilon_+}{\epsilon_-}, \qquad \kappa_{\mu,\epsilon} := \max\{\kappa_\mu, \kappa_\epsilon\}.$

**Relative conductivity:** $\tilde{\sigma} := \epsilon^{-1/2}\sigma\epsilon^{-1/2}, \qquad$ key scale: $\dfrac{\tilde{\sigma}_-}{\omega}.$

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# Continuity and (rotated) coercivity

**Sesquilinear form:** $a_\omega(\boldsymbol{E}, \boldsymbol{v}) := \left(\mu^{-1} \operatorname{curl} \boldsymbol{E},\, \operatorname{curl} \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2 \left(\varepsilon_c\, \boldsymbol{E},\, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}.$

**Energy norm:** $\|\boldsymbol{v}\|^2_{\mu,\varepsilon,\omega} := \left\|\mu^{-1/2} \operatorname{curl} \boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)} + \omega^2 \left\|\varepsilon^{1/2} \boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)}$

## Continuity

For all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}; \Omega)$

$$|a_\omega(\boldsymbol{u}, \boldsymbol{v})| \lesssim \|\boldsymbol{u}\|_{\mu,\varepsilon,\omega} \, \|\boldsymbol{v}\|_{\mu,\varepsilon,\omega},$$

## Rotated coercivity

For all $\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl}; \Omega)$

$$|a_\omega(\boldsymbol{v}, \boldsymbol{v})| \geq \operatorname{Im}\left(e^{i\theta_\omega} a_\omega(\boldsymbol{v}, \boldsymbol{v})\right) \geq c_\omega \, \|\boldsymbol{v}\|^2_{\mu,\varepsilon,\omega},$$

where $c_\omega := \dfrac{\tilde{\sigma}_-}{\sqrt{\tilde{\sigma}_-{}^2 + 4\omega^2}} \sim \dfrac{\tilde{\sigma}_-}{\omega}$

# Stable decomposition in the energy norm

**Weighted stable splitting**

For every $\boldsymbol{v}_h \in \boldsymbol{Q}_h$, there exists $\boldsymbol{v}^\ell \in \boldsymbol{Q}_h^\ell$, $\ell = 1, \ldots, N_{\text{sub}}$, $\boldsymbol{v}^0 \in \boldsymbol{Q}_H$, such that

$$\boldsymbol{v}_h = \boldsymbol{v}^0 + \sum_{\ell=1}^{N_{\text{sub}}} \boldsymbol{v}^\ell .$$

and moreoever,

$$\sum_{\ell=0}^{N_{\text{sub}}} \left\| \boldsymbol{v}^\ell \right\|^2_{\mathbb{\mu},\mathbb{\epsilon},\omega} \lesssim \kappa_{\mu,\epsilon} \left(1 + (H/\delta)^2\right) \|\boldsymbol{v}_h\|^2_{\mathbb{\mu},\mathbb{\epsilon},\omega} .$$

**Proof ingredients:**

- regular decomposition in $\boldsymbol{H}_0(\mathrm{curl};\Omega)$
- (unweighted) interpolation stability
- finite overlap

# GMRES proof strategy

We apply GMRES to the Schwarz-preconditioned Maxwell matrix

$$C_\omega := B_{\mathrm{AS}}^{-1} A_\omega .$$

$$\text{Field-of-values (FOV)} = W(C_\omega) := \left\{ \langle \mathbf{v}, C_\omega \mathbf{v} \rangle_D \; : \; \|\mathbf{v}\|_D = 1 \right\}.$$

$$\underbrace{\nu := \operatorname{dist}(0, W(C_\omega)),}_{\text{FOV lower bound}} \qquad \underbrace{\Lambda := \|C_\omega\|}_{\text{operator norm upper bound}} \quad .$$

$$\text{control of } \nu, \Lambda \Longrightarrow \text{GMRES convergence.}$$


Eisenstat–Stanley–Elman–Schultz, 1986
Starke 1997

# Schwarz projections: matrix bounds as PDE estimates

**Local/coarse Ritz projections for $a_\omega(\cdot,\cdot)$**

$$\ell \geq 1 : \boldsymbol{T}_\omega^\ell : \boldsymbol{Q}_h \to \boldsymbol{Q}_h^\ell, \qquad \boldsymbol{T}_\omega^0 : \boldsymbol{Q}_h \to \boldsymbol{Q}_H, \qquad \boldsymbol{T}_\omega := \sum_{\ell=0}^{N_{\text{sub}}} \boldsymbol{T}_\omega^\ell.$$

Correspondence $\boldsymbol{T}_\omega \leftrightarrow C_\omega$.

For the coefficient vector $\mathbf{v}$ of $\boldsymbol{v}_h$,

$$\langle \mathbf{v}, C_\omega \mathbf{v} \rangle_D \quad \text{bounded via} \quad (\boldsymbol{v}_h, \boldsymbol{T}_\omega \boldsymbol{v}_h)_{\mu,\epsilon,\omega}.$$

- FOV lower bound: estimate $(\boldsymbol{v}_h, \boldsymbol{T}_\omega \boldsymbol{v}_h)_{\mu,\epsilon,\omega}$ from below

**Remainder decomposition:**

$$(\boldsymbol{v}_h, \boldsymbol{T}_\omega \boldsymbol{v}_h)_{\mu,\epsilon,\omega} = \underbrace{\sum_{\ell=0}^{N} \left\| \boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h \right\|_{\mu,\epsilon,\omega}^2}_{\text{coercive/local terms}} + \underbrace{\sum_{\ell=0}^{N} \boldsymbol{R}_\omega^\ell(\boldsymbol{v}_h)}_{\text{remainders without fixed sign}}$$

**Main difficulty:** controlling remainder terms so positive terms dominate $\Longrightarrow$ *resolution conditions*

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# Main theorem

**Explicit GMRES contraction**

Under the resolution conditions:

$$\max\left\{C_{\text{sub}}\omega H_{\text{sub}},\, C_{\text{coarse}}\omega^2 H^s\right\} \lesssim \left[\left(1+(H/\delta)^2\right)\kappa_{\mu,\epsilon}\right]^{-1}\frac{\tilde{\sigma}_-{}^2}{\tilde{\sigma}_-{}^2+4\omega^2}.$$

the GMRES residual $\mathbf{r}_m := B_{\text{AS}}^{-1}(A\mathbf{x}_m-\mathbf{b})$ satisfies

$$\frac{\|\mathbf{r}_m\|}{\|\mathbf{r}_0\|} \le q^m, \quad 0<q<1,$$

where $q = q(\text{FOV control})$ monotone decreasing and

$$\text{FOV control} \approx \frac{1}{1+(H/\delta)^2}\frac{1}{\kappa_{\mu,\epsilon}}\left(\frac{\tilde{\sigma}_-{}^2}{\tilde{\sigma}_-{}^2+4\omega^2}\right)^{3/2}.$$

$\underbrace{\delta/H \to 0}_{\text{small overlap}}$ or $\underbrace{\kappa_{\mu,\epsilon}\to\infty}_{\text{large contrast}}$ or $\underbrace{\tilde{\sigma}_-/\omega \to 0}_{\text{high frequency}} \implies$ FOV control $\downarrow 0 \implies q \uparrow 1.$

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# Overview

# Numerical setup

**Problem setup**

- Unit cube: $\Omega = (0,1)^3$ uniform tetrahedral meshes
- 2nd order Nédélec elements (first kind)

**Solver**

- Right preconditioned GMRES (no restart)
- Reported quantity: GMRES iterations to reach $\|\mathbf{r}_m\| \,/\, \|\mathbf{r}_0\| \leq 10^{-6}$.
- Zero initial guess $\mathbf{x}_0 = \mathbf{0}$
- Minimal overlap $\delta$ one layer of elements

Tests run on the CLEPS cluster at Inria Paris

# Scalar reference experiment: code validation

**Scaling:** keep $\omega h$ and $\omega H$ approximately fixed as $\omega$ increases.
**Scalar absorptive reference regime:** $\mathbb{\mu} = \mathbb{\varepsilon} = \mathbb{I}_{3\times3}, \quad \mathbb{\sigma} = \omega \mathbb{I}_{3\times3}$

| | | | | | GMRES iterations: two-level AS | |
|---|---|---|---|---|---|---|
| $\omega/2\pi$ | $h$ | $H$ | $N_{\text{sub}}$ | $\text{DOFs}_h$ | Reference code | Tensor code |
| 2 | $1/40$ | $1/4$ | 64 | $2.48 \times 10^6$ | 37 | 37 |
| 3 | $1/60$ | $1/6$ | 216 | $8.32 \times 10^6$ | 39 | 39 |
| 4 | $1/80$ | $1/8$ | 512 | $19.6 \times 10^6$ | 39 | 39 |

**Take-aways:**

- reproduces the reference implementation before fully tensor coefficients.
- #iters remain essentially constant under the scalar wave-resolution scaling

# Normalized anisotropy scaling

**Determinant-normalized anisotropy**

$$\varepsilon_\eta = \operatorname{diag}\left(\eta^{2/3}, \eta^{-1/3}, \eta^{-1/3}\right),$$

$$\det(\varepsilon_\eta) = 1, \qquad \kappa_\epsilon = \eta.$$

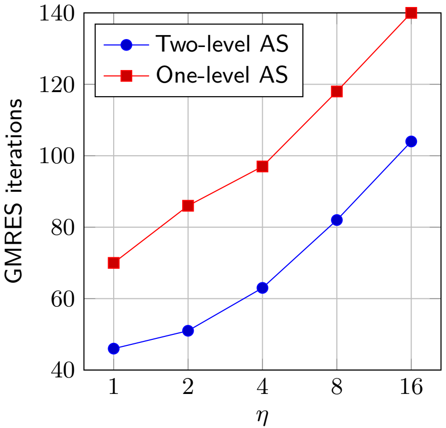
$$\varepsilon_c = (1 + \tfrac{i}{2})\varepsilon_\eta, \qquad \mu = \mathbb{I}_{3\times 3}.$$

Equivalently,

$$\tilde{\sigma} = \varepsilon_\eta^{-1/2} \sigma \varepsilon_\eta^{-1/2} = \tfrac{1}{2}\omega\, \mathbb{I}_{3\times 3},$$

i.e. the scale $\tilde{\sigma}_- / \omega$ is independent of $\eta$

$\omega/2\pi = 2$ and the mesh/DD parameters are fixed.



# Conclusion

**Summary:**

- Two-level Schwarz for absorptive Maxwell extends to tensor media.
- GMRES estimate tracks DD parameters, coefficient parameters and frequency
- HAL preprint [Bonazzoli–Ciarlet–Modave–R, 2026]

**Perspectives:**

- Extend the tensor theory to absorbing boundary conditions.
- Design anisotropy-aware coarse spaces.
- Separate anisotropy effects from spatial heterogeneity.

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Thank you for your attention!

# Frequency scaling at fixed anisotropy

**Question:** for nontrivial anisotropy and fixed absorption, do GMRES iterations remain stable as $\omega$ increases?

Fixed tensor contrast and absorption: $\kappa_\varepsilon = 8, \quad \tilde{\sigma} = \frac{1}{2}\omega \mathbb{I}_{3\times 3}$.

| | | | | | GMRES iterations | |
|---|---|---|---|---|---|---|
| $\omega/2\pi$ | $h$ | $H$ | $N_{\text{sub}}$ | $\text{DOFs}_h$ | 2L AS | 1L AS |
| 1 | $1/40$ | $1/4$ | 64 | $2.48 \times 10^6$ | 52 | 127 |
| 2 | $1/80$ | $1/8$ | 512 | $1.96 \times 10^7$ | 62 | $> 200$ |

**Take-away:**

- Doubling $\omega$ increases the fine problem size by a factor of about $8$
- two-level AS only increases from $52$ to $62$ GMRES iterations.

# Right preconditioning and the adjoint

Left-preconditioned theory gives FOV bounds for

$$C_L := B_{\mathrm{AS}}^{-1} A_\omega$$

in the energy inner product induced by $D_{\mu,\epsilon,\omega}$:

$$\|C_L\|_{D_{\mu,\epsilon,\omega}} \le \Lambda, \qquad \operatorname{dist}\big(0, W_{D_{\mu,\epsilon,\omega}}(C_L)\big) \ge \nu.$$

For the right-preconditioned matrix

$$C_R := A_\omega B_{\mathrm{AS}}^{-1},$$

we instead use the $D_{\mu,\epsilon,\omega}^{-1}$-inner product. With $\mathbf{w} = D_{\mu,\epsilon,\omega}^{-1}\mathbf{v}$,

$$\frac{\left|\left\langle \mathbf{v}, A_\omega B_{\mathrm{AS}}^{-1}\mathbf{v} \right\rangle_{D_{\mu,\epsilon,\omega}^{-1}}\right|}{\|\mathbf{v}\|^2_{D_{\mu,\epsilon,\omega}^{-1}}} = \frac{\left|\left\langle \mathbf{w}, (B_{\mathrm{AS}}^*)^{-1} A_\omega^* \mathbf{w} \right\rangle_{D_{\mu,\epsilon,\omega}}\right|}{\|\mathbf{w}\|^2_{D_{\mu,\epsilon,\omega}}}.$$

**Key point**

Right preconditioning for $A_\omega$ is reduced to left preconditioning for the adjoint problem.

The adjoint form corresponds to changing the sign of the absorptive part,

$$\epsilon_c = \epsilon + \mathrm{i}\omega^{-1}\sigma \quad \rightsquigarrow \quad {\epsilon_c}^* = \epsilon - \mathrm{i}\omega^{-1}\sigma.$$

The rotated coercivity estimate is unchanged, up to reversing the rotation angle. Hence the same FOV and GMRES bounds apply.

# Elman criterion and field of values

**Elman-type criterion**

$$\operatorname{dist}(0, W(C)) \geq \nu > 0, \qquad \|C\| \leq \Lambda,$$

then GMRES for $Cx = \mathbf{b}$ satisfies

$$\frac{\|\mathbf{r}_m\|}{\|\mathbf{r}_0\|} \leq \left(1 - \frac{\nu^2}{\Lambda^2}\right)^{m/2}.$$

**Proof sketch:** GMRES minimizes over residual polynomials:

$$\|\mathbf{r}_m\| = \min_{\substack{p \in \mathcal{P}_m \\ p(0)=1}} \|p(C)\mathbf{r}_0\|.$$

Since $\operatorname{dist}(0, W(C)) \geq \nu$, after a rotation one has

$$\operatorname{Re}\langle \mathbf{v}, C\mathbf{v}\rangle_D \geq \nu \|\mathbf{v}\|_D^2.$$

For $p(z) = (1 - \lambda z)^m$,

$$\|(I - \lambda C)\mathbf{v}\|^2 \leq \|\mathbf{v}\|^2 - 2\nu \operatorname{Re}\langle \mathbf{v}, C\mathbf{v}\rangle_D + \lambda^2 \|C\|^2 \leq \left(1 - 2\lambda\nu + \lambda^2\Lambda^2\right) \|\mathbf{v}\|^2.$$

Choosing $\lambda = \nu/\Lambda^2$ gives the stated contraction.

# Where scalar orthogonality is lost

$$(\boldsymbol{v}_h,\, \boldsymbol{T}_\omega \boldsymbol{v}_h)_{\mu,\epsilon,\omega} = \sum_{\ell=0}^{N} \left\| \boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h \right\|_{\mu,\epsilon,\omega}^2 + \sum_{\ell=0}^{N} R_\omega{}^\ell(\boldsymbol{v}_h).$$

The anisotropic remainder contains

$$R_\omega{}^\ell(\boldsymbol{v}_h) = \omega^2 \left( (2\epsilon + \mathrm{i}\omega^{-1}\sigma)(\boldsymbol{I} - \boldsymbol{T}_\omega{}^\ell)\boldsymbol{v}_h,\, \boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h \right)_{\boldsymbol{L}^2(\Omega_\ell)}.$$

Projection orthogonality is naturally available for the full tensor $\epsilon_c = \epsilon + \mathrm{i}\omega^{-1}\sigma$, not separately for $\epsilon$ and $\sigma$. The rewrite $2\epsilon + \mathrm{i}\omega^{-1}\sigma = 2\epsilon_c - \mathrm{i}\omega^{-1}\sigma$ isolates a conductivity remainder controlled by $\tilde{\sigma}_+ \le C_{\mathrm{wa}}\omega$.
More explicitly, for $\boldsymbol{e}_h = (\boldsymbol{I} - \boldsymbol{T}_\omega{}^\ell)\boldsymbol{v}_h$ and $\boldsymbol{w}_h = \boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h$,

$$\omega \left| (\sigma \boldsymbol{e}_h,\, \boldsymbol{w}_h)_{\boldsymbol{L}^2(\Omega_\ell)} \right| \le C_{\mathrm{wa}} \left\| \boldsymbol{e}_h \right\|_{\mu,\epsilon,\omega} \left\| \boldsymbol{w}_h \right\|_{\mu,\epsilon,\omega}.$$

## Where duality and regularity enter

For the coarse defect $\boldsymbol{e}_{h,0} := (\boldsymbol{I} - \boldsymbol{T}_{\omega}{}^{0})\boldsymbol{v}_h$, introduce an adjoint/dual problem:

$$a_{\omega}{}^{*}(\boldsymbol{u}, \boldsymbol{z}) = (\boldsymbol{s},\, \boldsymbol{z})_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{z} \in \boldsymbol{H}_0(\mathrm{curl}; \Omega).$$

- Regularity gives $\boldsymbol{u}$ enough smoothness for coarse approximation.
- The coarse projection uses Galerkin orthogonality against $\boldsymbol{Q}_H$.
- This recovers the small factor

$$\left\|(\boldsymbol{I} - \boldsymbol{T}_{\omega}{}^{0})\boldsymbol{v}_h\right\|_{\mu,\epsilon,\omega} \quad \text{with coefficient} \quad \omega^2 H^s C_{\mathrm{coarse}}.$$

- Duality and regularity are the technical ingredients behind the coarse-scale smallness in the FOV remainder estimate.

# From tensor estimates to a GMRES angle

Under the technical absorption and resolution assumptions stated in backup,

$$\frac{\left|(\boldsymbol{v}_h,\, \boldsymbol{T}_\omega \boldsymbol{v}_h)_{\mathbb{\mu},\mathbb{\epsilon},\omega}\right|}{\|\boldsymbol{v}_h\|^2_{\mathbb{\mu},\mathbb{\epsilon},\omega}} \gtrsim \left[\left(1 + (H/\delta)^2\right) \kappa_{\mu,\epsilon}\right]^{-1} \frac{\tilde{\sigma}_-{}^2}{\tilde{\sigma}_-{}^2 + 4\omega^2}.$$

Together with

$$\Lambda_D = \|C_\omega\|_D \lesssim \frac{\sqrt{\tilde{\sigma}_-{}^2 + 4\omega^2}}{\tilde{\sigma}_-},$$

this gives the GMRES angle estimate

$$\gamma := \frac{\nu_D}{\Lambda_D} \gtrsim \frac{1}{1 + (H/\delta)^2} \frac{1}{\kappa_{\mu,\epsilon}} \left(\frac{\tilde{\sigma}_-{}^2}{\tilde{\sigma}_-{}^2 + 4\omega^2}\right)^{3/2}.$$

# Weak absorption and resolution conditions

The conductivity remainder is controlled by

$$\tilde{\sigma}_+ \le C_{\mathrm{wa}}\omega, \qquad C_{\mathrm{wa}} < 1.$$

The technical angle bound also assumes the resolution smallness condition

$$\max\left\{\omega H_{\mathrm{sub}} C_{\mathrm{sub}},\ \omega^2 H^s C_{\mathrm{coarse}}\right\} + 2C_{\mathrm{wa}} \le C_1 \left[\left(1 + (H/\delta)^2\right)\kappa_{\mu,\epsilon}\right]^{-1} \frac{\tilde{\sigma}_-^{\,2}}{\tilde{\sigma}_-^{\,2} + 4\omega^2}.$$

Here, schematically,

$$C_{\mathrm{sub}} = \epsilon_+ \sqrt{\frac{\mu_+}{\epsilon_-}}, \qquad C_{\mathrm{coarse}} = \max\left\{C_{\mathrm{dual}}, \frac{C_{\mathrm{wa}}\sqrt{\mu_+}}{\tilde{\sigma}_+}\right\} \frac{\epsilon_+}{\sqrt{\epsilon_-}}.$$

**Relation to scalar theory:** In the scalar constant setting, the corresponding assumptions reduce to the Bonazzoli et al. absorption/resolution regime, with $\tilde{\sigma}_-/\omega$ playing the role of the absorption-to-frequency scale.

# Interpretation: what the theory explains

Let

$$\rho := \frac{\tilde{\sigma}_-}{\omega}.$$

The GMRES angle estimate contains

$$\left(\frac{\tilde{\sigma}_-{}^2}{\tilde{\sigma}_-{}^2 + 4\omega^2}\right)^{3/2} = \left(\frac{\rho^2}{\rho^2 + 4}\right)^{3/2}.$$

**Weak absorption relative to frequency:** If $\rho \to 0$,

$$\left(\frac{\rho^2}{\rho^2 + 4}\right)^{3/2} \sim \frac{\rho^3}{8}.$$

The guaranteed GMRES angle collapses cubically.

**Large material contrast:** The estimate deteriorates at least like

$$\kappa_{\mu,\epsilon}{}^{-1}.$$

This includes spatial heterogeneity and pointwise anisotropy.

**The estimate separates the roles of weak absorption and material contrast.**

# Scalar reference point: Bonazzoli et al. 2019

**Reference setting**

$$-\operatorname{curl}(\operatorname{curl}\boldsymbol{E}) - (k^2 + \mathrm{i}\xi)\boldsymbol{E} = \boldsymbol{F}, \qquad k, \xi \in \mathbb{R}, \quad |\xi| \lesssim k^2.$$

- Constant scalar coefficients.
- Conforming Nédélec discretization (edge-elements)
- Two-level overlapping additive Schwarz preconditioner for GMRES.

**What the scalar theory gives**

Under resolution conditions on $H_{\text{sub}}$ and $H$,

$$|\xi| \sim k^2, \qquad \delta \sim H, \qquad H_{\text{sub}} \sim H \sim k^{-1} \quad \Longrightarrow \quad k\text{-independent GMRES iteration bound.}$$

Bonazzoli, Dolean, Graham, Spence, Tournier (2019).

# Why Ritz projections give the preconditioned matrix

Let $\boldsymbol{v}_h = \sum_j v_j \boldsymbol{\varphi}_j \in \boldsymbol{Q}_h$, with coefficient vector $\mathbf{v}$.
For each subspace $\boldsymbol{Q}_h^\ell$, define the Ritz projection

$$a_\omega(\boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h, \boldsymbol{w}_h^\ell) = a_\omega(\boldsymbol{v}_h, \boldsymbol{w}_h^\ell) \qquad \forall \boldsymbol{w}_h^\ell \in \boldsymbol{Q}_h^\ell.$$

In coordinates, this is just

$$A_\omega{}^\ell \mathbf{t}^\ell = R^\ell A_\omega \mathbf{v}, \qquad \mathbf{t}^\ell = (A_\omega{}^\ell)^{-1} R^\ell A_\omega \mathbf{v}.$$

Injecting back to the global space gives

$$\boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h \quad \longleftrightarrow \quad (R^\ell)^T (A_\omega{}^\ell)^{-1} R^\ell A_\omega \mathbf{v}.$$

Therefore, after summing local and coarse corrections,

$$\boldsymbol{T}_\omega \boldsymbol{v}_h = \sum_{\ell=0}^{N_{\mathrm{sub}}} \boldsymbol{T}_\omega{}^\ell \boldsymbol{v}_h \quad \longleftrightarrow \quad \left[ \sum_{\ell=0}^{N_{\mathrm{sub}}} (R^\ell)^T (A_\omega{}^\ell)^{-1} R^\ell \right] A_\omega \mathbf{v} = B_{\mathrm{AS}}^{-1} A_\omega \mathbf{v}.$$

$$\boxed{(\boldsymbol{v}_h, \boldsymbol{T}_\omega \boldsymbol{v}_h)_{\mu,\varepsilon,\omega} = \left\langle \mathbf{v},\, B_{\mathrm{AS}}^{-1} A_\omega \mathbf{v} \right\rangle_D}$$

Thus the PDE estimates on $\boldsymbol{T}_\omega$ are exactly estimates on the Schwarz-preconditioned matrix.