Numerical simulations of a quasilinear Gross-Pitaevskii equation with vanishing and nonvanishing conditions at infinity

Sebastián Tapia Mandiola & André de Laire & Guillaume Dujardin

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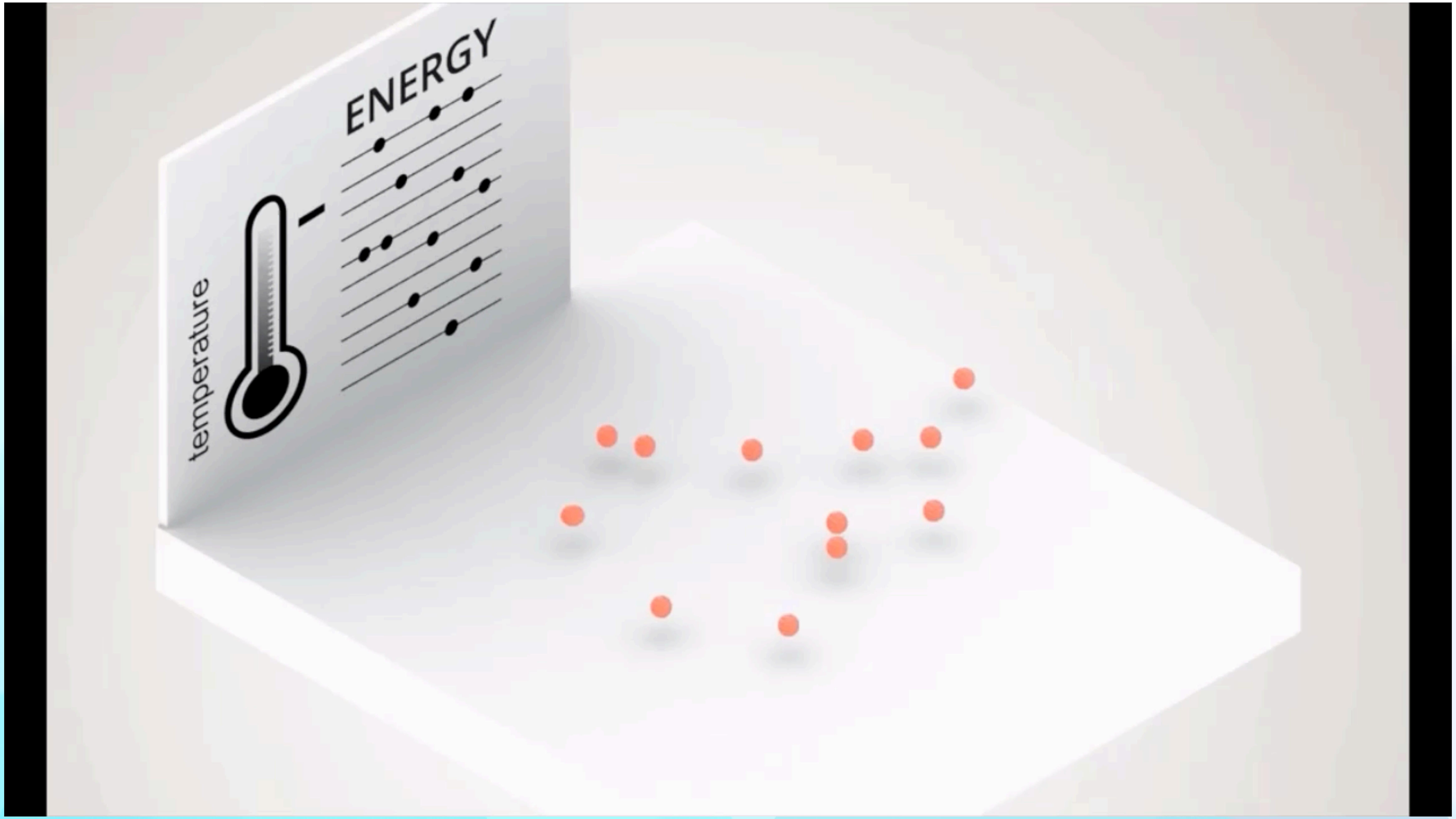
CANUM

Laboratoire Paul Painlevé
Université de Lille
Région Hauts-de-France
Inria

Bose-Einstein condensate



Quasilinear Gross–Pitaevskii equation

We consider the following one dimensional quasilinear Gross–Pitaevski equation:

$$i\partial_t \Psi = \partial_{xx}\Psi + \mathfrak{s}\Psi|\Psi|^2 + \kappa\Psi\partial_{xx}|\Psi|^2, \quad \text{(QGP)}$$

$$\lim_{|x|\to\infty} |\Psi(x,\cdot)| = \mathfrak{b}_\infty, \quad \mathfrak{b}_\infty \geq 0,$$

- $\Psi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$, $\mathfrak{s}, \kappa \in \mathbb{R}$.
- $\mathfrak{s} = 1$: **Focusing,** $\mathfrak{s} = -1$: **Defocusing** (Rescaling)
- $\mathfrak{b}_\infty = 0$: **Zero background**
- $\mathfrak{b}_\infty = 1$: **Non-zero background**
- Applications: Nonlinear optics, Bose-Enstein condensate, superfluids…

Preserved quantities:

- $N(t) = \int_{\mathbb{R}} |\Psi(x,t)|^2 \, dx$
- $E(t) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x \Psi|^2 dx - \frac{\mathfrak{s}}{4}\left(|\Psi|^2 - \mathfrak{b}_\infty\right)^2 dx + \frac{\kappa}{4}\left(\partial_x(|\Psi|^2)\right)^2 dx$

Solitons in QGP equation

In (QGP) equation there exists bright solitons when $\mathfrak{b}_\infty = 0$ (and $\mathfrak{s} = 1$), and dark solitons with $\mathfrak{b}_\infty = 1$ ($\mathfrak{s} = -1$). These solitons correspond to **smooth finite-energy traveling wave solutions**:

$$\Psi(x,t) = u(x - ct - x_0)e^{i\vartheta(t)}$$

Bright solitons (Focusing – zero background)

For $\kappa > 0$ we have an explicit formula for bright solitons:

$$u(x) = F^{-1}_{\omega,\kappa}(|x|)e^{-icx/2} \quad \text{and} \quad \vartheta(c,\omega) = -\left(\frac{c^2}{4} + \omega\right)t,$$

for all $x, t, c \in \mathbb{R}$, $\omega > 0$ where

$$F_{\omega,\kappa}(y) = \frac{1}{\sqrt{\omega}}\text{atanh}\left(\frac{1}{\sqrt{2\omega}}\sqrt{\frac{2\omega - y^2}{1 + 2\kappa y^2}}\right) + 2\sqrt{\kappa}\ \text{atan}\left(\sqrt{2\kappa}\sqrt{\frac{2\omega - y^2}{1 + 2\kappa y^2}}\right).$$

Dark solitons (Defocusing – nonzero background)

$\Psi \mapsto \Psi e^{it}$ gives: $i\partial_t\Psi = \partial_{xx}\Psi + \Psi(1 - |\Psi|^2) + \kappa\Psi\partial_{xx}(1 - |\Psi|^2)$.

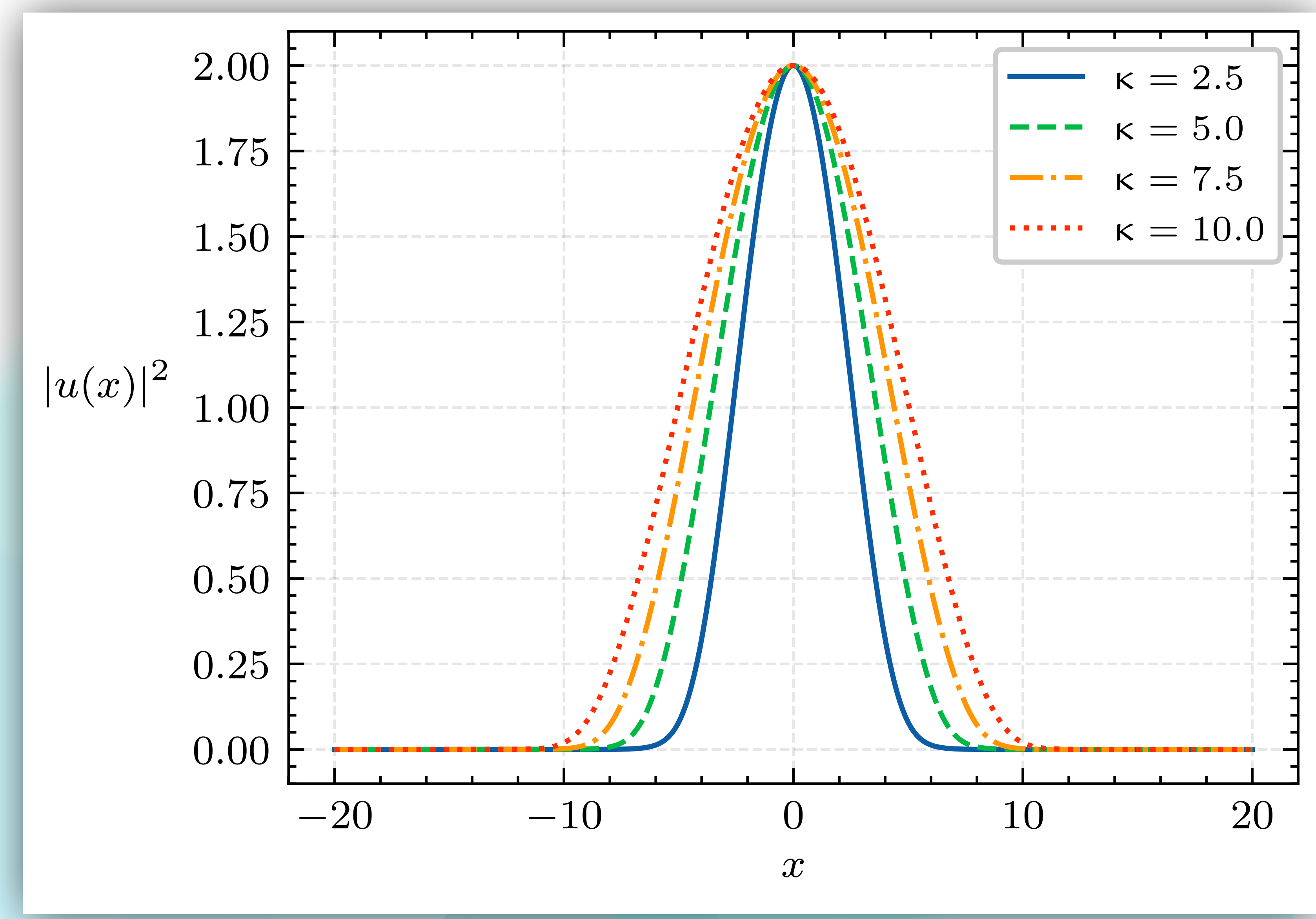
$\kappa \in (-\infty, 1/2)$ and $c \in (0, \sqrt{2})$ dark solitons are given by:

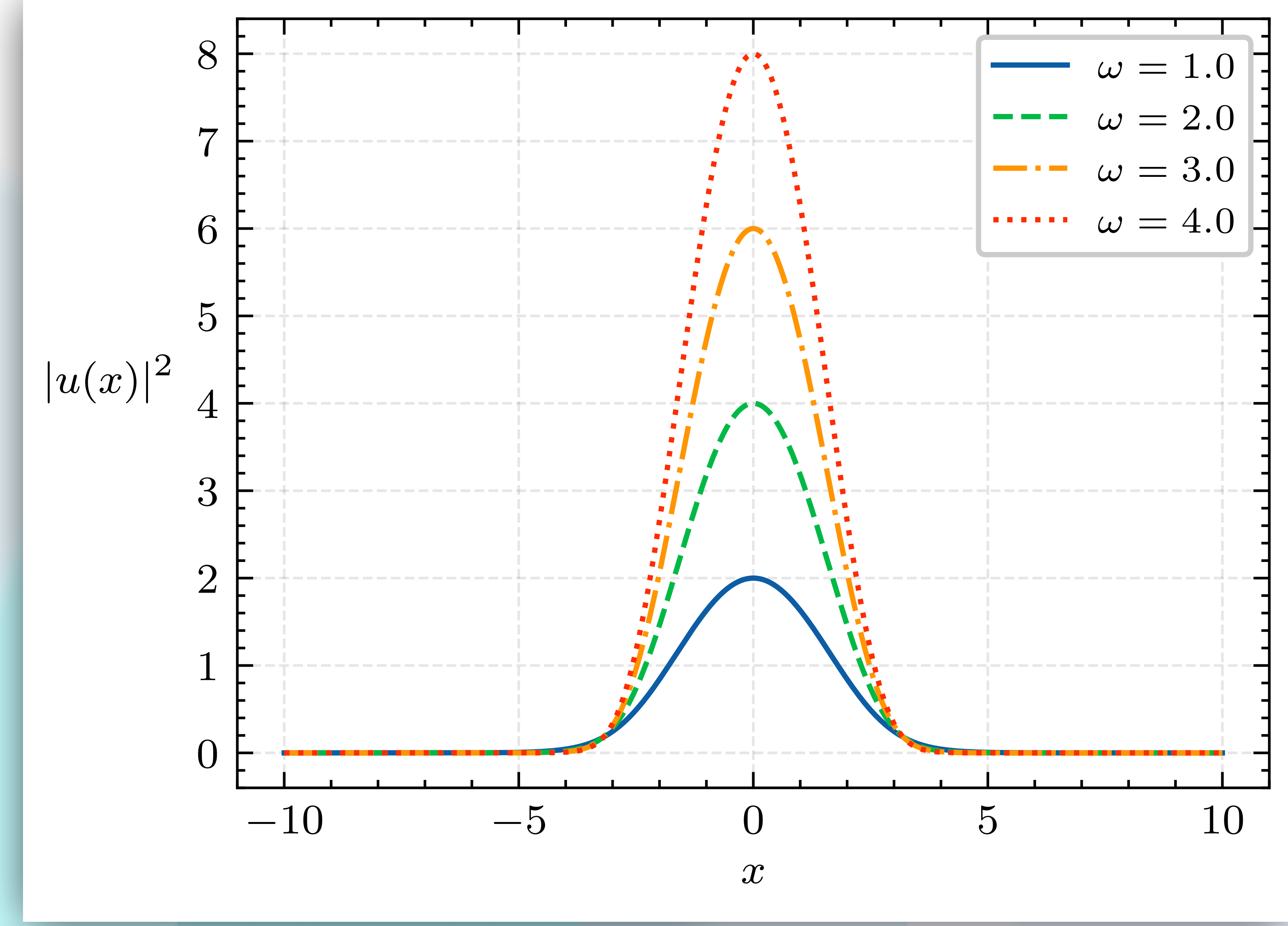
$$u(x) = \sqrt{1 - \eta_{c,\kappa}(x)}\,\exp\left(\frac{c}{2}\int_0^x \frac{\eta_{c,\kappa}(y)}{1 - \eta_{c,\kappa}(y)}dy\right), \quad \text{and} \quad \vartheta(t) = -t$$

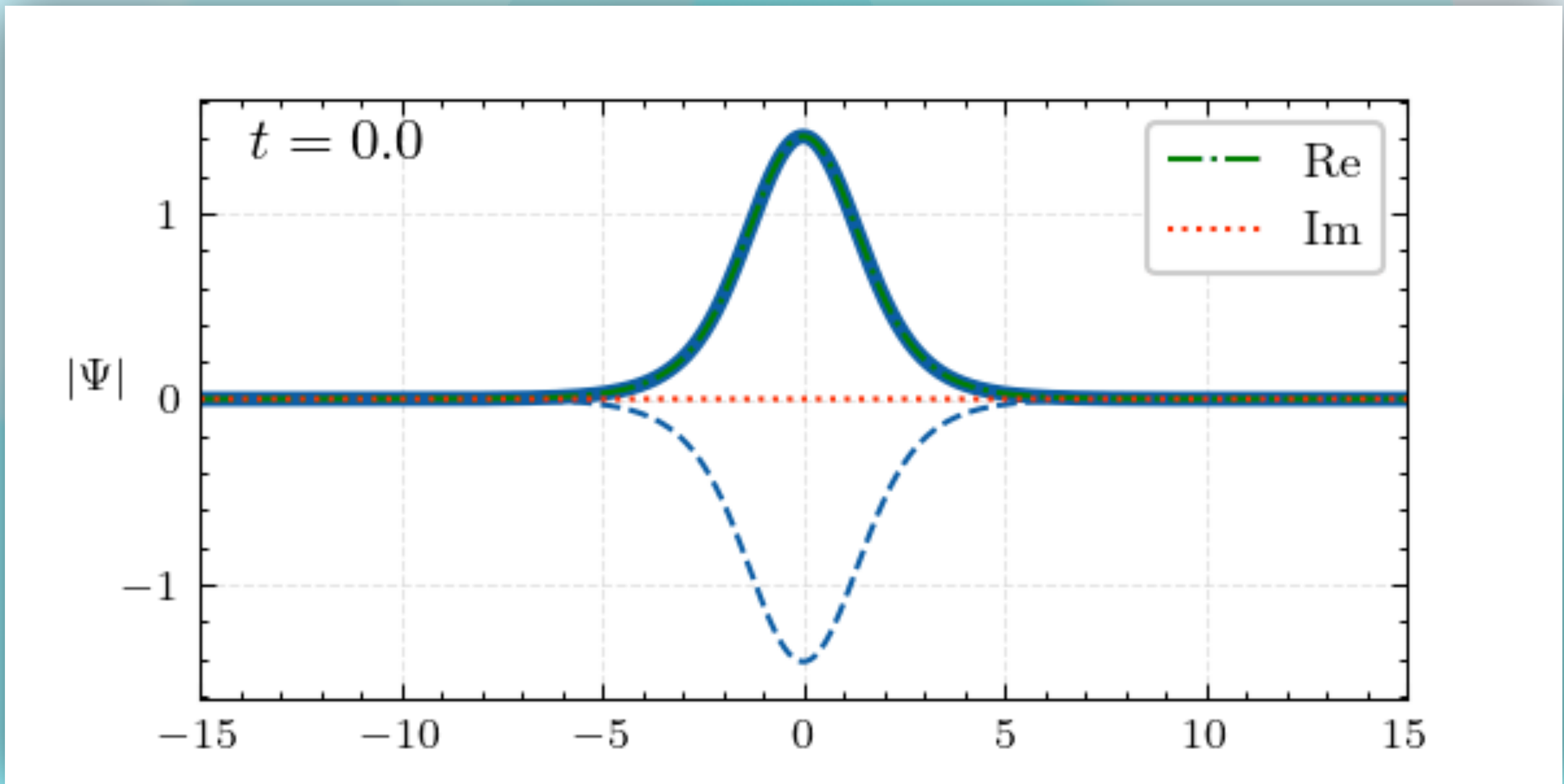
for all $y \in \mathbb{R}$, with $\eta_{c,\kappa} = G^{-1}_{c,\kappa}$ and

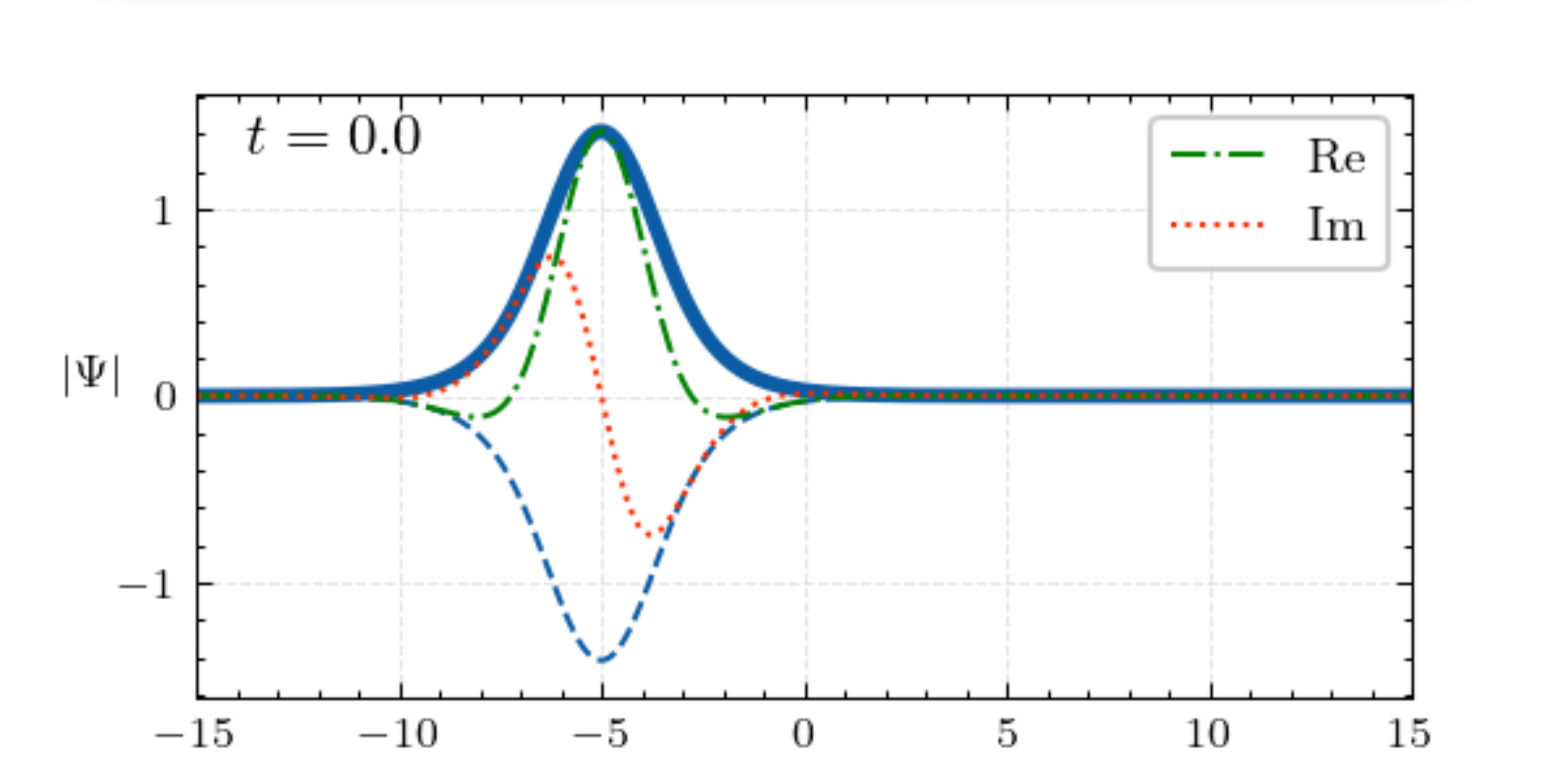
$$G_{c,\kappa}(y) = -2\sqrt{-\kappa}\,\text{atanh}\left(\sqrt{-\kappa}\sqrt{\frac{2 - c^2 - 2y}{1 - 2\kappa + 2\kappa y}}\right) + 2\sqrt{\frac{1 - 2\kappa}{2 - c^2}}\text{atanh}\left(\sqrt{\frac{(1 - 2\kappa)(2 - c^2 - 2y)}{(2 - c^2)(1 - 2\kappa + 2\kappa y)}}\right).$$

Bright solitons









Dark solitons

We identify stability regions for dark solitons:

- In $D_1 = \{(c, \kappa) : 0 \leq c < \sqrt{2} \text{ and } 0 < \kappa < 1/2\}$ there exist **stable dark solitons**
- In $D_2 = \{(c, \kappa) : 0 \leq c < \sqrt{2} \text{ and } \kappa \leq 0\}$ there exist **stable and unstable dark solitons**

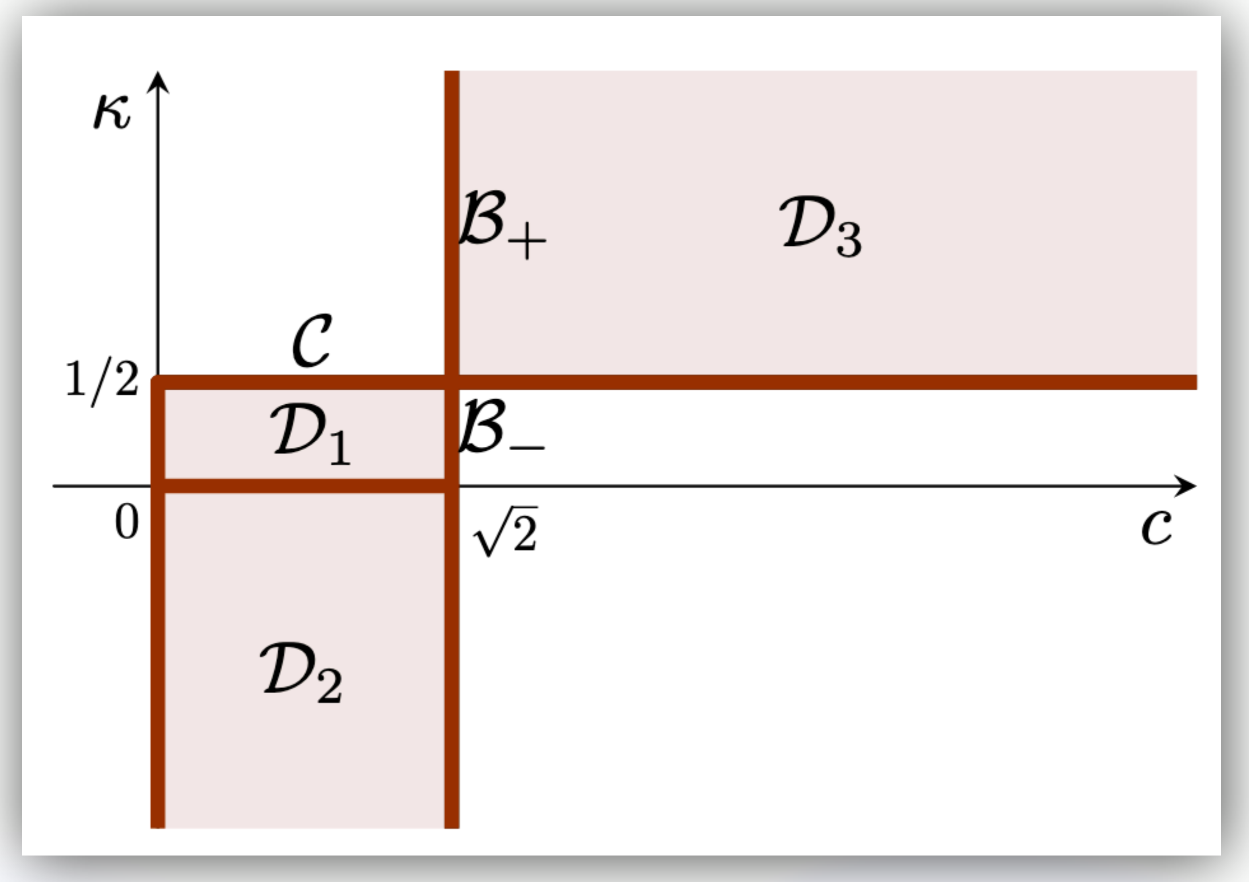
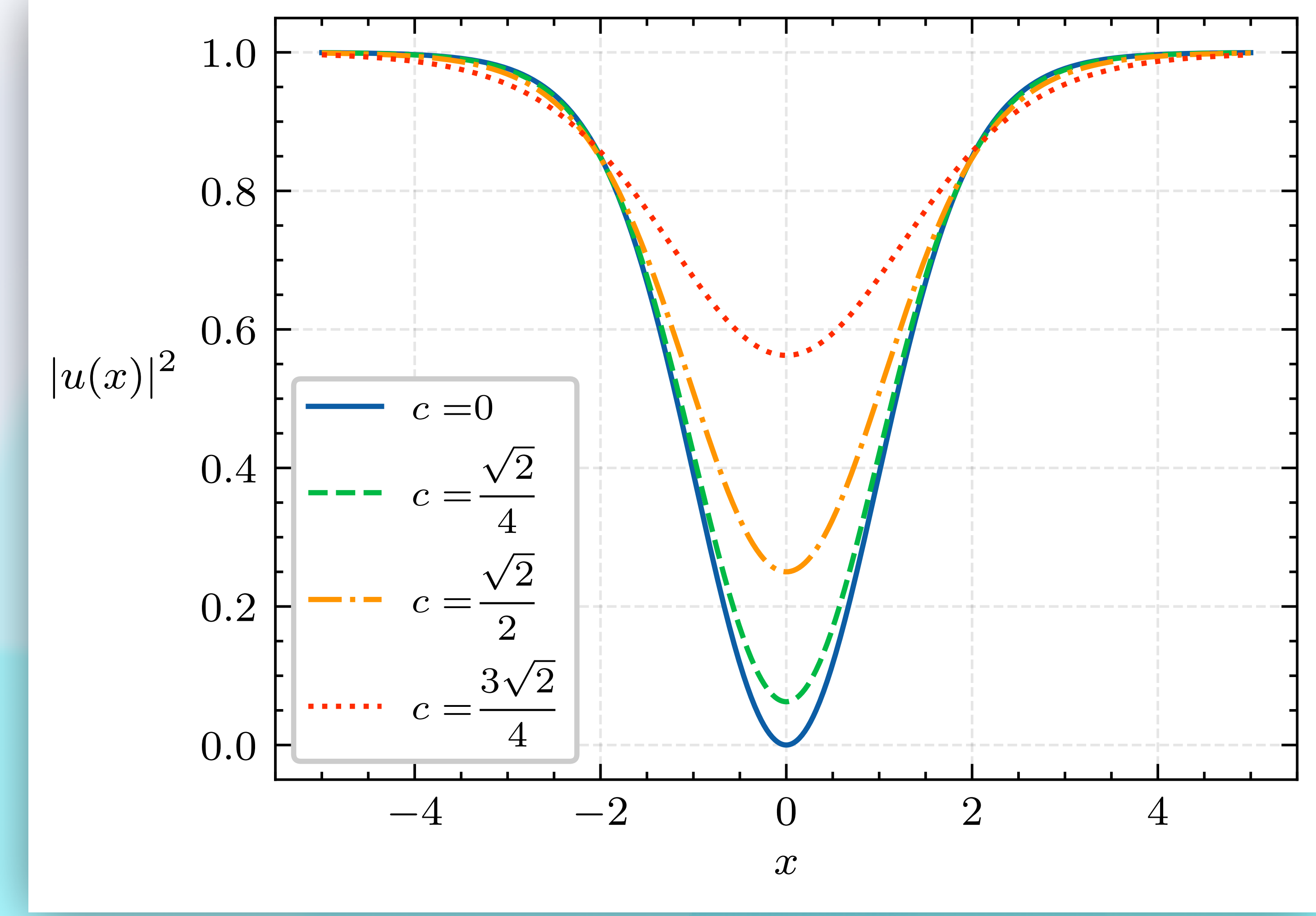
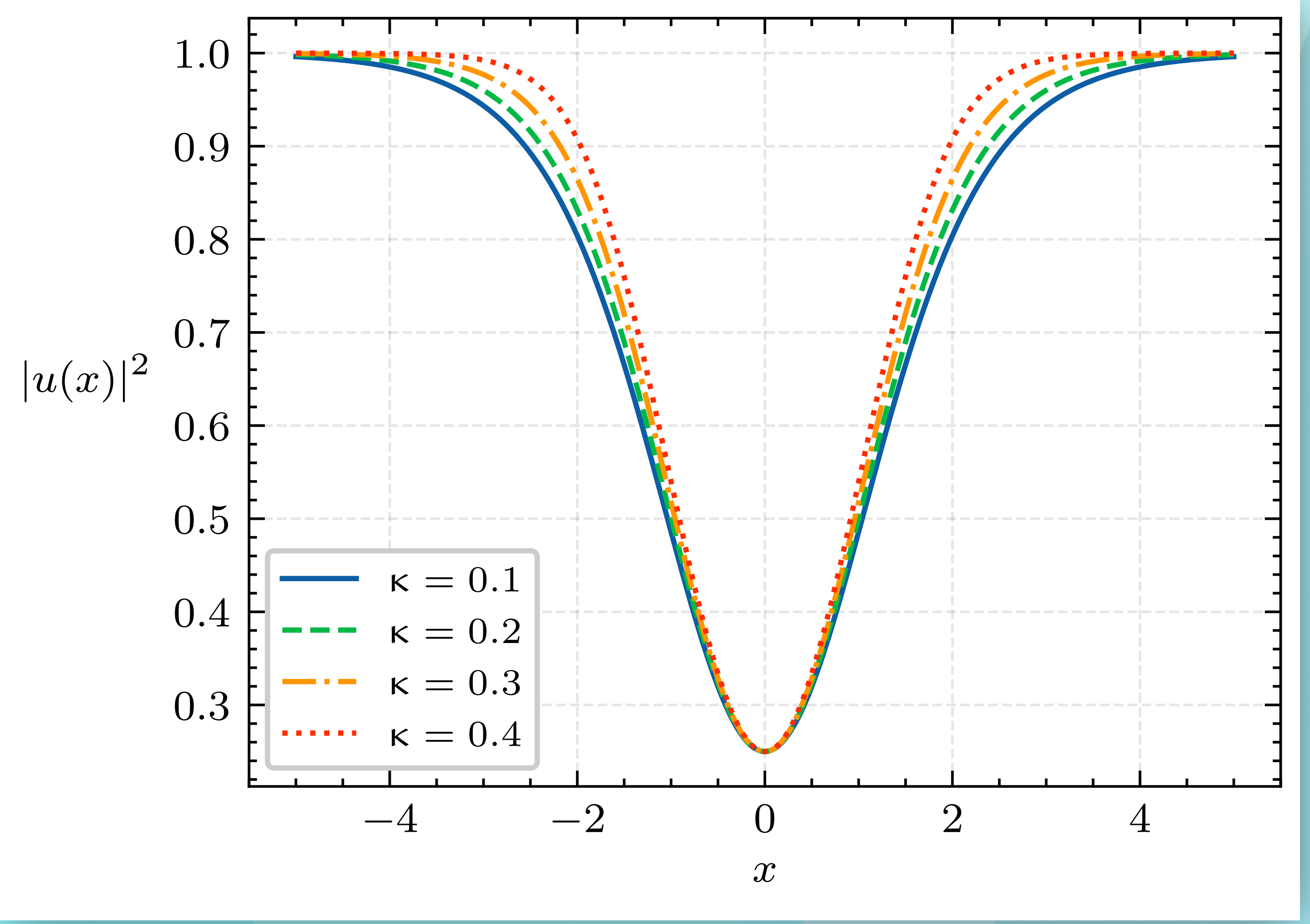
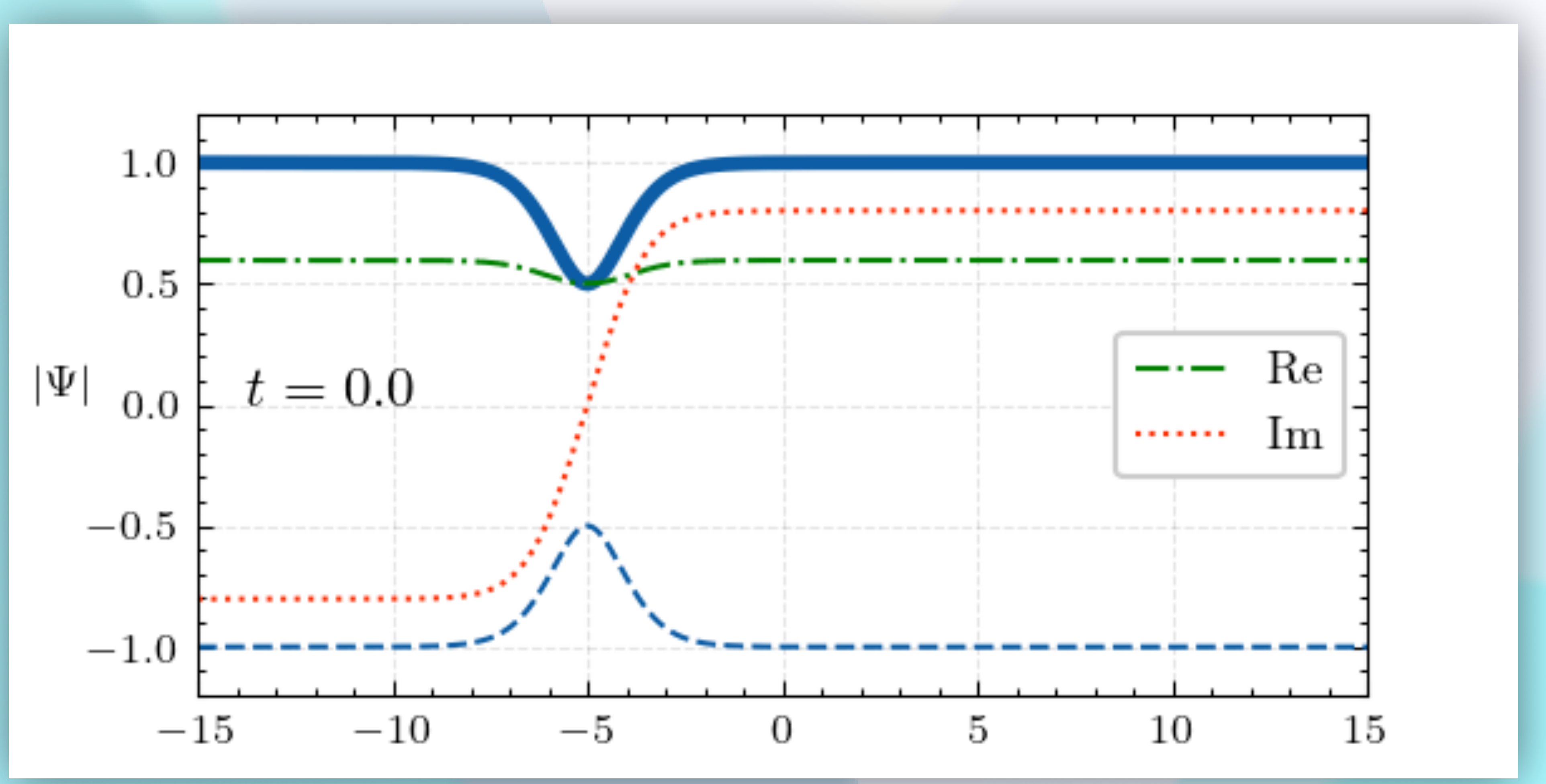


Image from A. de Laire (2025)

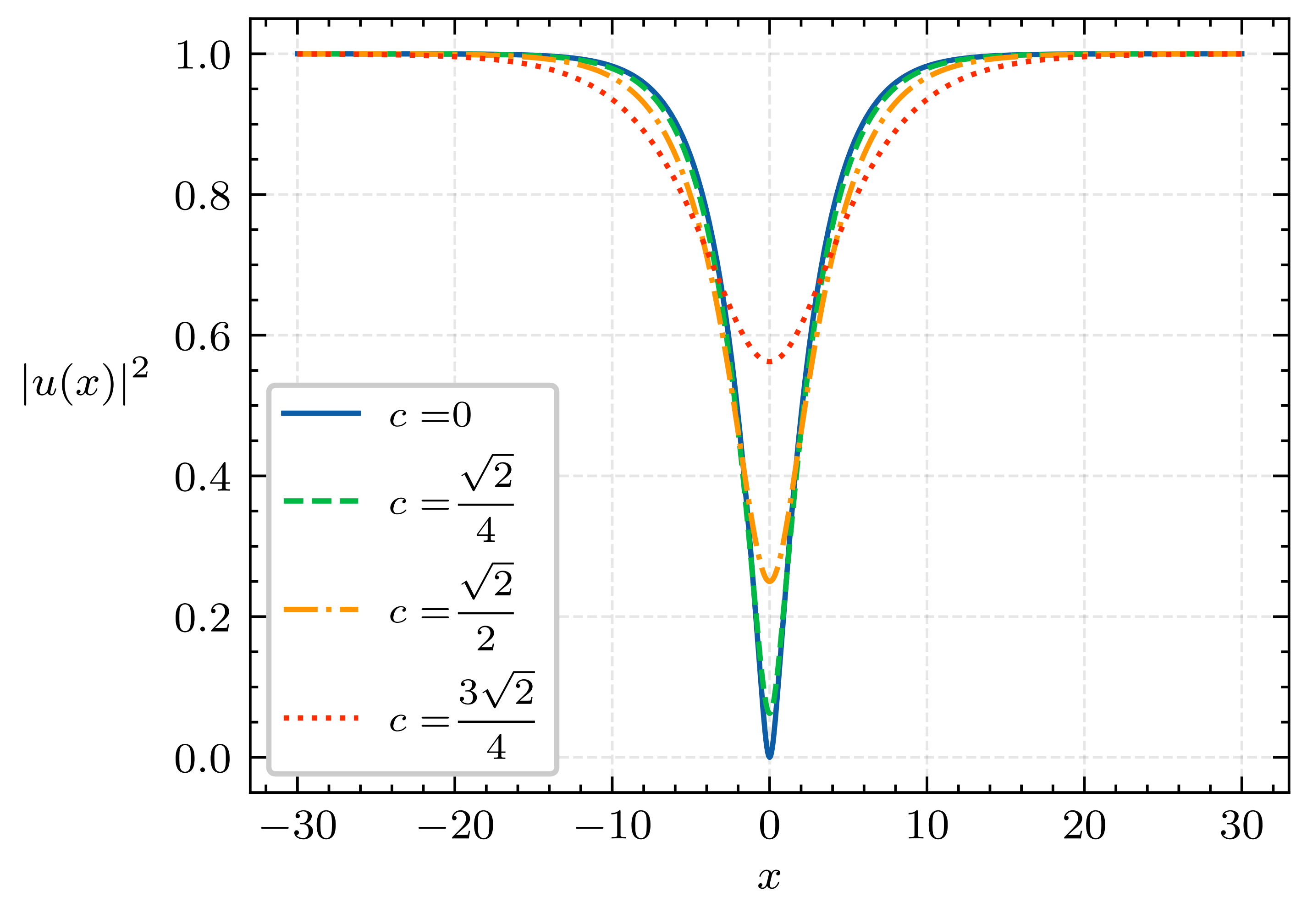


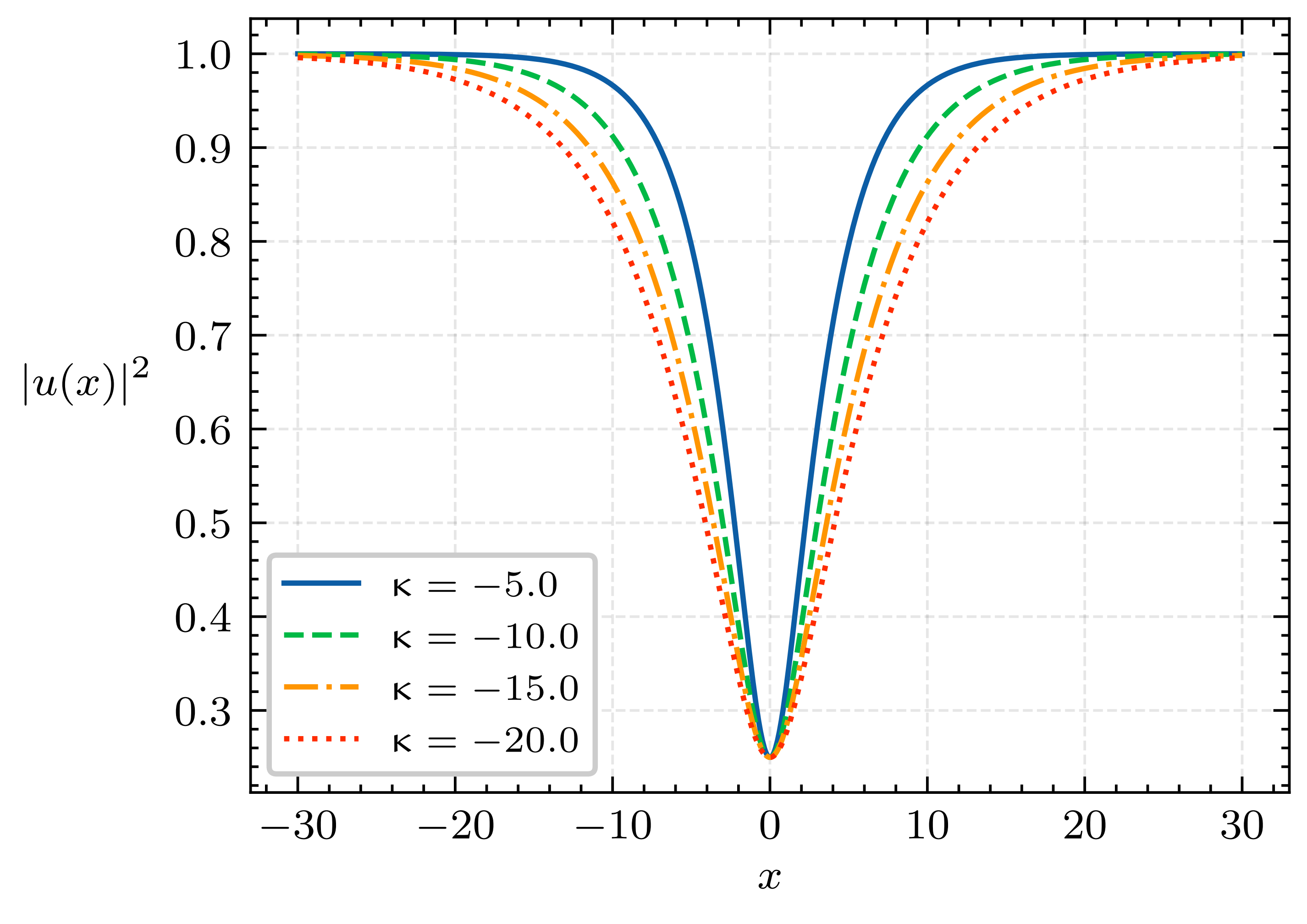


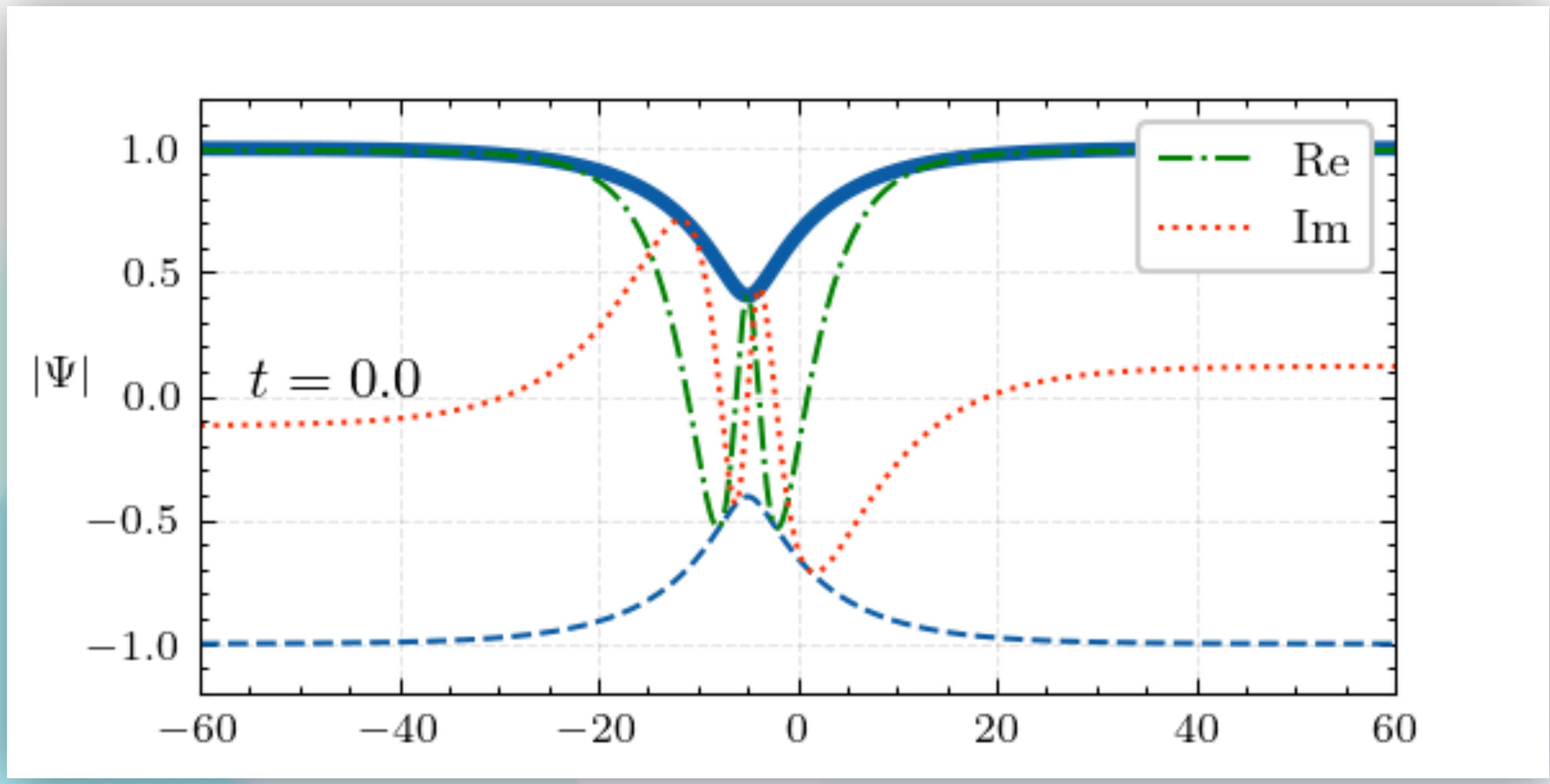


Dark solitons in D_1

Dark solitons in D_2





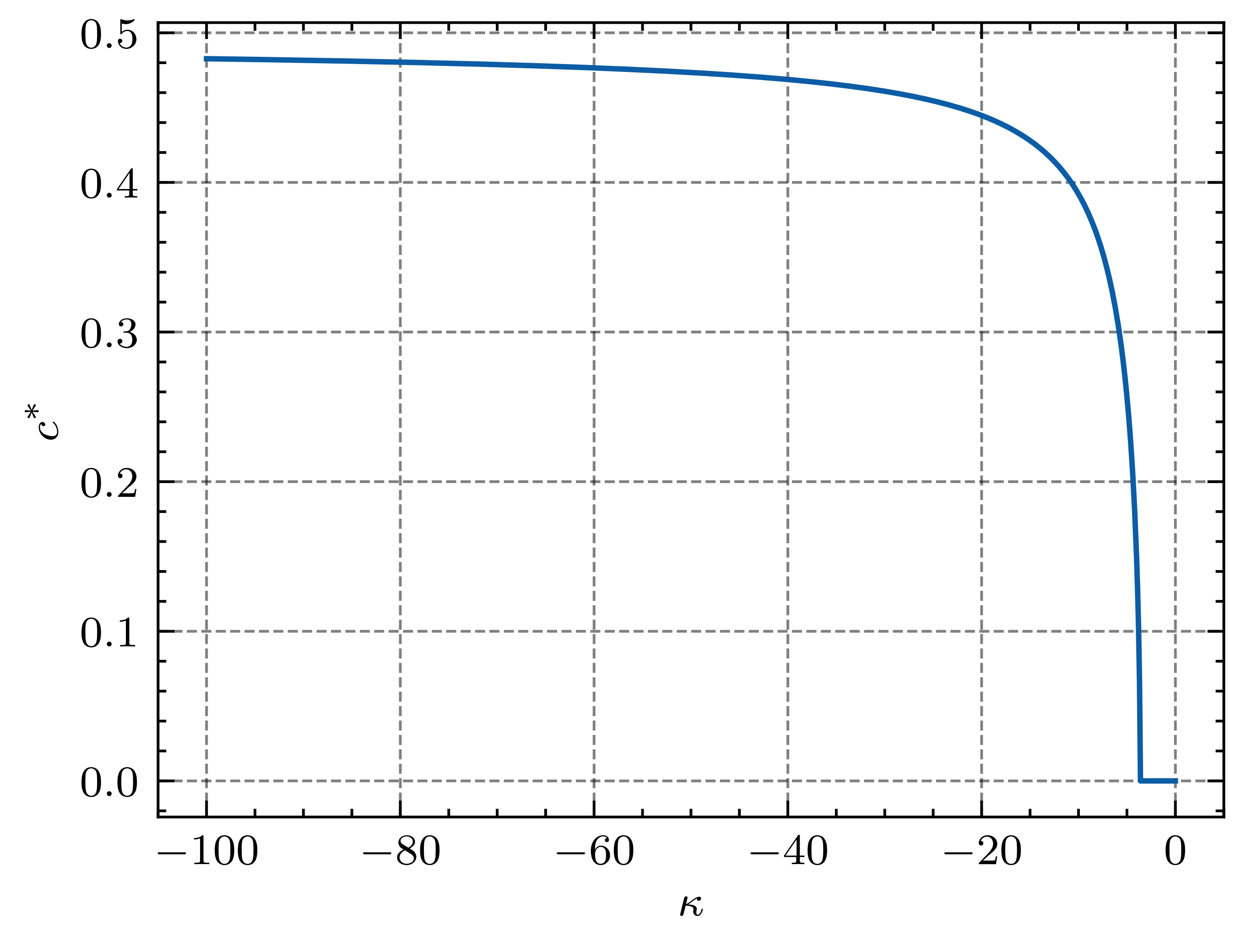


Dark solitons in D_2

Depending on values of c and κ these solitons are stable or unstable (Le Quiniou, 2025):

There exists $\kappa_0 \approx 3.636$ such that,

- If $\kappa \in [\kappa_0,\ 1/2]$, then the soliton is stable for all $c \in (0, \sqrt{2})$
- If $\kappa < \kappa_0$ the soliton is unstable if $c \in (0,\ c^*(\kappa))$ or stable if $c \in (c^*(\kappa),\ \sqrt{2})$



Goals

Our main goal is to define a fully implicit Crank–Nicolson finite–difference scheme for quasilinear Schrödinger equations with cubic nonlinearity that:

- Preserve the mass and the energy
- Has second order convergence in time and space

To validate our scheme, we perform several experiments:

- Simulation of **bright solitons** to check convergence.
- Simulation of **bright soliton collisions**, varying κ.
- Replication of finite time **blow-ups** in the defocusing case with zero background.
- Simulation of dark solitons to check convergence.
- Explore instability of dark solitons.

Numerical scheme

Let $T > 0$ the final time, $I_T = [0,T]$ the time interval, $N > 0$ the total time points, $\tau = T/(N+1)$ and $t_n = n \cdot \Delta t, n \in \{0,\ldots,N+1\}$. Let $J > 0$ the total of space points, $L > 0$, the interval $I_L = [-L, L]$, $h = 2L/J$ and $x_j = j \cdot h - L,\ j \in \{0,\ldots,J\}$

The fully discrete scheme reads as

$$\frac{i\left(\Psi^{n+1} - \Psi^{n}\right)}{\Delta t} = D_{xx}^{h}\Psi^{n+1/2} + \mathfrak{s}\left(\frac{|\Psi^{n}|^{2} + |\Psi^{n+1}|^{2}}{2}\right)\Psi^{n+1/2} + \kappa\left(D_{xx}^{h}\frac{|\Psi^{n}|^{2} + |\Psi^{n+1}|^{2}}{2}\right)\Psi^{n+1/2},$$

where $\Psi^{n+\frac{1}{2}} = (\Psi^{n} + \Psi^{n+1})/2$ and $D_{xx}^{h} = \frac{1}{h^{2}}\begin{pmatrix} -2 & 2 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & 1 & -2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & 0 & 2 & -2 \end{pmatrix}$.

- We consider homogeneous Neumann boundary conditions
- It preserves the mass and energy (discrete level) and is H^1 stable.
- We perform a complex Newton method to solve a nonlinear system on each iteration.

Second order consistency

Proposition.

Let $\Psi : \mathbb{R} \times I_T \to \mathbb{C}$ be a solution of equation (QGP) with $\Psi \in C^3(I_T, H^5(\mathbb{R}))$. Let $\Psi_j^n := \Psi(x_j, t_n)$, $\Psi_j^{n+\frac{1}{2}} = \dfrac{\Psi_j^n + \Psi_j^{n+1}}{2}$

and define the local truncation error:

$$\mathscr{R}_j^{n+\frac{1}{2}} := \frac{i\left(\Psi_j^n - \Psi_j^{n+1}\right)}{\Delta t} + \left(D_{xx}^h \Psi^{n+\frac{1}{2}}\right)_j + \mathfrak{s}\left(\frac{|\Psi_j^n|^2 + |\Psi_j^{n+1}|^2}{2}\right)\Psi_j^{n+\frac{1}{2}} + \kappa\left(D_{xx}^h \frac{|\Psi^n|^2 + |\Psi^{n+1}|^2}{2}\right)_j \Psi_j^{n+\frac{1}{2}}$$

where $h, \tau < 1$. Then, for $t_{n+\frac{1}{2}} \leq T$,

$$\max_{1 \leq j \leq J-1} |\mathscr{R}_j^{n+\frac{1}{2}}| \leq C(\tau^2 + h^2)$$

where C does not depend on h nor Δt. At the boundary, if we assume that there is $\rho > 0$ such that for all $t > 0$ and for $j \in \{0, J\}$,

$$(|\partial_x \Psi(x_j, t)| + |\partial_x |\Psi(x_j, t)|^2|)h^{-1} + (|\partial_{xxx}\Psi(x_j, t)| + |\partial_{xxx}|\Psi(x_j, t)|^2|) \cdot h \leq \rho, \text{ then}$$

$$|\mathscr{R}_0^{n+\frac{1}{2}}| + |\mathscr{R}_J^{n+\frac{1}{2}}| \leq C(\tau^2 + h^2 + \rho).$$

In particular, if $\rho \leq h^2$, then the scheme behaves like second-order in time and space.

Experiment 1: Bright soliton simulations

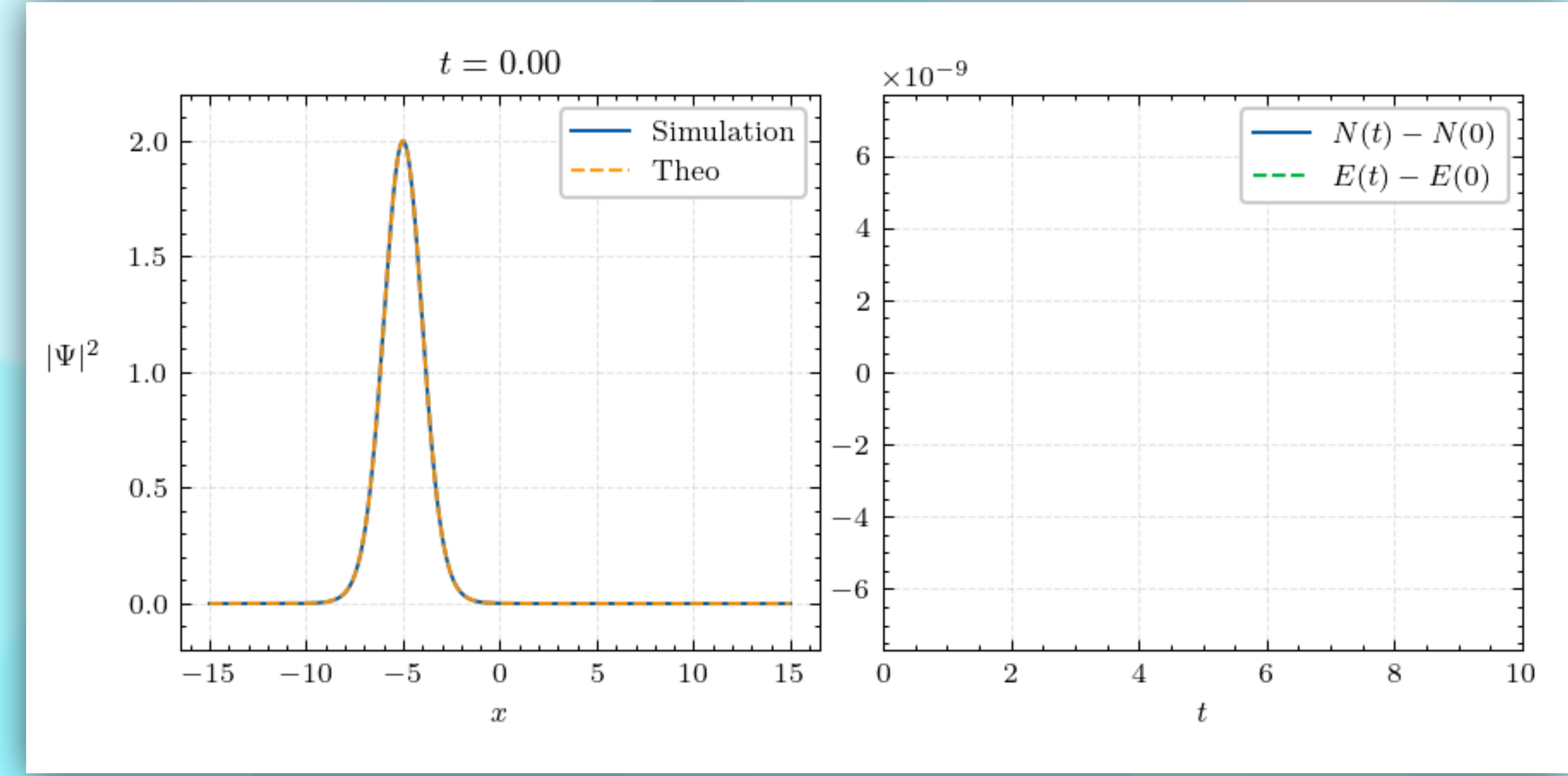
Second order convergence

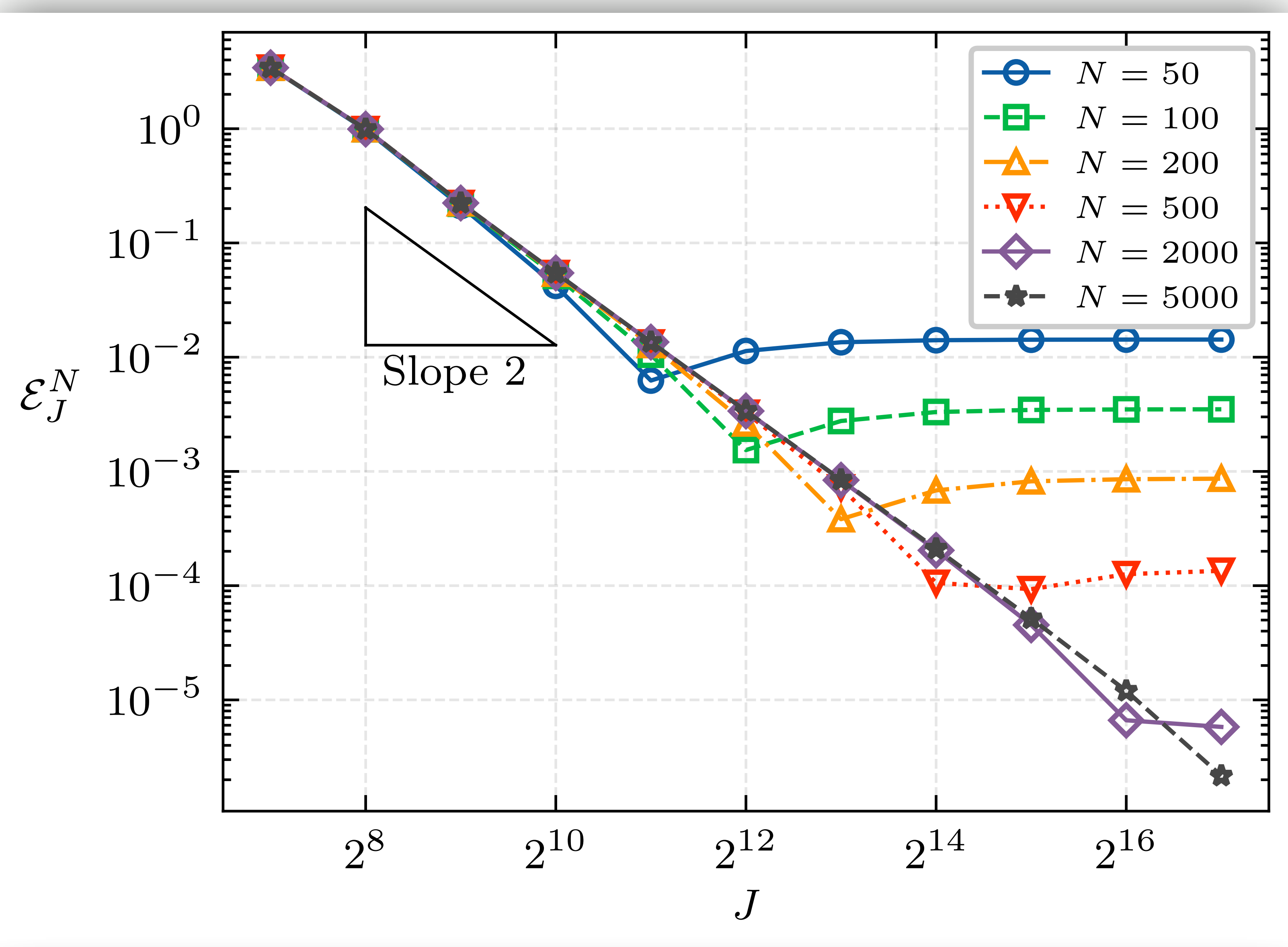
We use the discrete L^2 error:

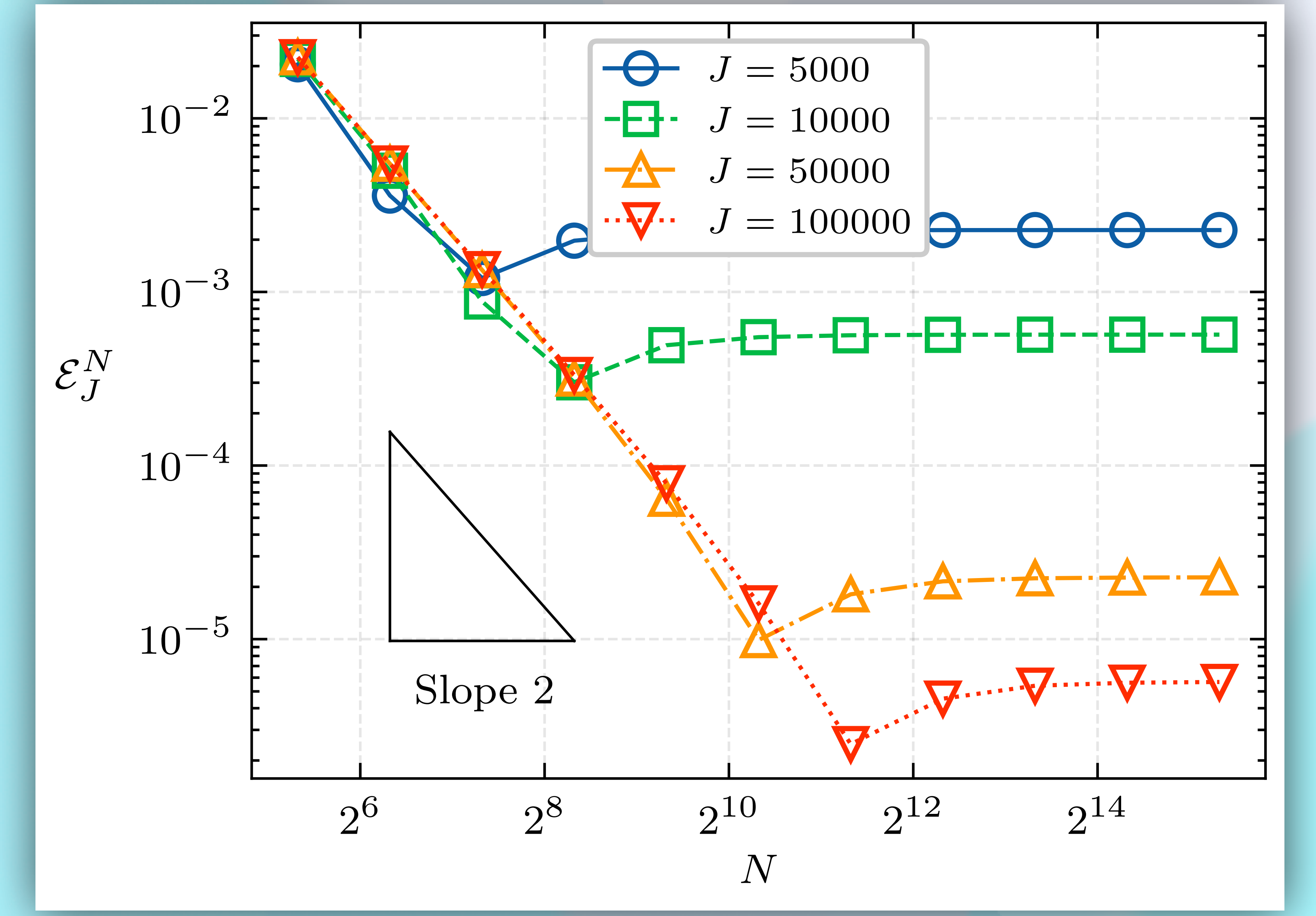
$$\mathscr{E}_J^N = \left(h \sum_{j=1}^{J} |\Psi_j^N - \Psi_{\text{sol}}(x_j, T)|^2 \right)^{1/2},$$

where Ψ_{sol} is the theoretical bright soliton.

All parameters were chosen such that the soliton does not reach the boundaries.







Experiment 2: Blow up

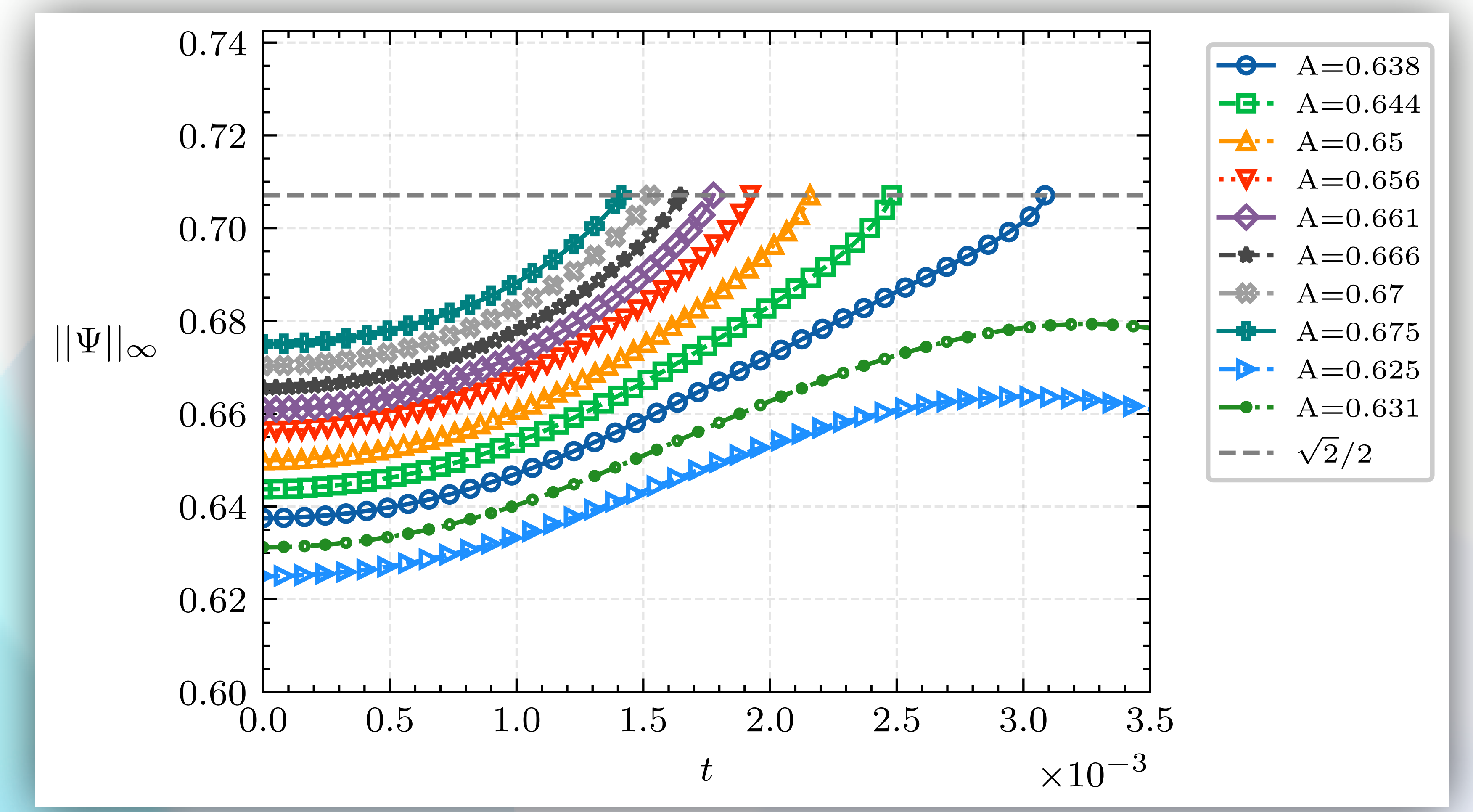
$$i\partial_t\Psi = \partial_{xx}\Psi + \mathfrak{s}\Psi|\Psi|^2 + \kappa\Psi\partial_{xx}|\Psi|^2$$

in defocusing QGP with zero background

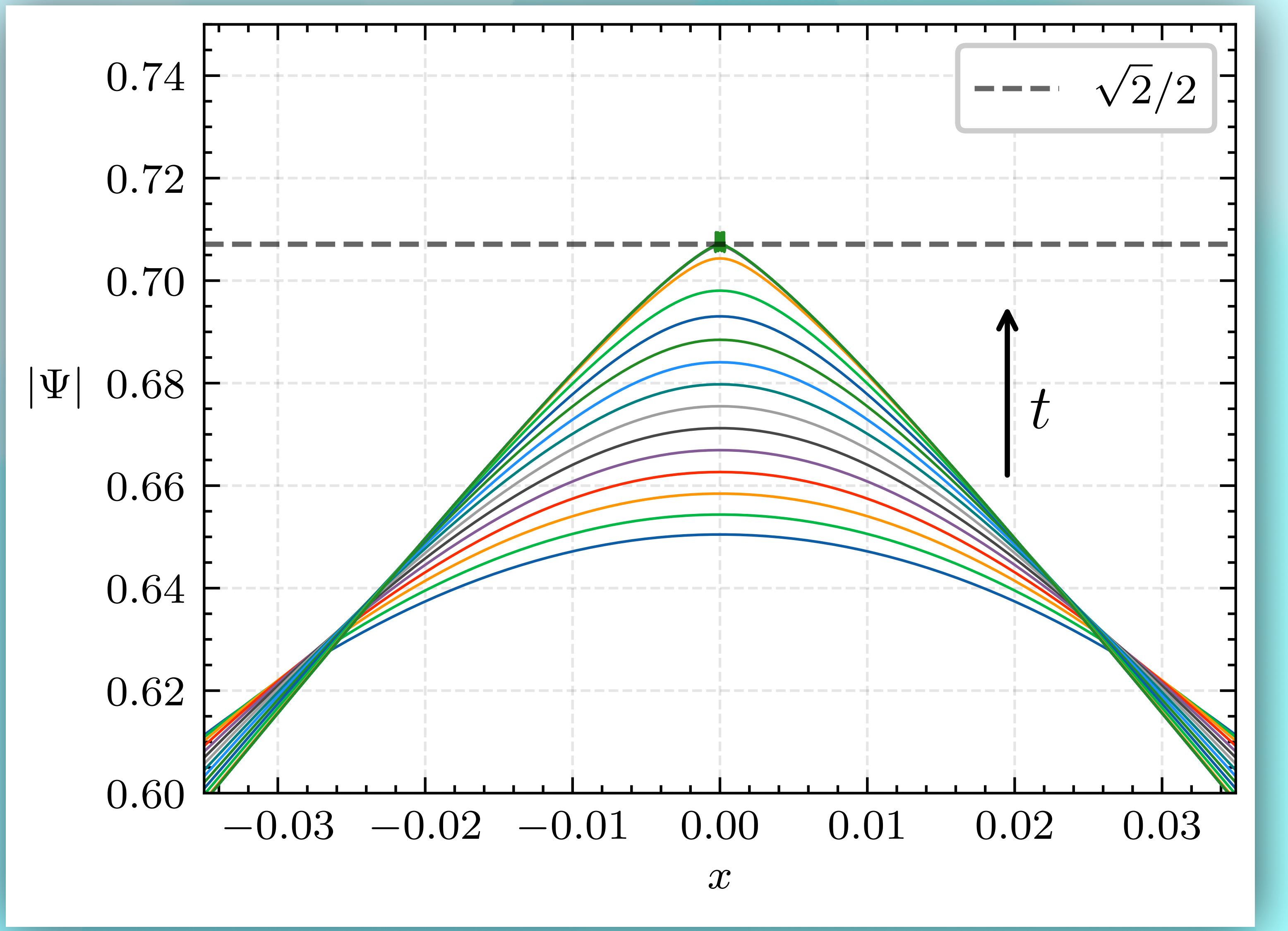
For $\mathfrak{s} = \kappa = -1$ and $\mathfrak{b}_\infty = 0$, a short time blow up was observed when a gaussian profile:

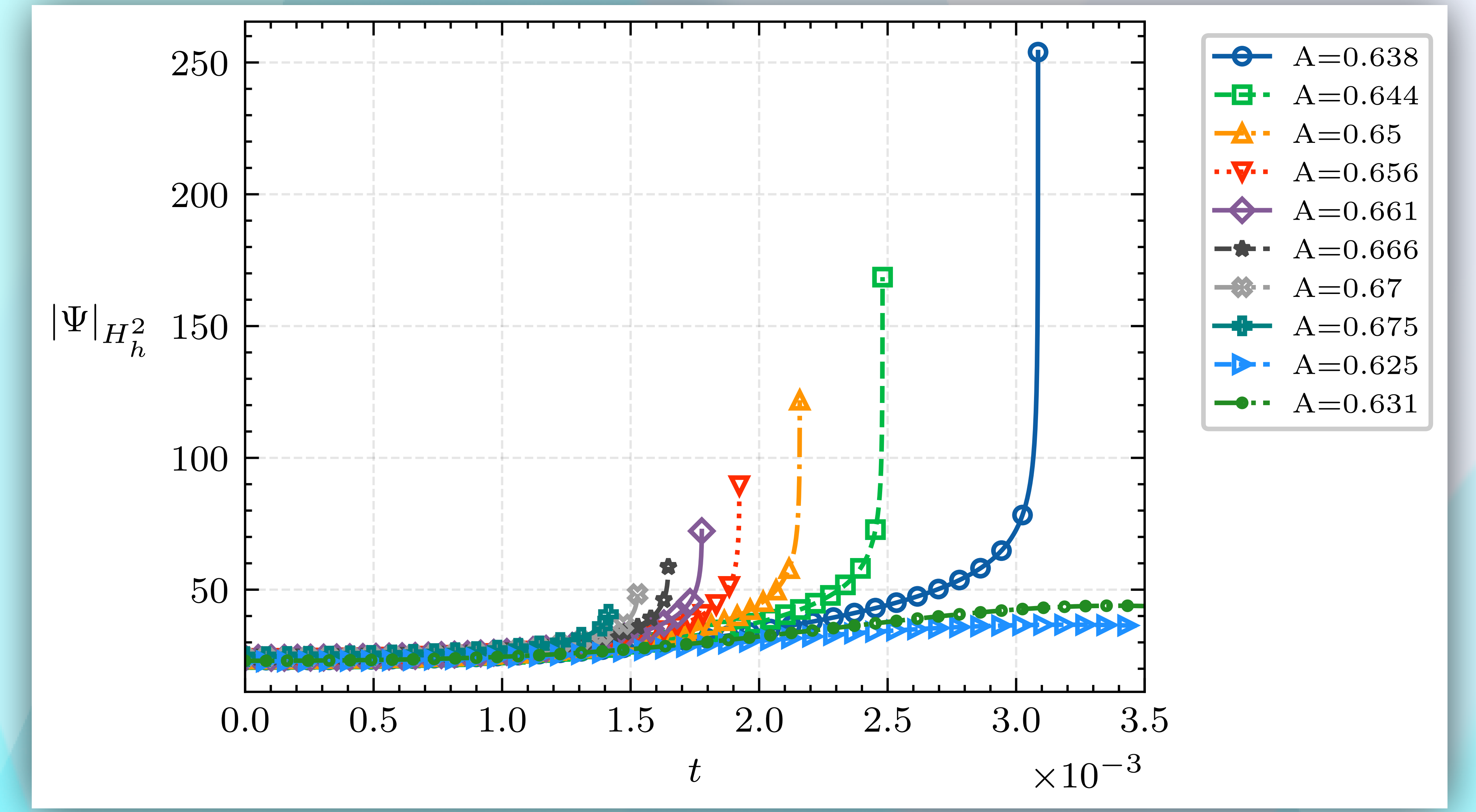
$$\Psi_{\text{gauss}}(x,0) = Ae^{-x^2/(2\sigma^2)},$$

is imposed as initial condition (Lu, J., & Marzuola, J. L. (2015)).



$\sigma = 1/10$





Experiment 2: Blow up

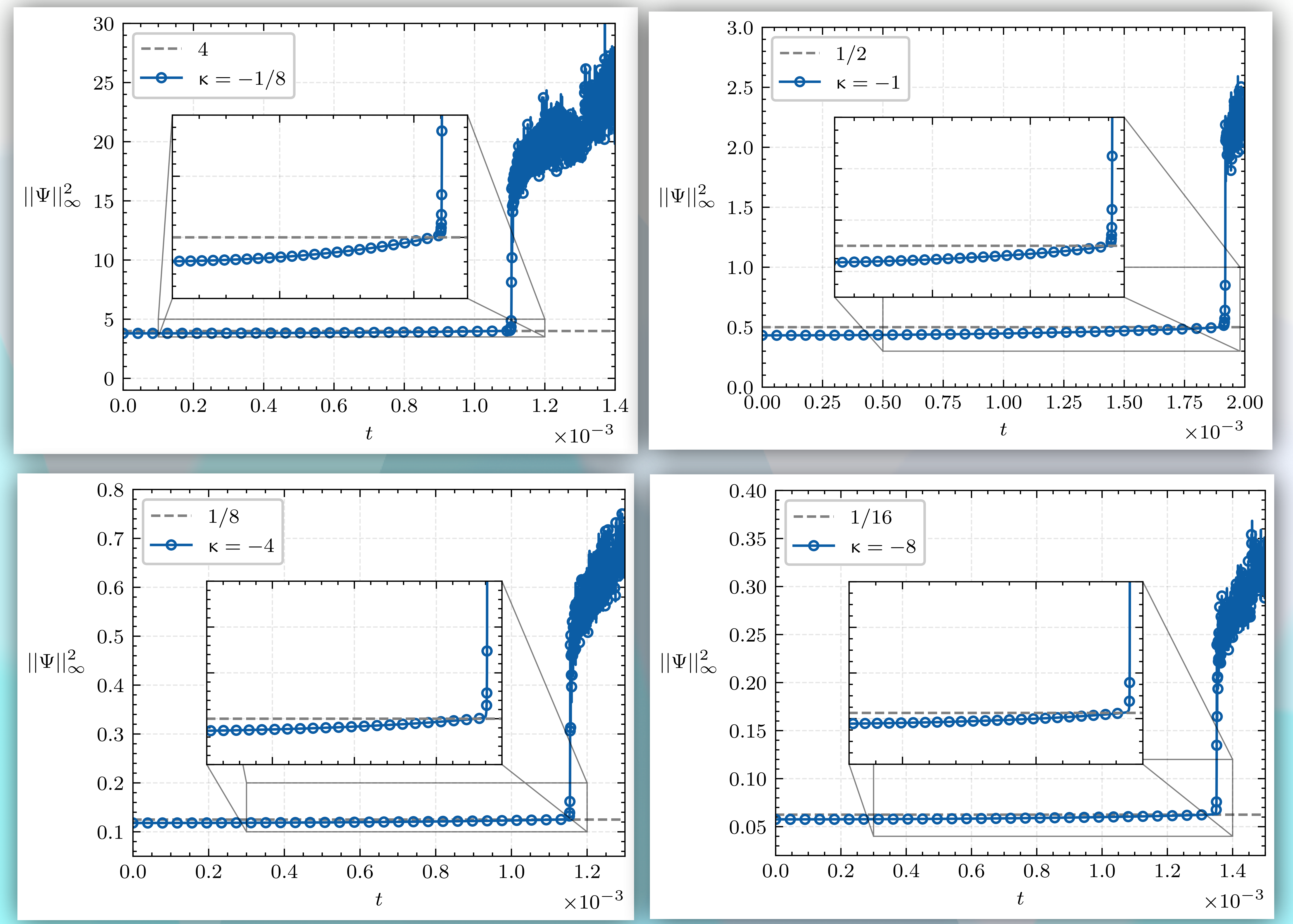
Degenerated value

In all simulations, blow-up occurs when the simulation reaches the value:

$$A_{\text{deg}} = 1/\sqrt{-2\kappa}.$$

The Cauchy problem is not well posed due to violation of the ellipticity condition:

$$1 + 2\kappa|\Psi|^2 > 0, \quad \text{in } \mathbb{R}.$$

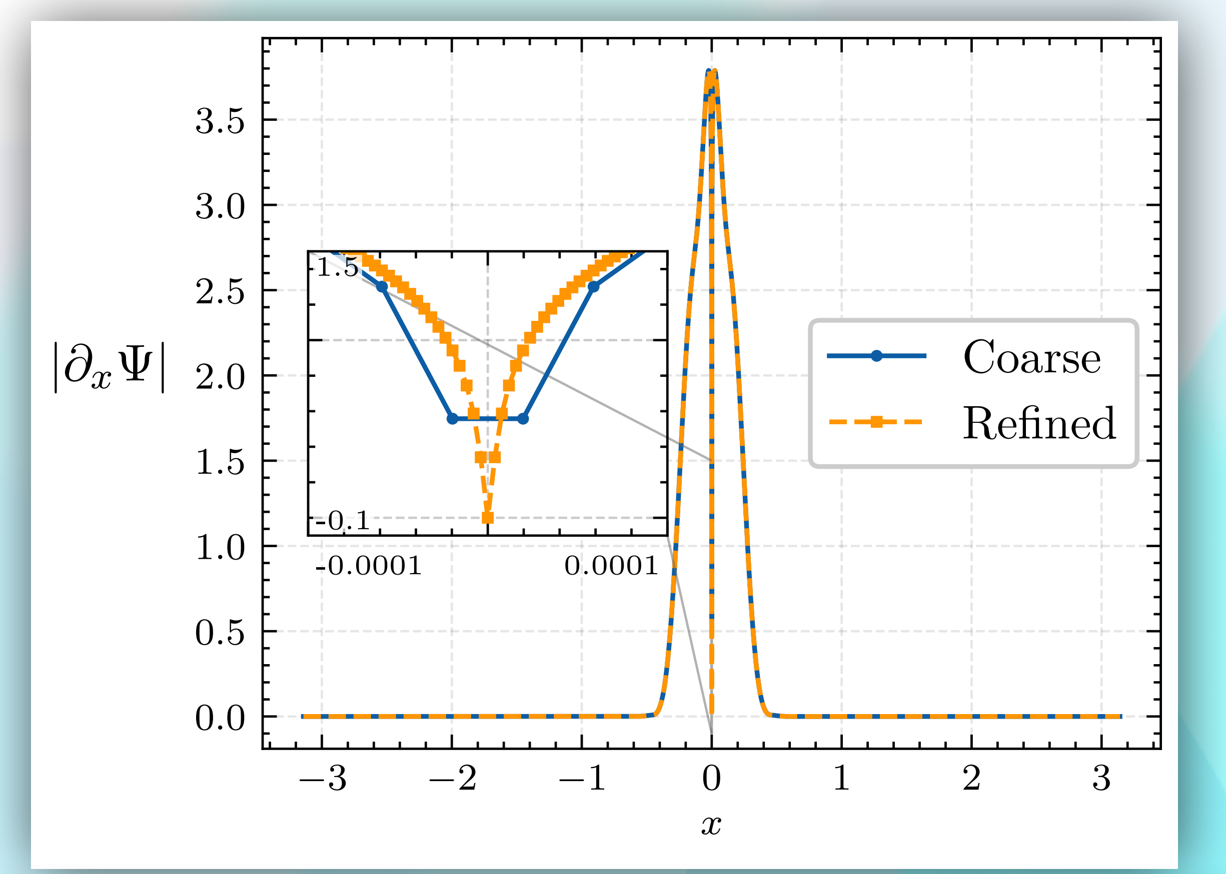


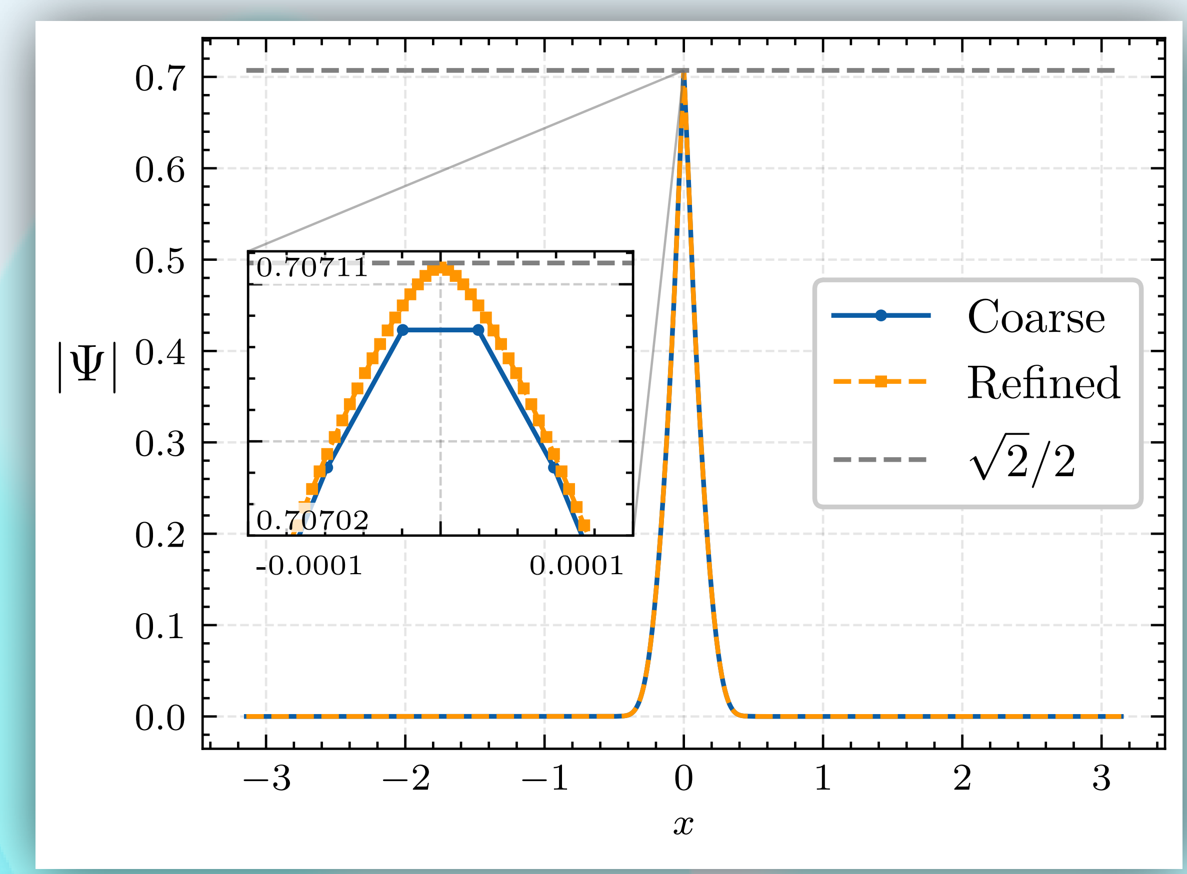
Experiment 2: Blow up

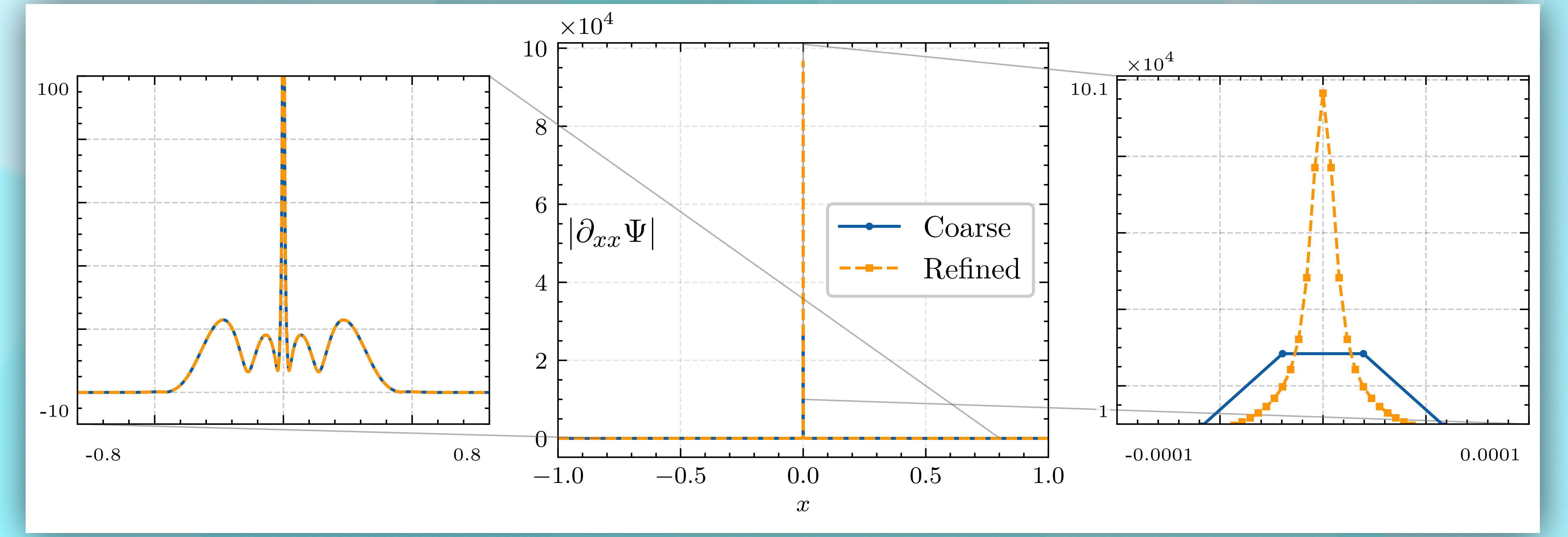
Refined mesh

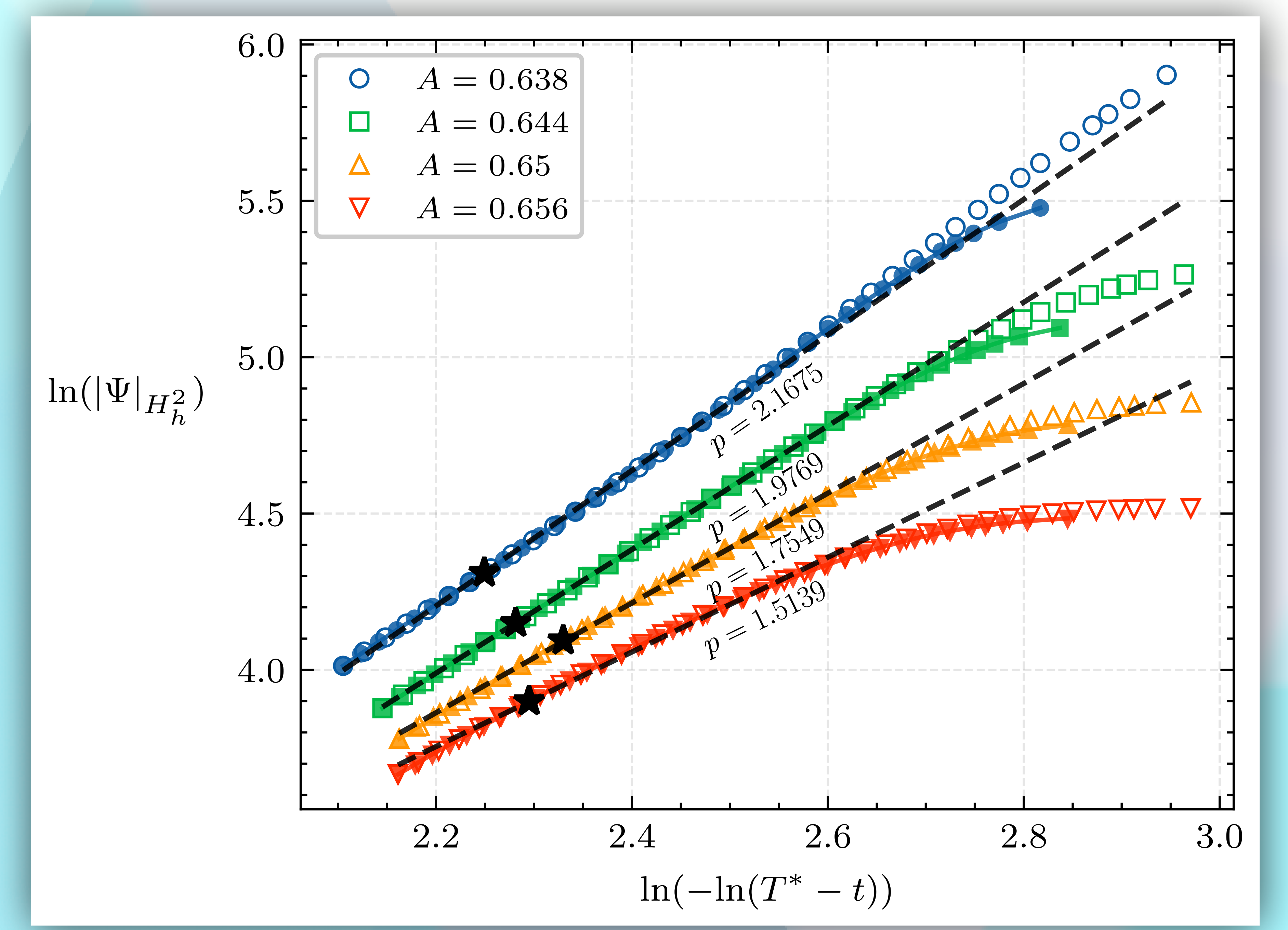
We refine the mesh near the explosion time by taking $\tau/10$ and $h/10$ to improve the performance near the blow up.

We aim to see the increase ratio near the explosion time.







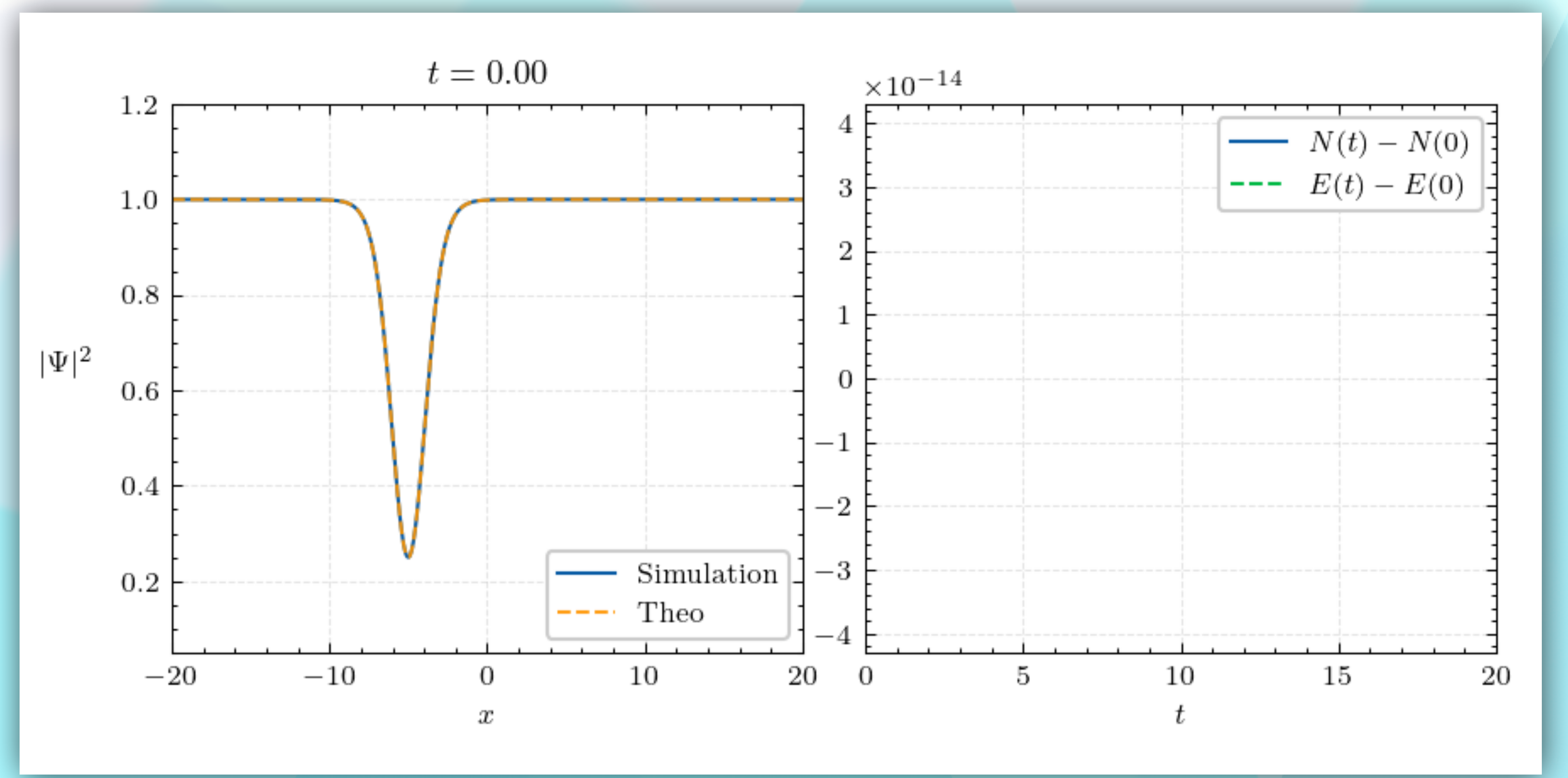


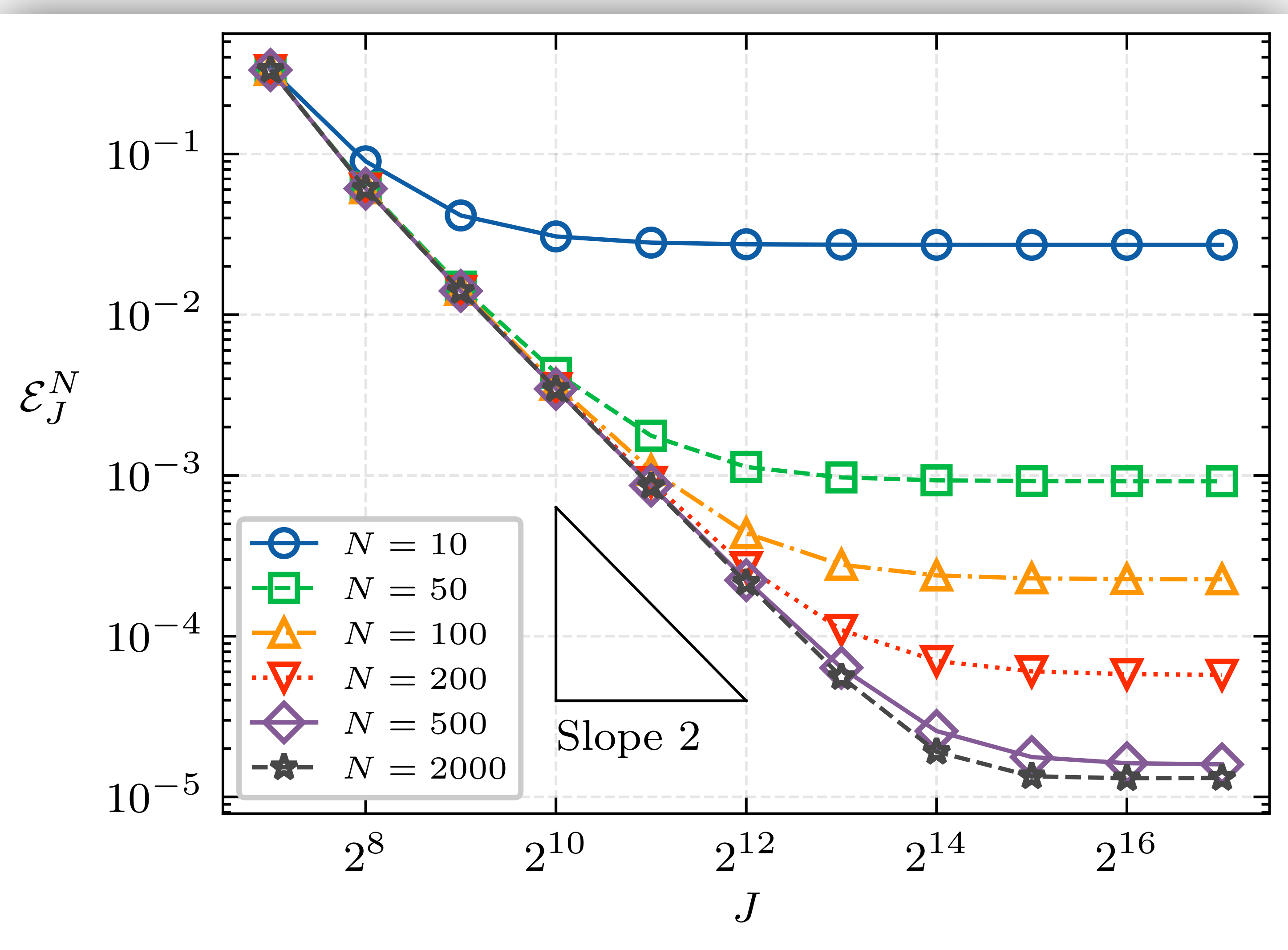
Empty points represents the refined mesh

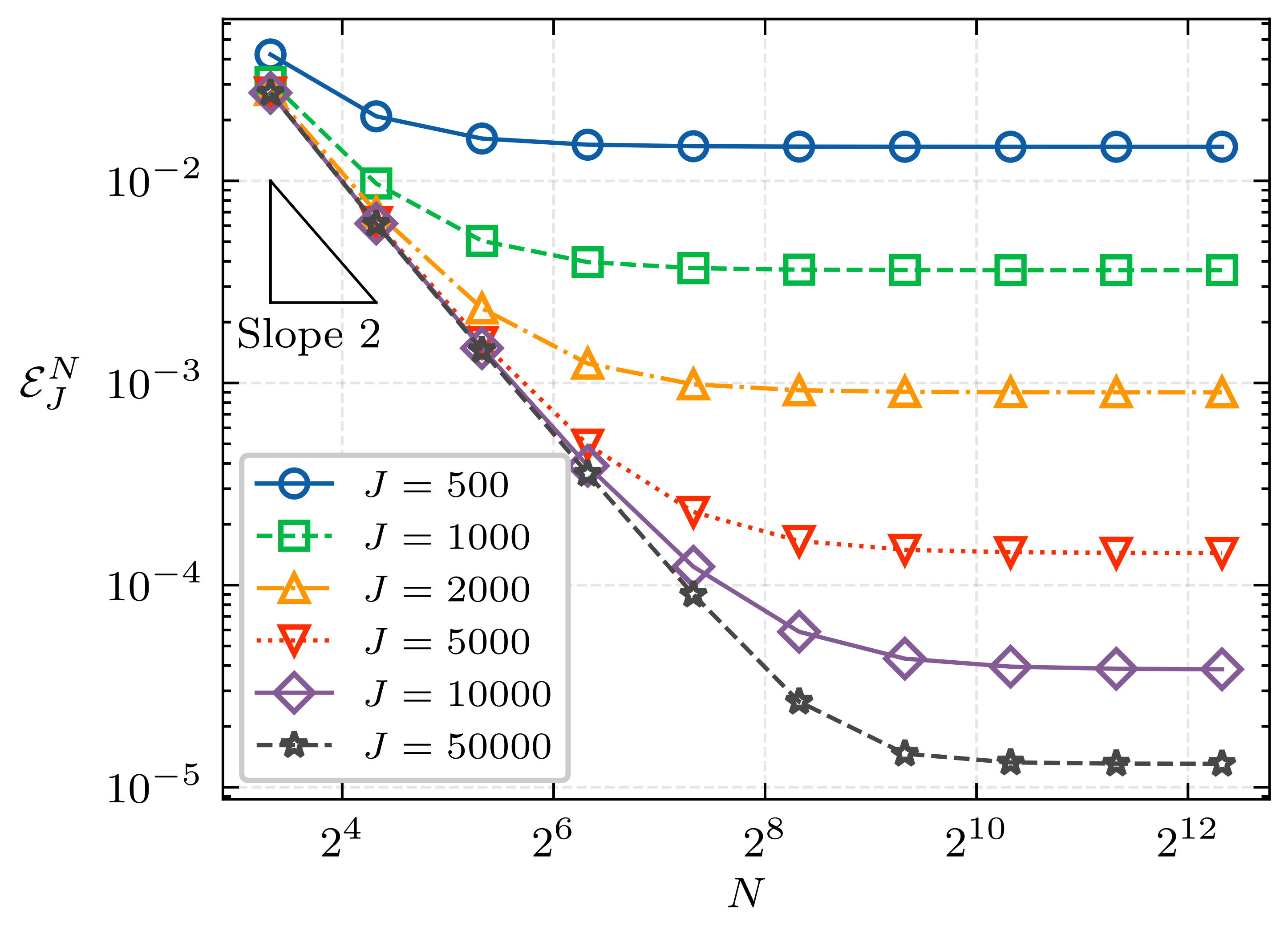
Experiment 3: Dark soliton simulations

Second order convergence in D_1

All parameters were chosen such that the soliton does not reach the boundaries.



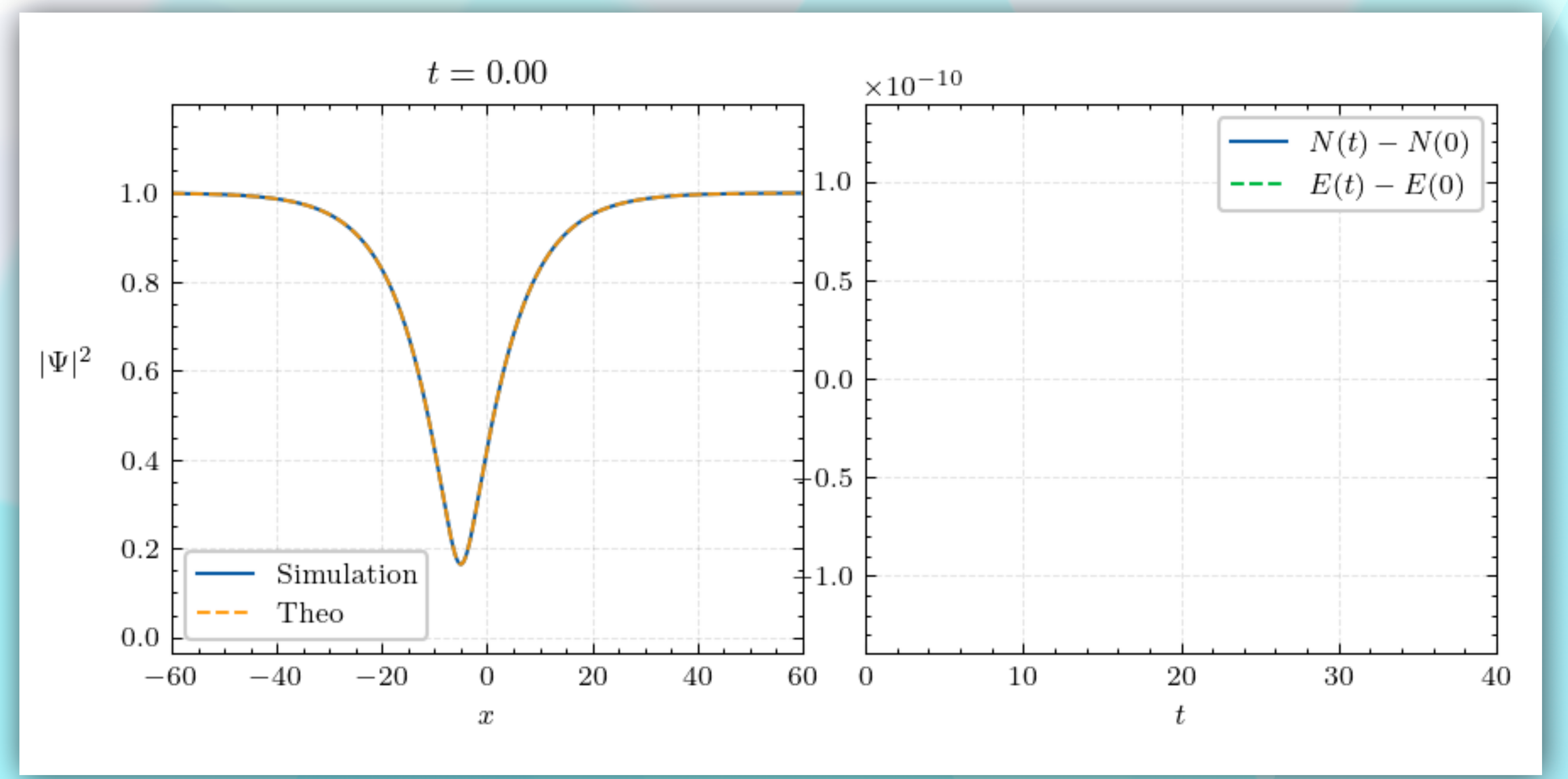


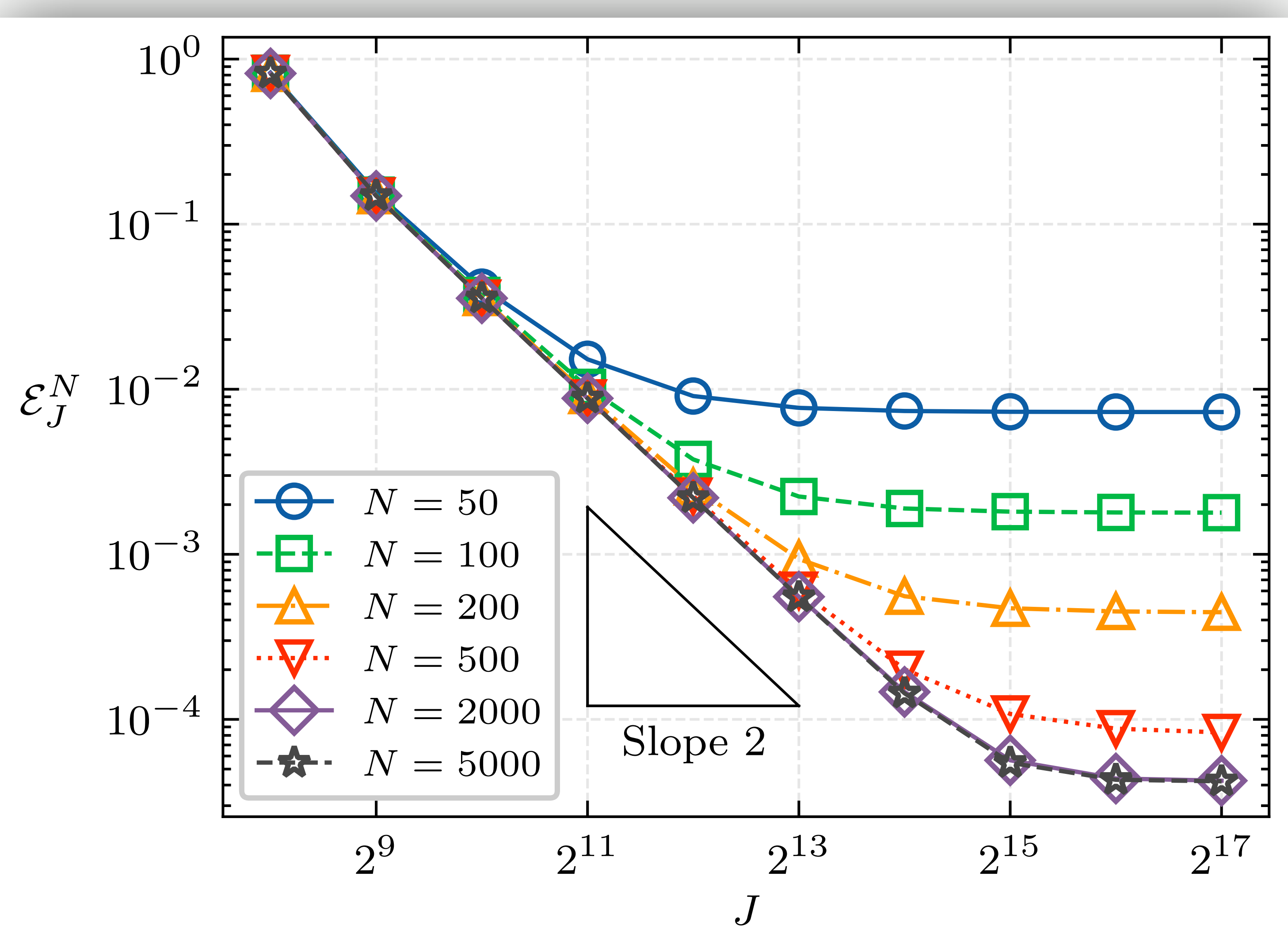


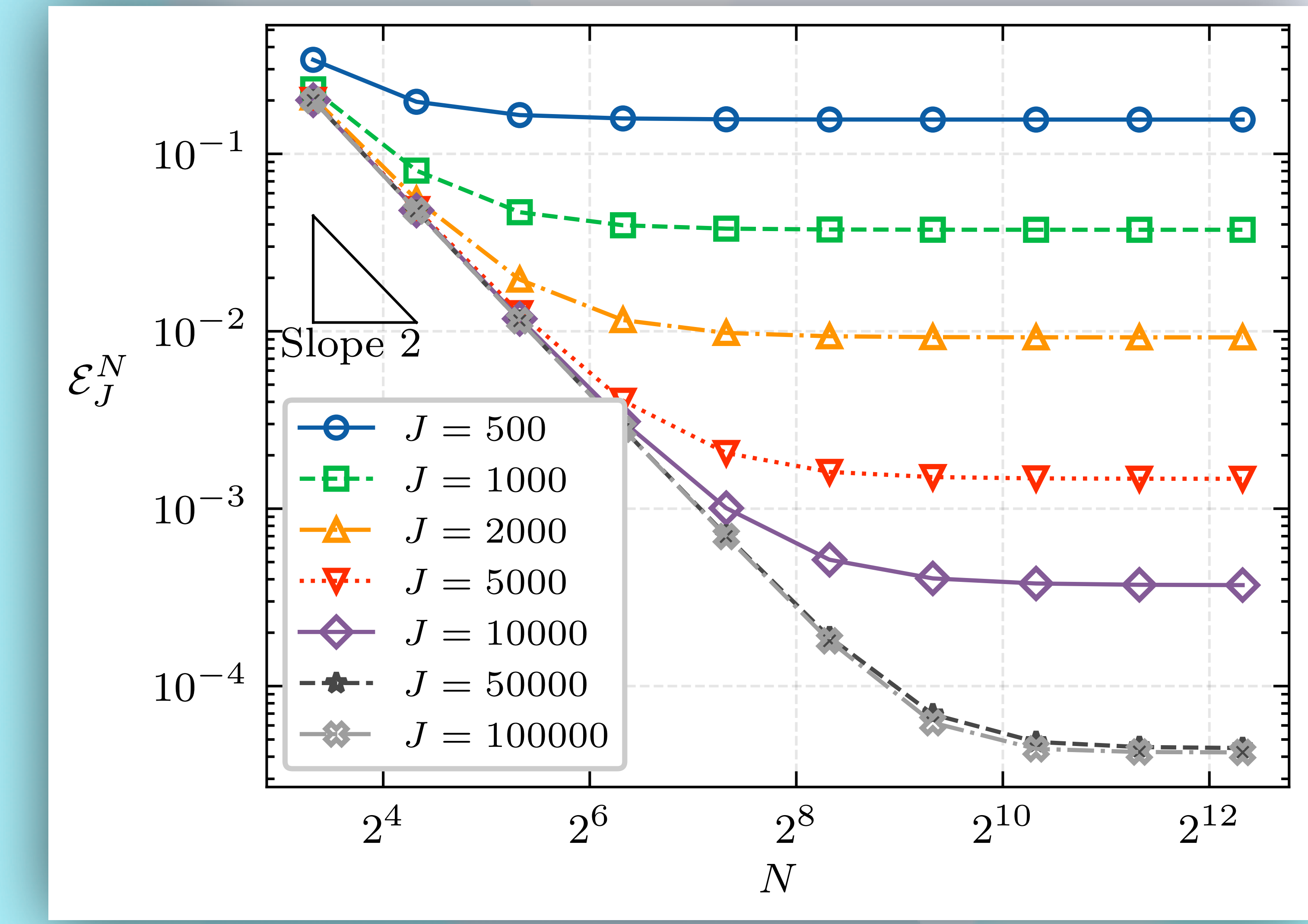
Experiment 3: Dark soliton simulations

Second order convergence in D_2

All parameters were chosen such that the soliton does not reach the boundaries.





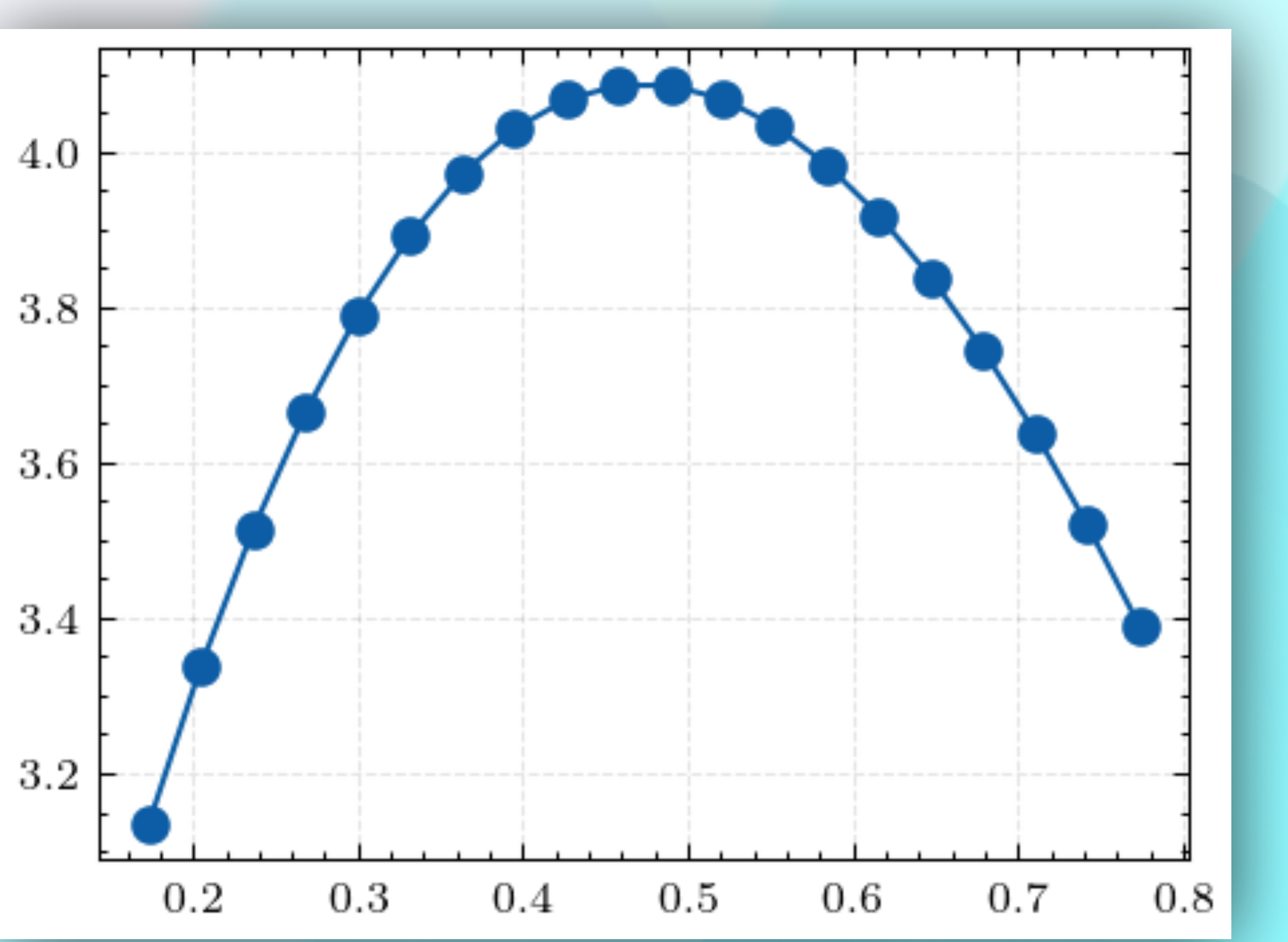


Experiment 3: Dark soliton simulations

Unstable dark solitons in D_2

(Le Quiniou, 2025).: There exists $\kappa_0 \approx 3.636$ such that, if $\kappa < \kappa_0$ the soliton is unstable if $c \in (0, c^*(\kappa))$ or stable if $c \in (c^*(\kappa), \sqrt{2})$

$$P(\Psi) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{Re}\left(i\partial_x \Psi \overline{\Psi}\right)\left(1 - \frac{1}{|\Psi|^2}\right)dx,$$



Momentum vs c for $\kappa = -50$

We study the linear stability. Let $u_{c,\kappa}$ an unstable soliton. Traveling frame $(x - ct, t)$:

$$i\partial_t U = \partial_{xx} U + ic\partial_x U + U(1 - |U|^2) - \kappa U \partial_{xx}|U|^2.$$

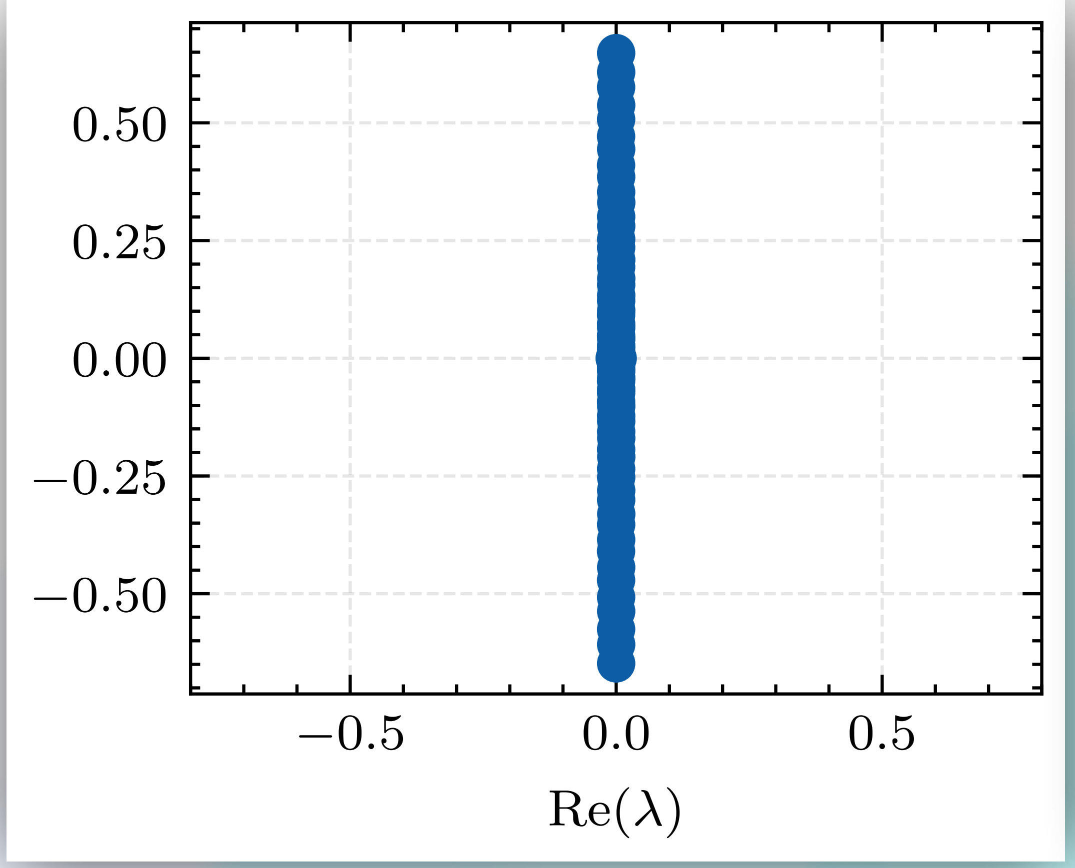
We perturb around the soliton: $U(x,t) = u(x) + \epsilon v(x,t)$, $v = p + iq$ and linearize the equation.

- The eigenvalues of the linearized system correspond to the exponential increase factor for instabilities in short time.
- In fact, the regime will be dominated by the eigenvalue with the largest real part.

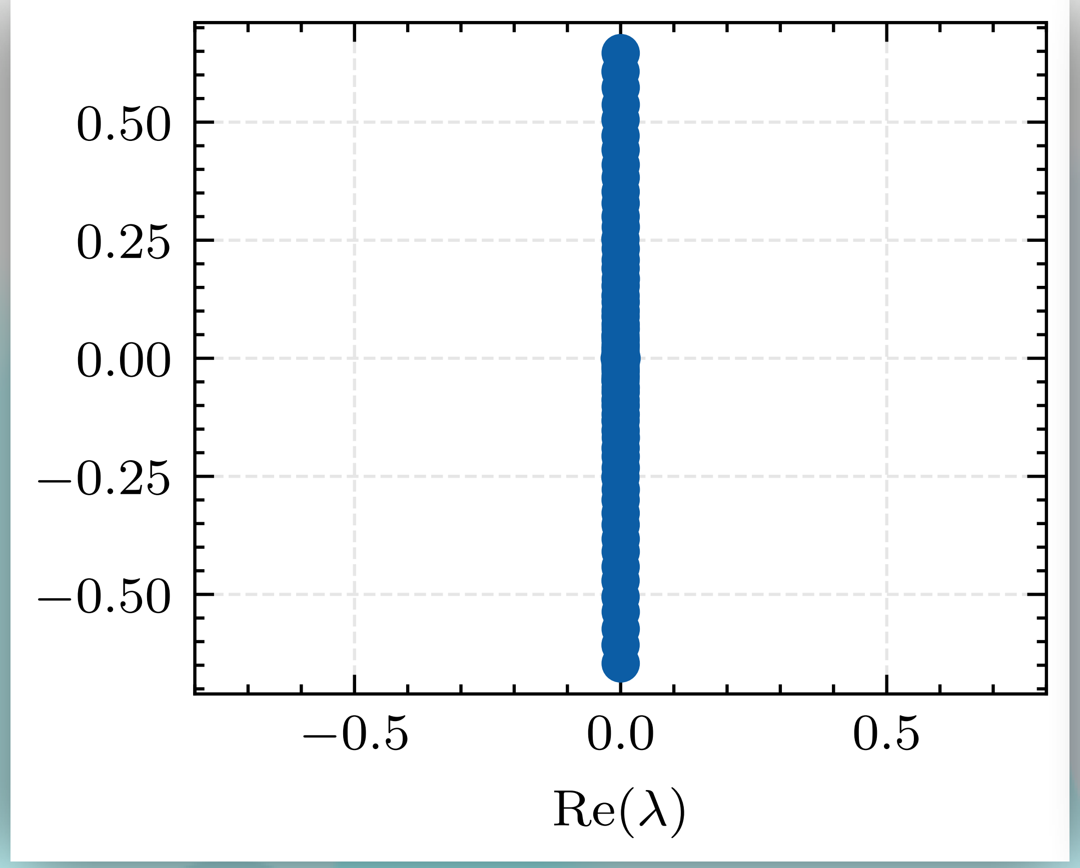
Experiment 3: Dark soliton simulations

Unstable dark solitons in D_2

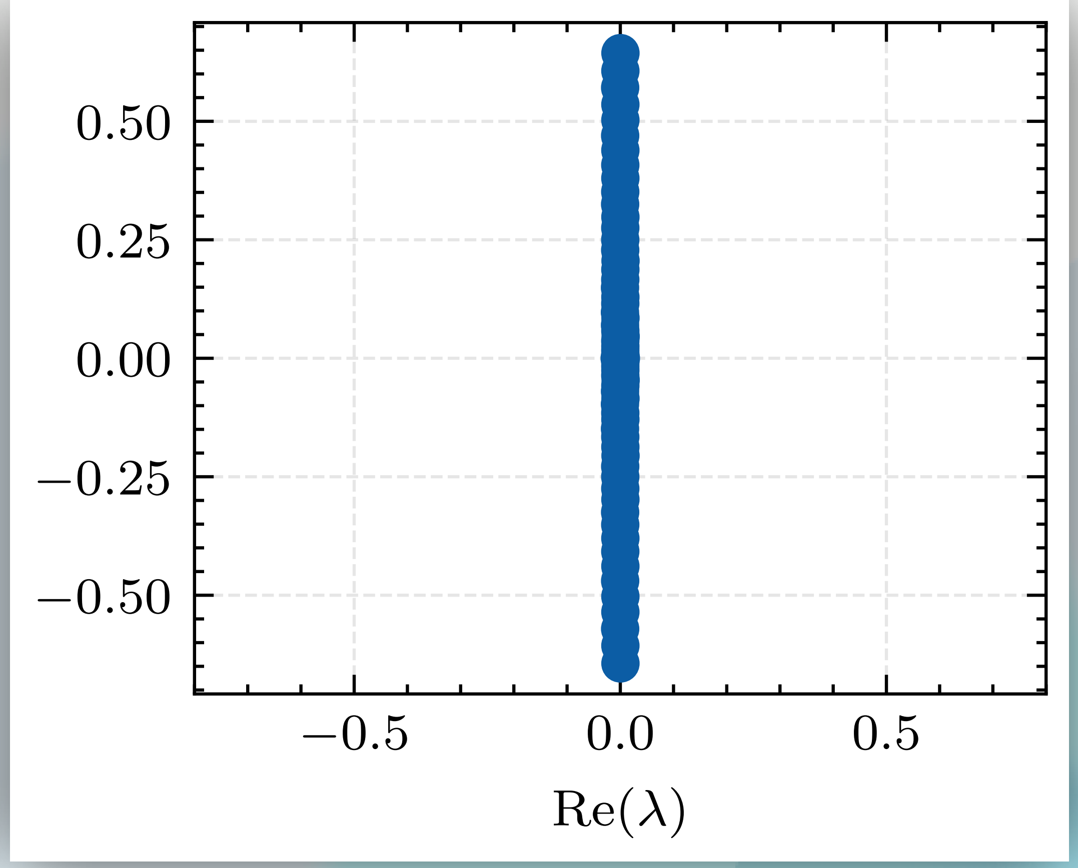
$c^* = c^*(-50)$



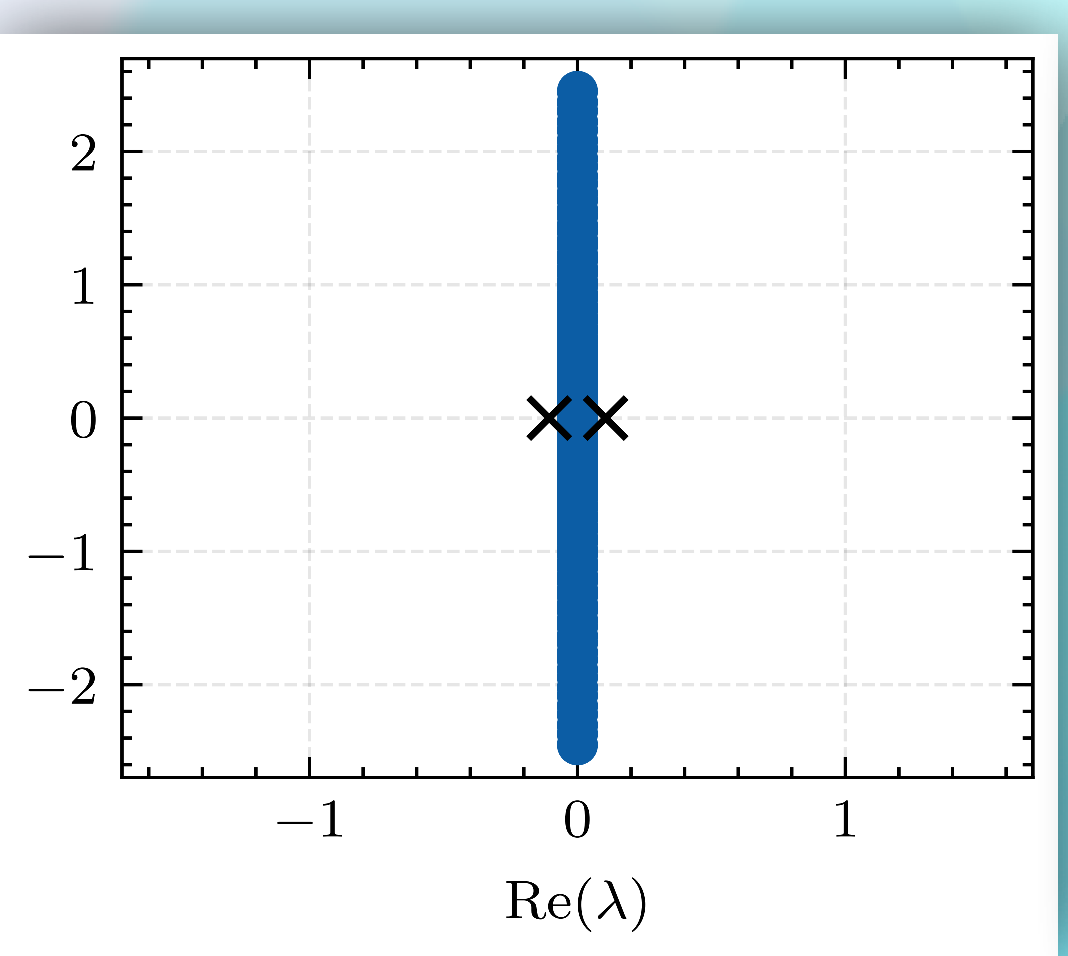
$c = c^* + 0.1$



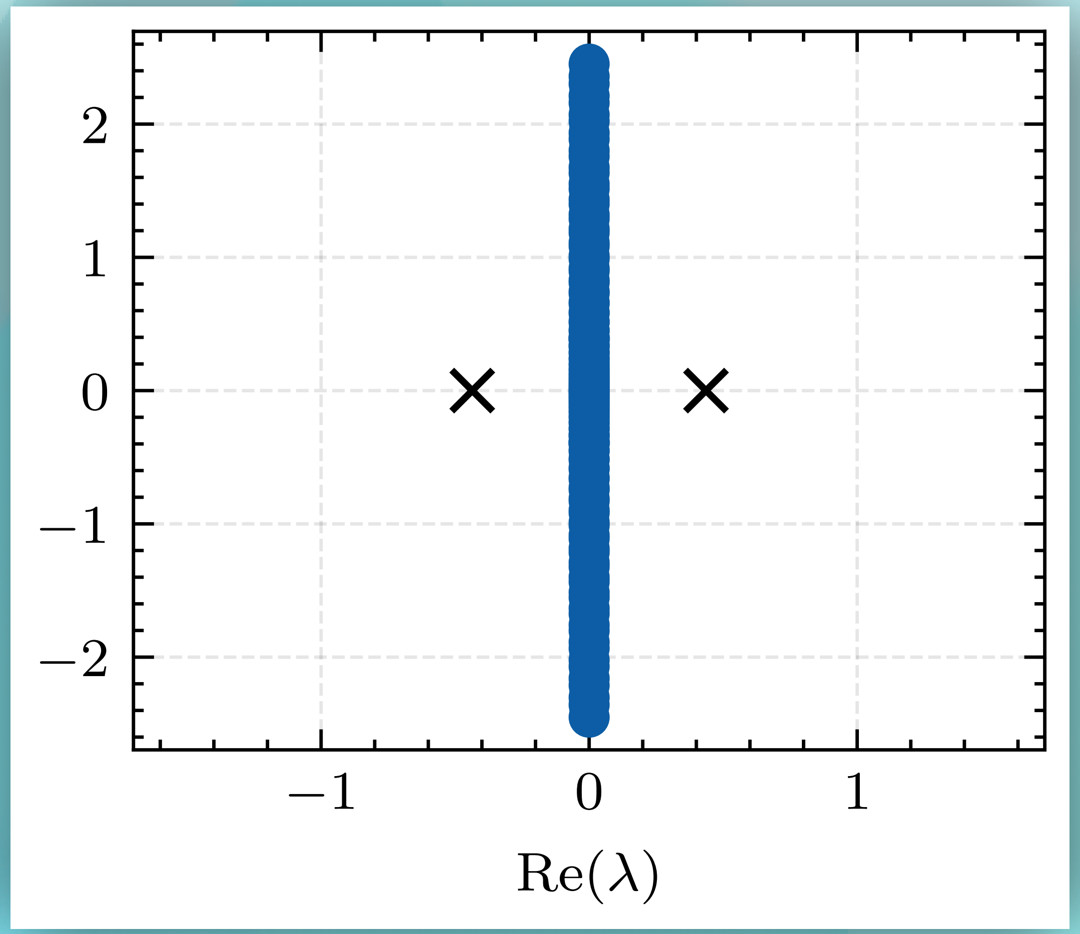
$c = c^* + 0.2$



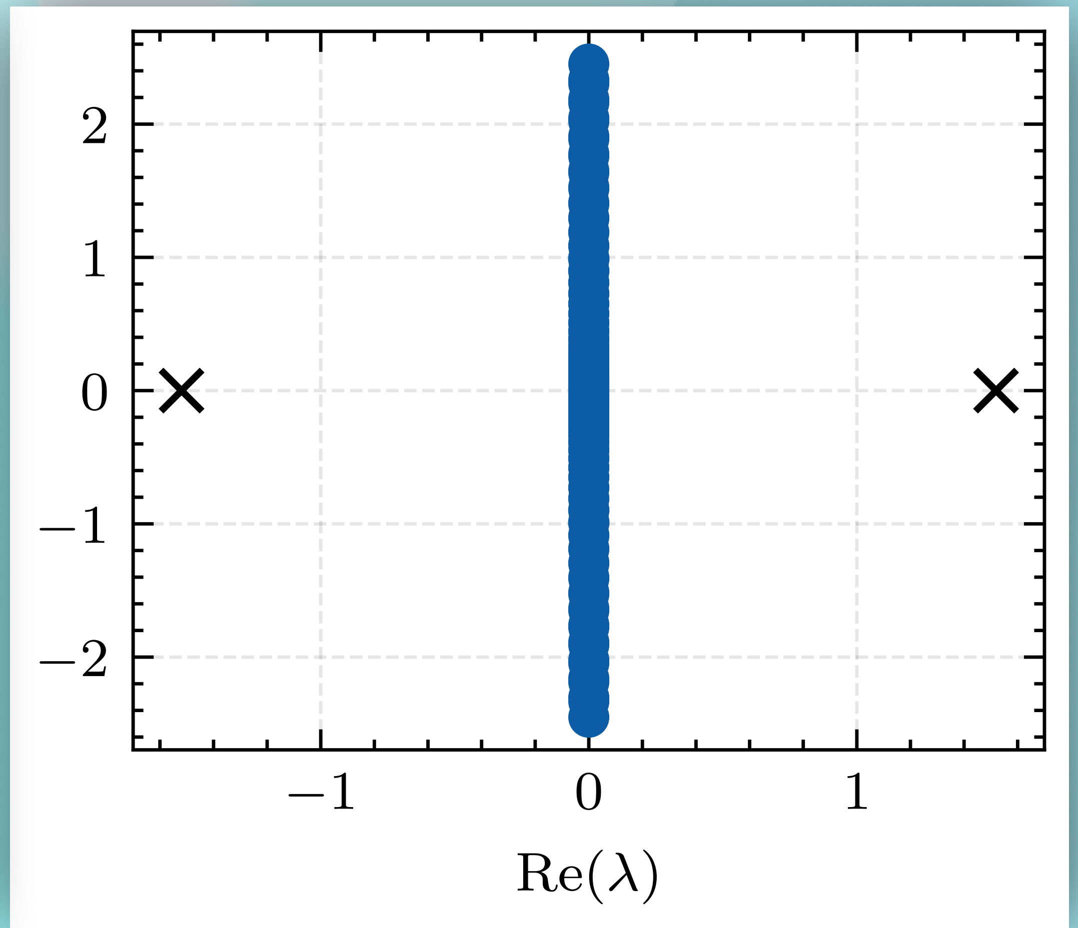
$c = c^* + 0.3$



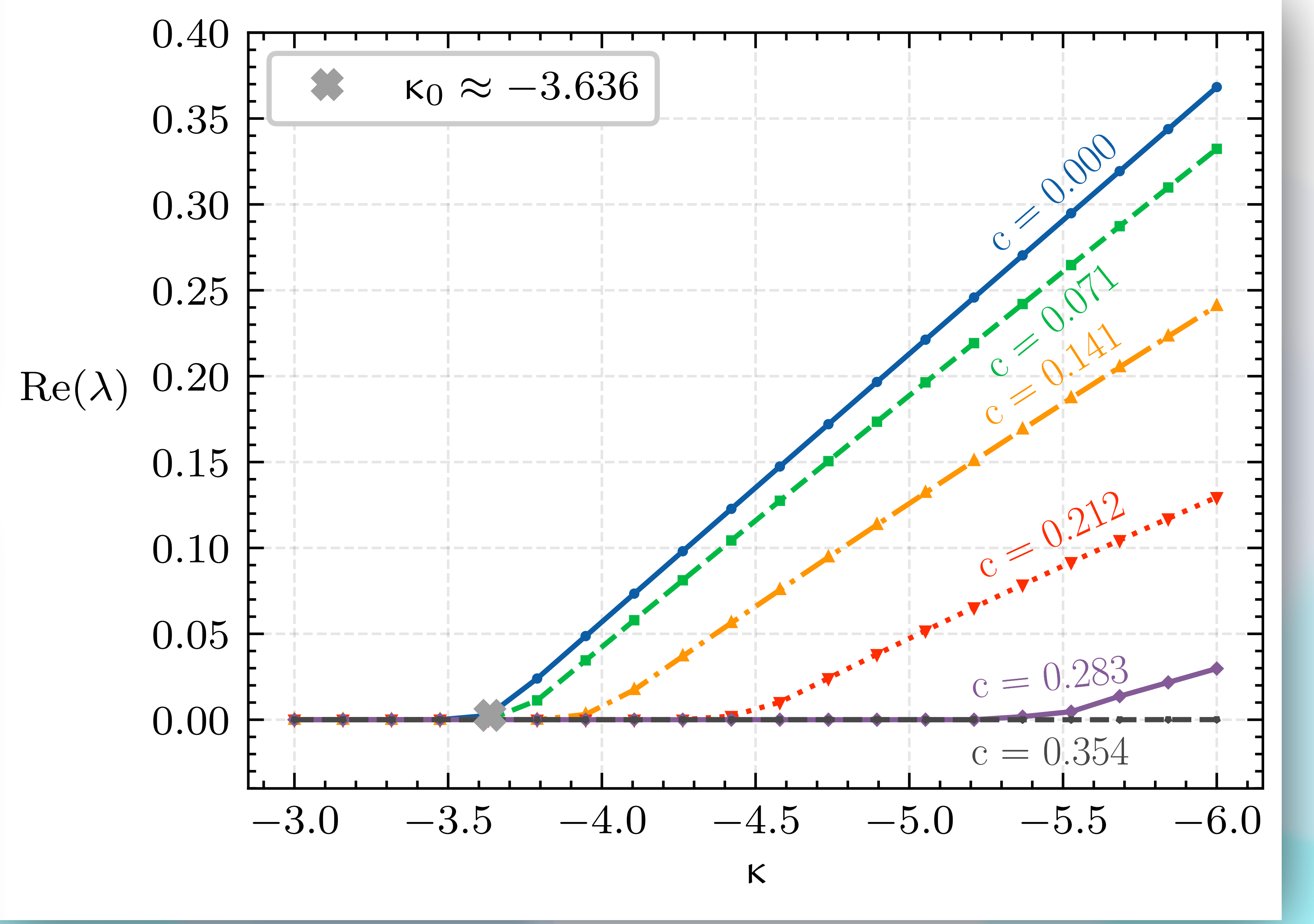
$c = c^* - 0.1$



$c = c^* - 0.2$



$c = c^* - 0.3$



Experiment 3: Dark soliton simulations

Unstable dark solitons in D_2

We perturb an unstable dark soliton with a gaussian:

$$\Psi(x,0) = u_{c,\kappa}(0) + 10^{-2}e^{-x^2},$$

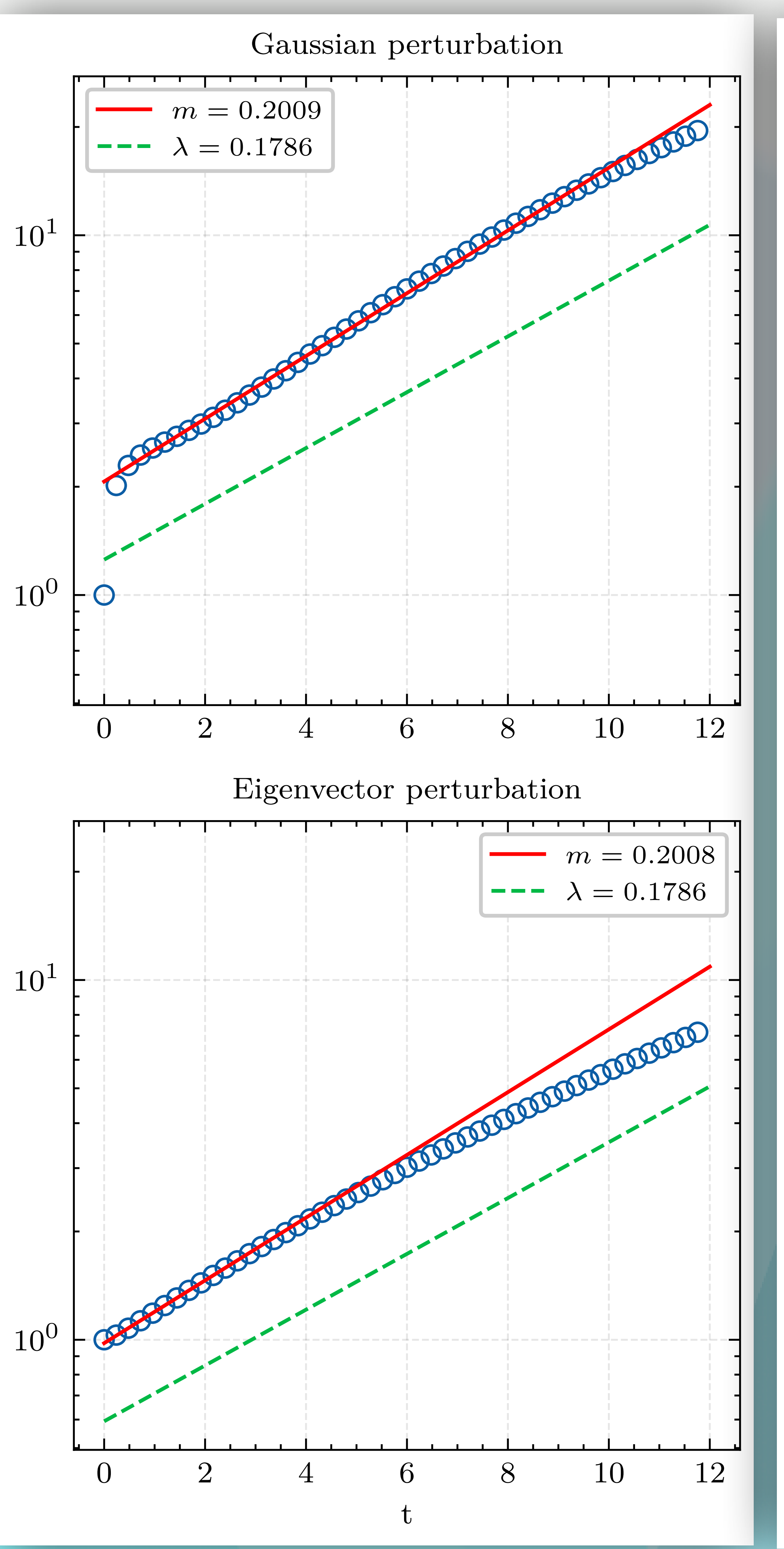
and with the associated eigenvector:

$$\Psi(x,0) = u_{c,\kappa}(0) + 10^{-2}v\,.$$

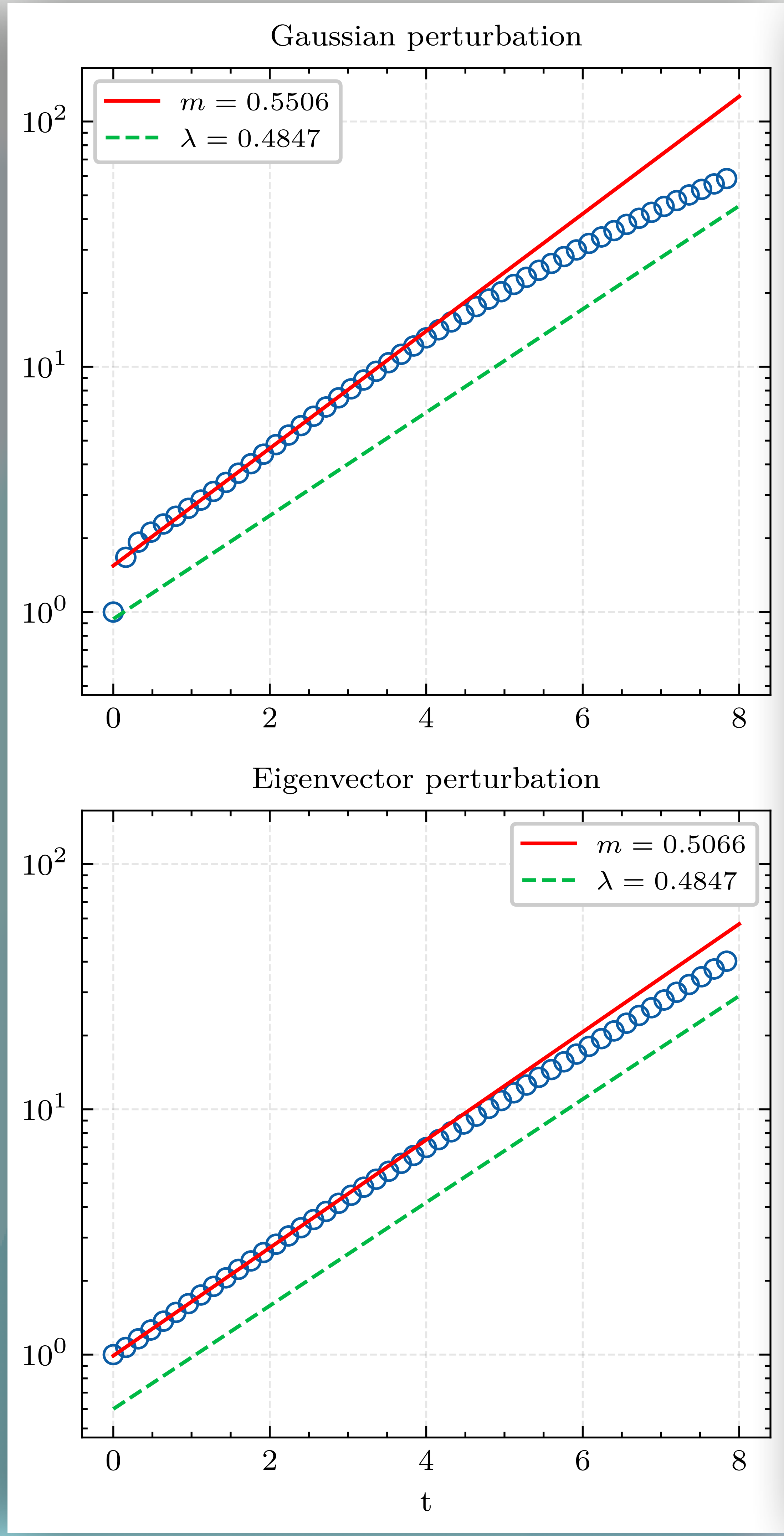
Problem: Scheme has a small error.

Solution: Choose T, J and N such that:

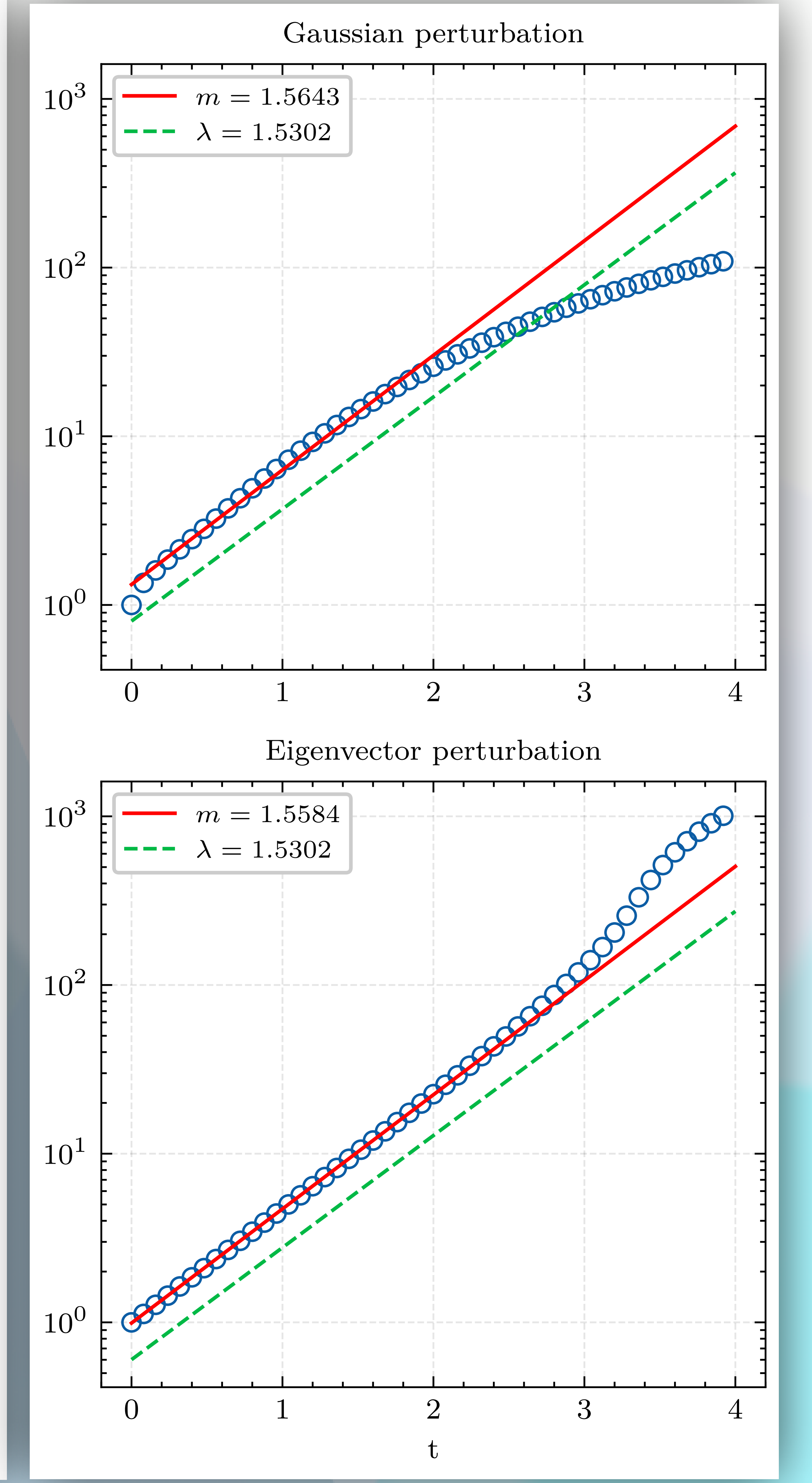
$$||\Psi_{theo}(x,t_n) - \Psi^n_{num}||_{L^2} < 10^{-2}, \text{ for } 1 \leq n \leq N.$$



$c = c^* - 0.1$



$c = c^* - 0.2$



$c = c^* - 0.3$

Conclusion and perspectives

Conclusion

- We proposed a fully implicit Crank-Nicolson finite-difference scheme for the QGP equation.
- The scheme preserves mass and energy at the discrete level.
- Numerical experiments confirm second-order convergence in time and space.
- Bright and dark solitons provide reliable benchmarks for the method.
- Blow-up simulations suggest a strong link with the loss of the ellipticity condition.
- The instability of dark solitons is consistent with the spectral predictions.

Perspectives

- Establish a rigorous convergence theory for the fully discrete scheme.
- Extend the numerical method to higher-dimensional settings, in particular to 2D and 3D problems.
- Develop suitable schemes for solutions with sharp localized structures or low-regularity profiles (antidark solitons, compactons, cuspons)
- Design adaptive mesh and time-stepping strategies for a more precise analysis of blow-up dynamics.

Thanks for your attention!

¡Gracias por su atención!

Merci de votre attention !

Appendix 1: Physical motivation I

Nonlinear optics

The evolution of an optical beam of intensity $|\Psi|^2$ in a nonlocal Kerr-like medium :

$$i\partial_t \Psi = \partial_{xx}\Psi + \mathfrak{s}\Psi\left(\mathscr{W} * |\Psi|^2\right).$$

where $\mathscr{W}$ characterizes the nonlocal response of the medium. In a weakly nonlocal medium, we can replace $\mathscr{W}$ by $\mathscr{W}_\varepsilon = \mathscr{W}(\,\cdot\,/\varepsilon)/\varepsilon$, for small $\varepsilon > 0$.

Performing a Taylor expansion on $\eta(x-y) = |\Psi|^2(x-y,t)$, we get

$$\eta(x-y) = \eta(x) - y\eta'(x) + \frac{y^2}{2}\eta''(x) + O(y^3).$$

Then, using that $\mathscr{W}_\varepsilon$ is even,

$$(\mathscr{W}_\varepsilon * \eta)(x) = \int_{\mathbb{R}} \mathscr{W}_\varepsilon(y)\eta(x-y)dy = \eta(x) + \kappa_\varepsilon \eta''(x) + O(\varepsilon^3), \text{ with } \kappa_\varepsilon = \frac{\varepsilon^2}{2}\int_{\mathbb{R}} y^2 \mathscr{W}(y).$$

QGP appears in the regime ε small, neglecting $\mathcal{O}(\varepsilon^3)$ (weakly nonlocal GP equation)

Appendix 2: Physical motivation II

Rutledge et al. [1] proposed a model for superfluidity in a thin helium film:

$$i\hbar\partial_t\varphi = -\underbrace{\frac{\hbar^2}{2m}\Delta\varphi}_{\text{Kinetic energy}} - \underbrace{\frac{A\varphi}{(a+|\varphi|^2)^3}}_{\text{Van der Walls force}} - \underbrace{\mu\varphi}_{\text{Chemical potential}} - \underbrace{B\varphi\Delta|\varphi|^2}_{\text{Surface tension}},$$

Normalized form proposed by Kurihara [2] (one spatial variable dependance):

$$i\partial_t\varphi = -\partial_{xx}\varphi - \varphi\left(\left(\frac{1+d_0}{1+d_0|\varphi|^2)}\right)^3 - 1\right) - d_1\varphi\partial_{xx}|\varphi|^2.$$

The regime where $|\varphi|$ is close to 1, Kurihara deduced:

$$i\partial_t\varphi = -\partial_{xx}\varphi - d_2(1-|\varphi|^2)\varphi - d_3\varphi\partial_{xx}|\varphi|^2,$$

where $d_2 = 3d_0/(1+d_0)$. We recover (QGP) by doing the change of variable $\Psi = \overline{\varphi}e^{-id_2t}$.

[1] J.E. Rutledge, W.L. McMillan, J.M. Mochel, and T.E. Washburn. Third sound, two-dimensional hydrodynamics, and elementary excitations in very thin helium films. Physical Review B, 18(5):2155, 1978.
[2] S. Kurihara. Large-amplitude quasi-solitons in superfluid films. Journal of the Physical Society of Japan, 50(10):3262–3267, 1981.

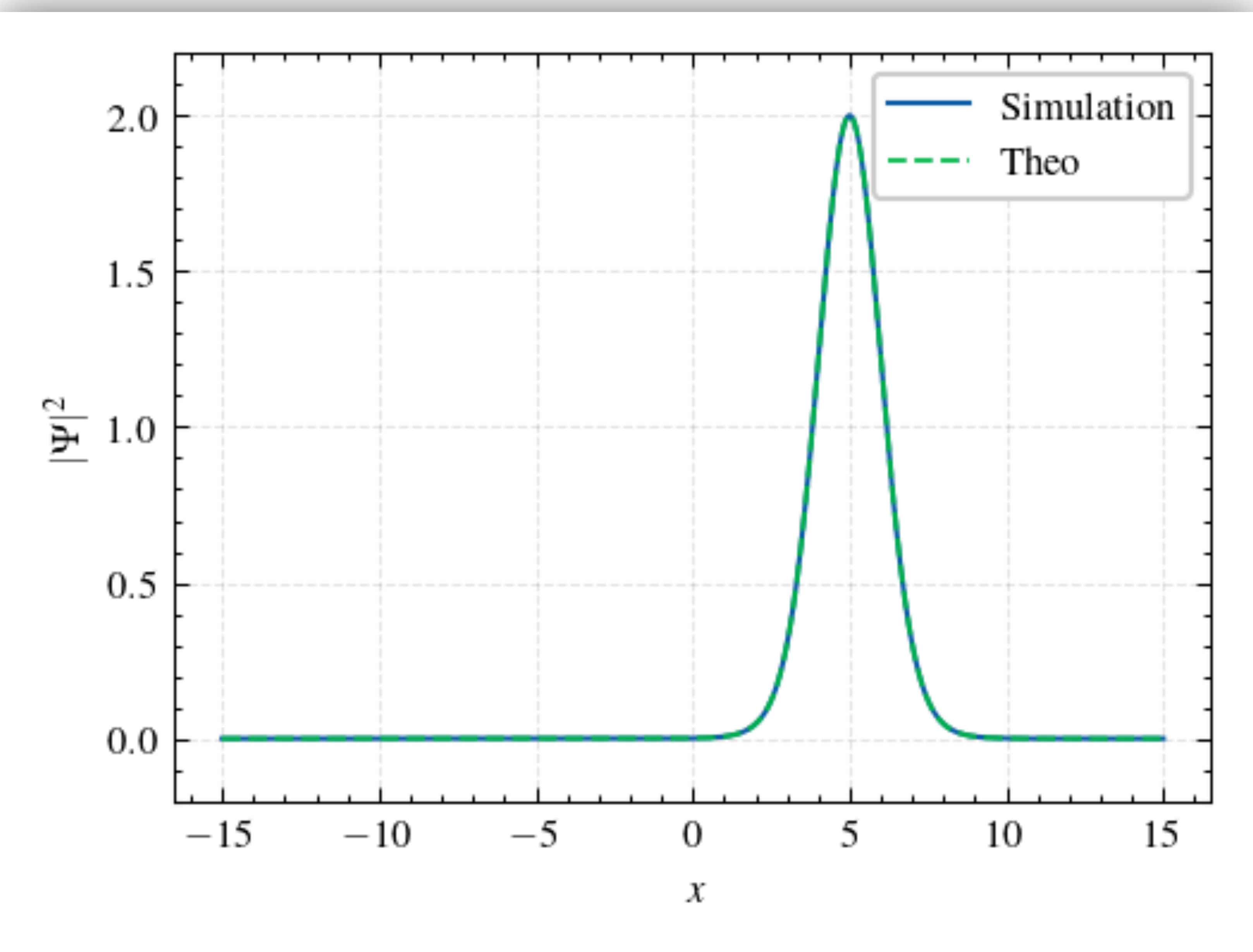
Appendix 3: Comments on boundary condition

As mentioned, we consider homogeneus Neumann boundary conditions. However, this **does not** allow us to simulate solutions for all reals due to **reflections** of the equation.

Some « solutions »:

- **Large enough L and stopping criteria when simulation reaches the boundary**
- Artificial boundary conditions (NLS with zero background)
- Absorbing layers (NLS with zero background)

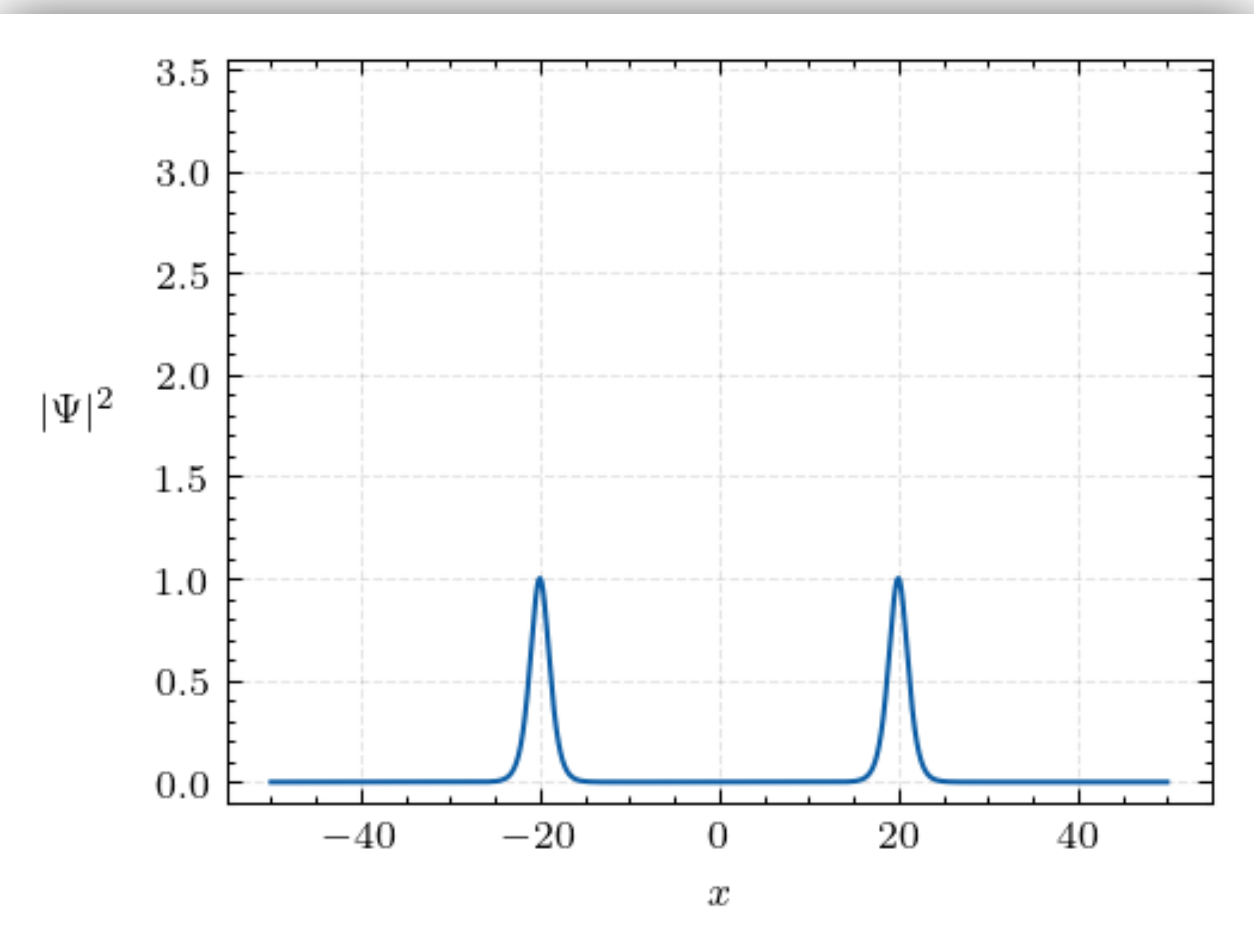
No artificial boundary conditions are known for non-zero background (even for NLS)

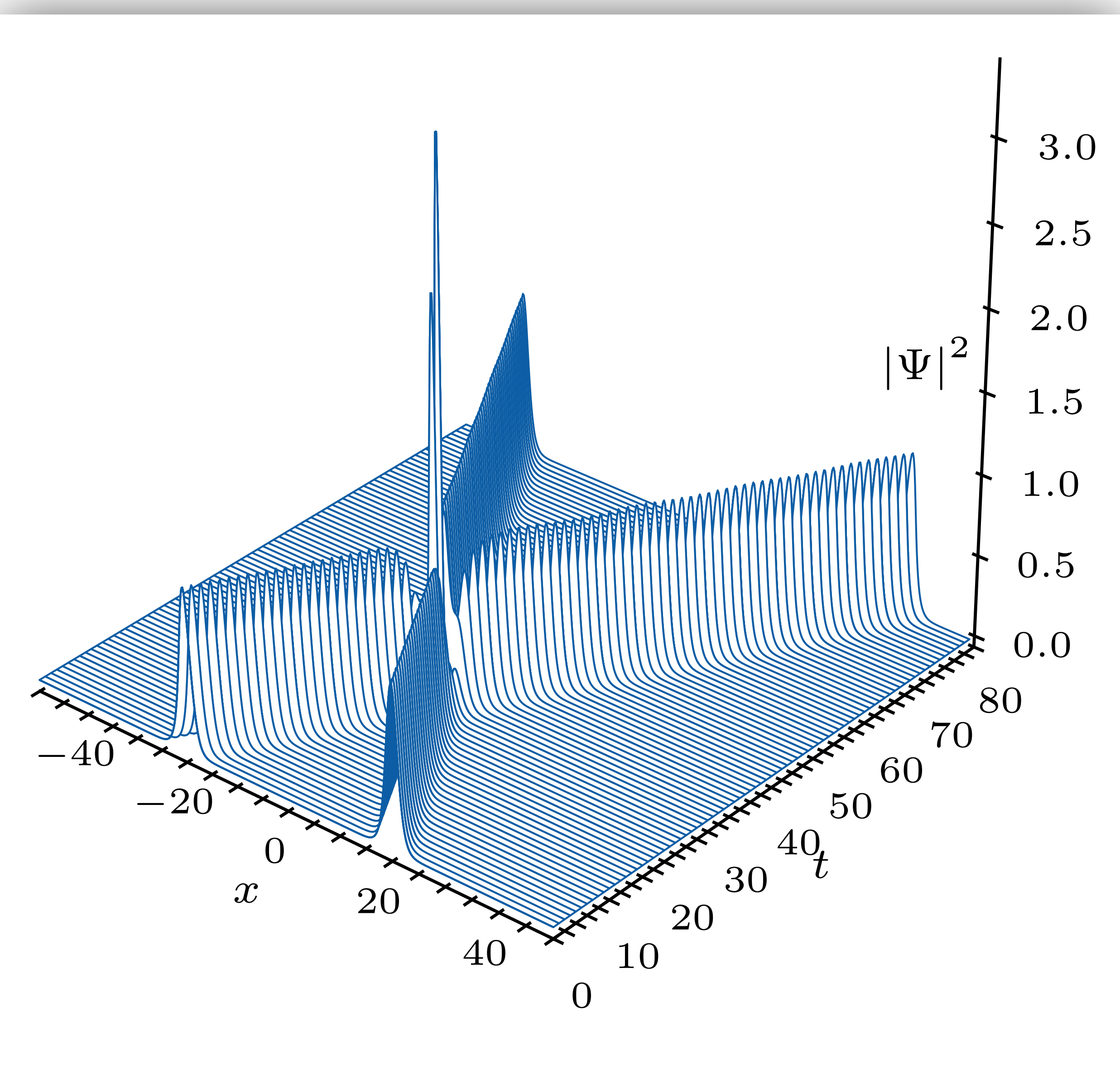


Reflection example in QGP ($\kappa = 1/4$)

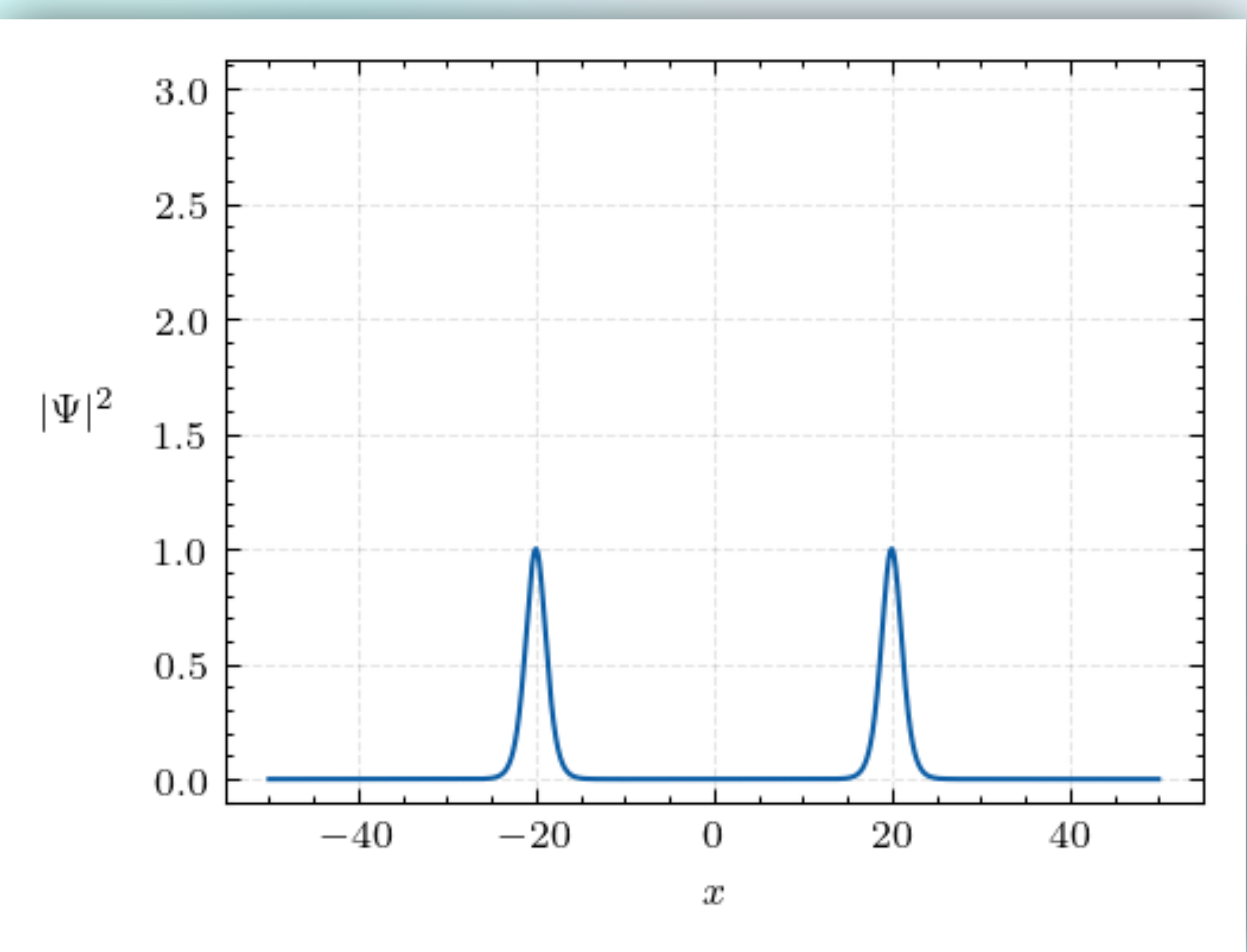
Appendix 4: Collisions

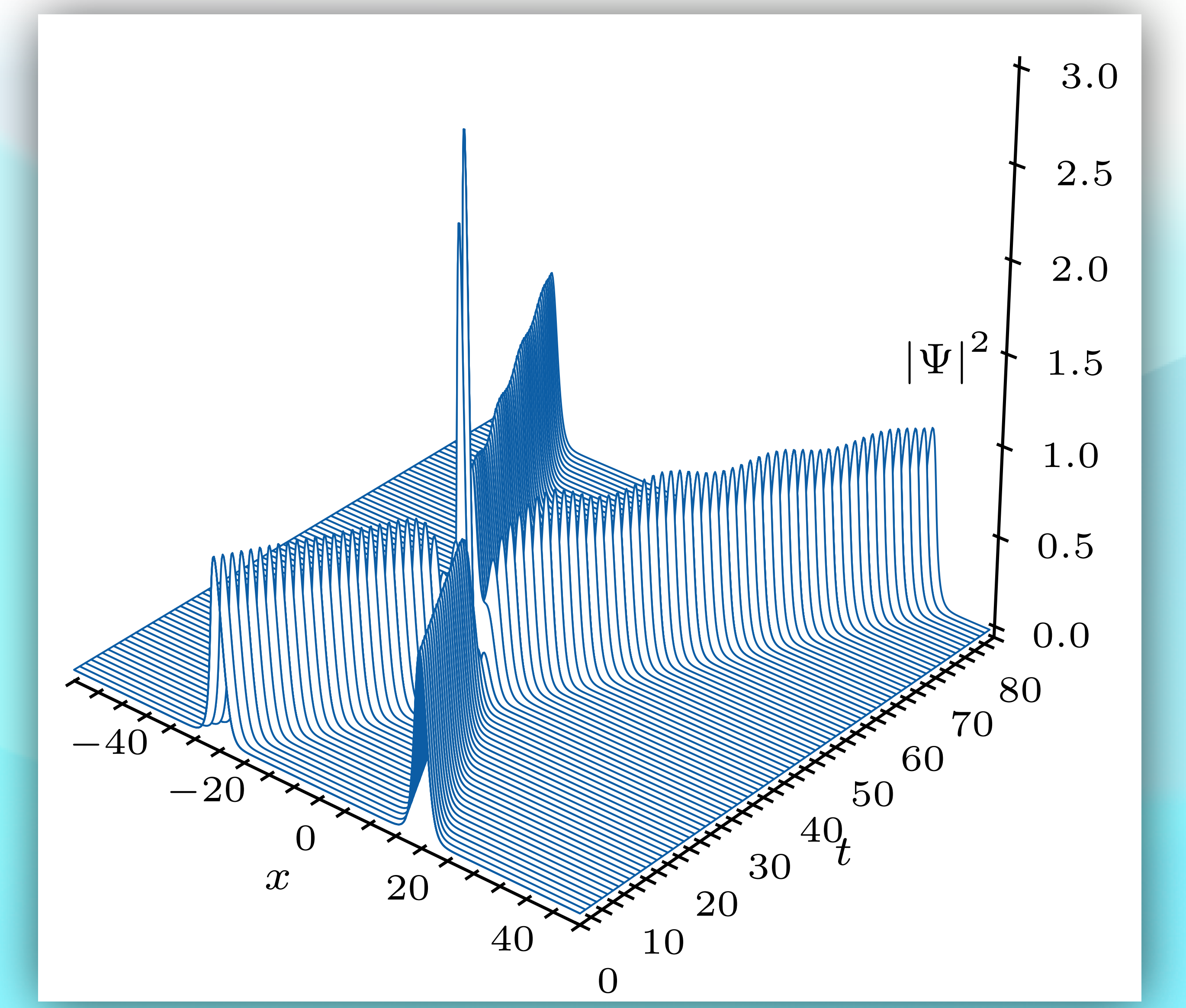
$\kappa = 0.01$





$\kappa = 0.1$

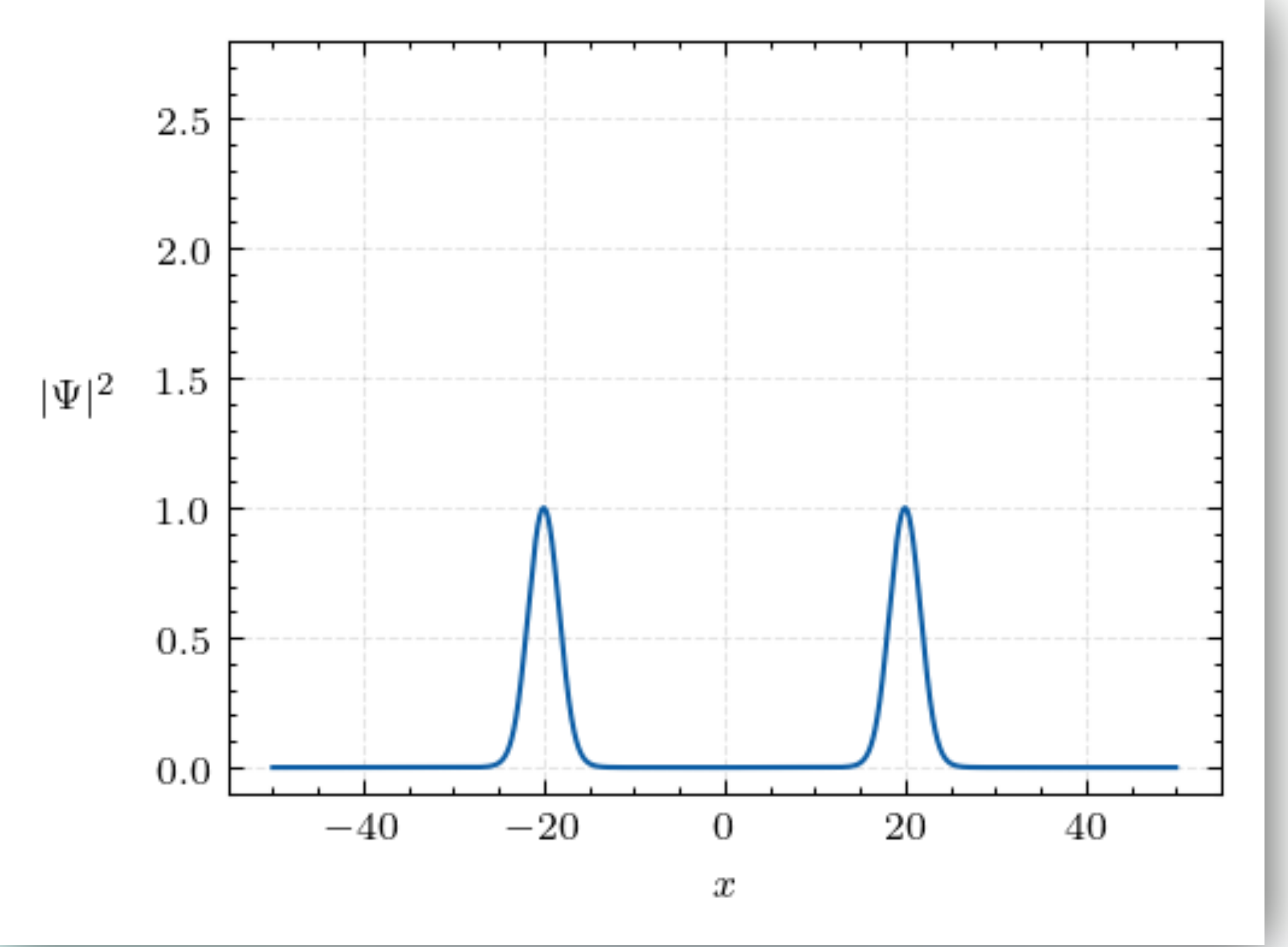


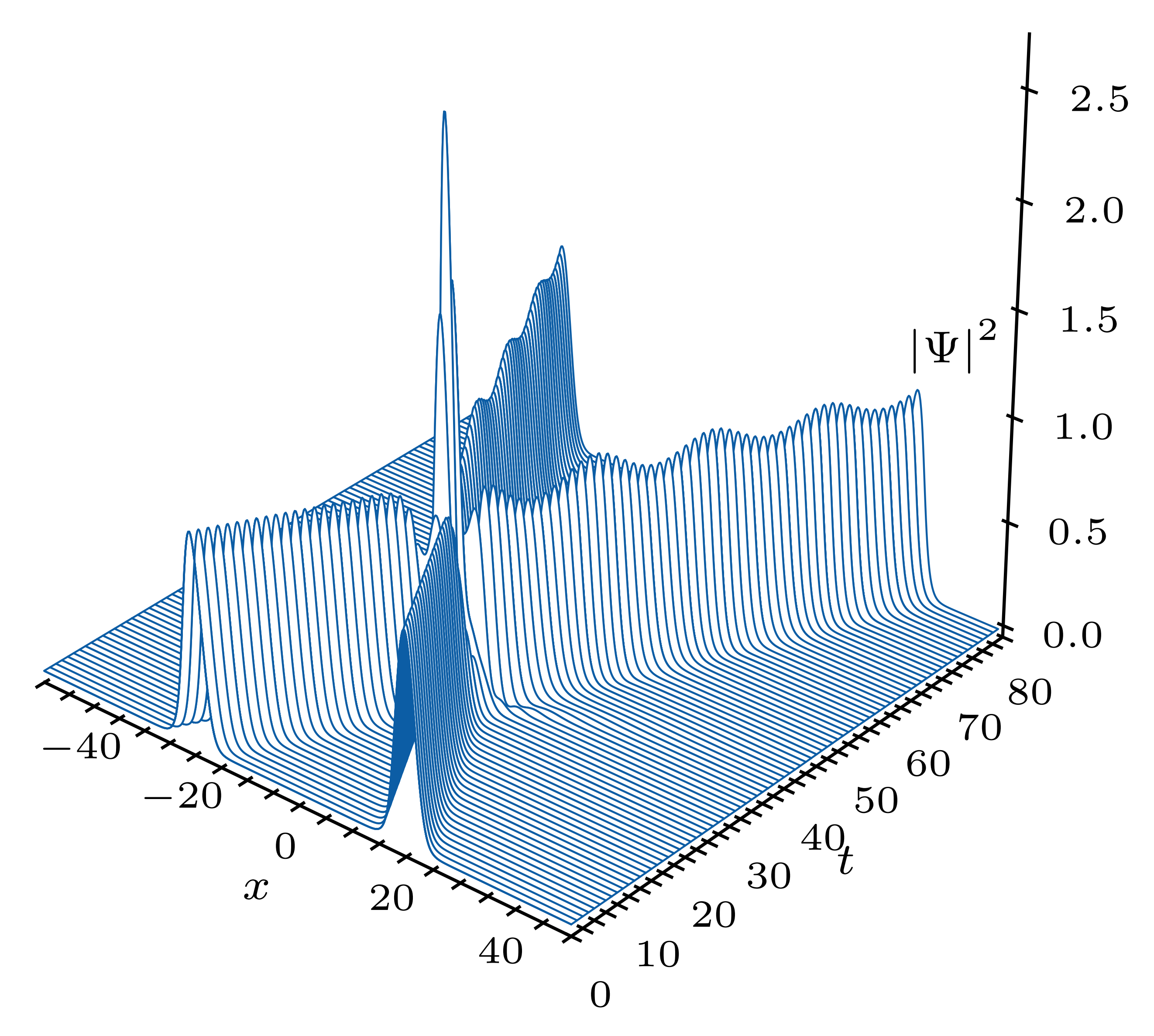


We recover (Krolikowski, W., & Bang, O. (2000)) collisions.

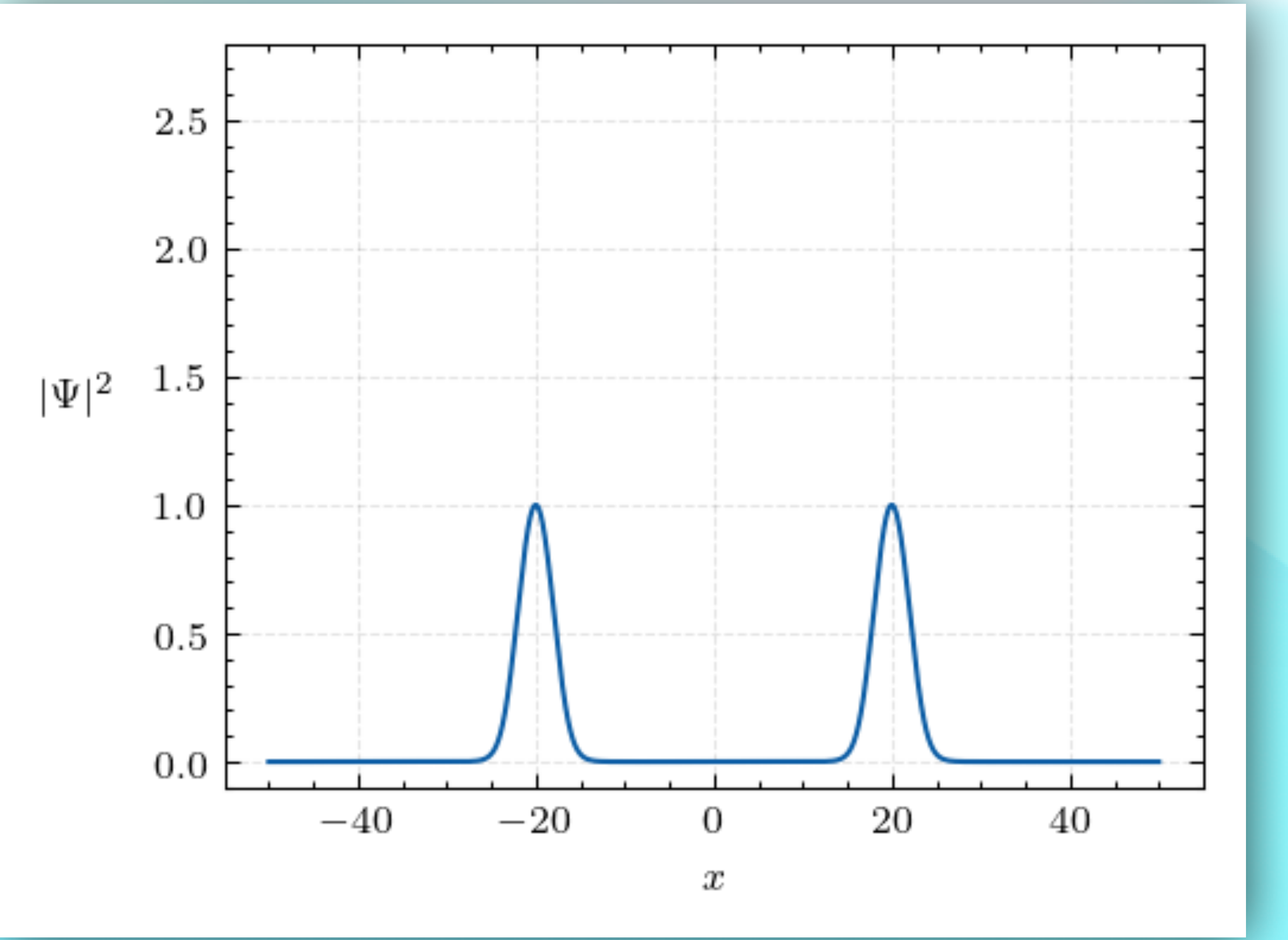
More collisions

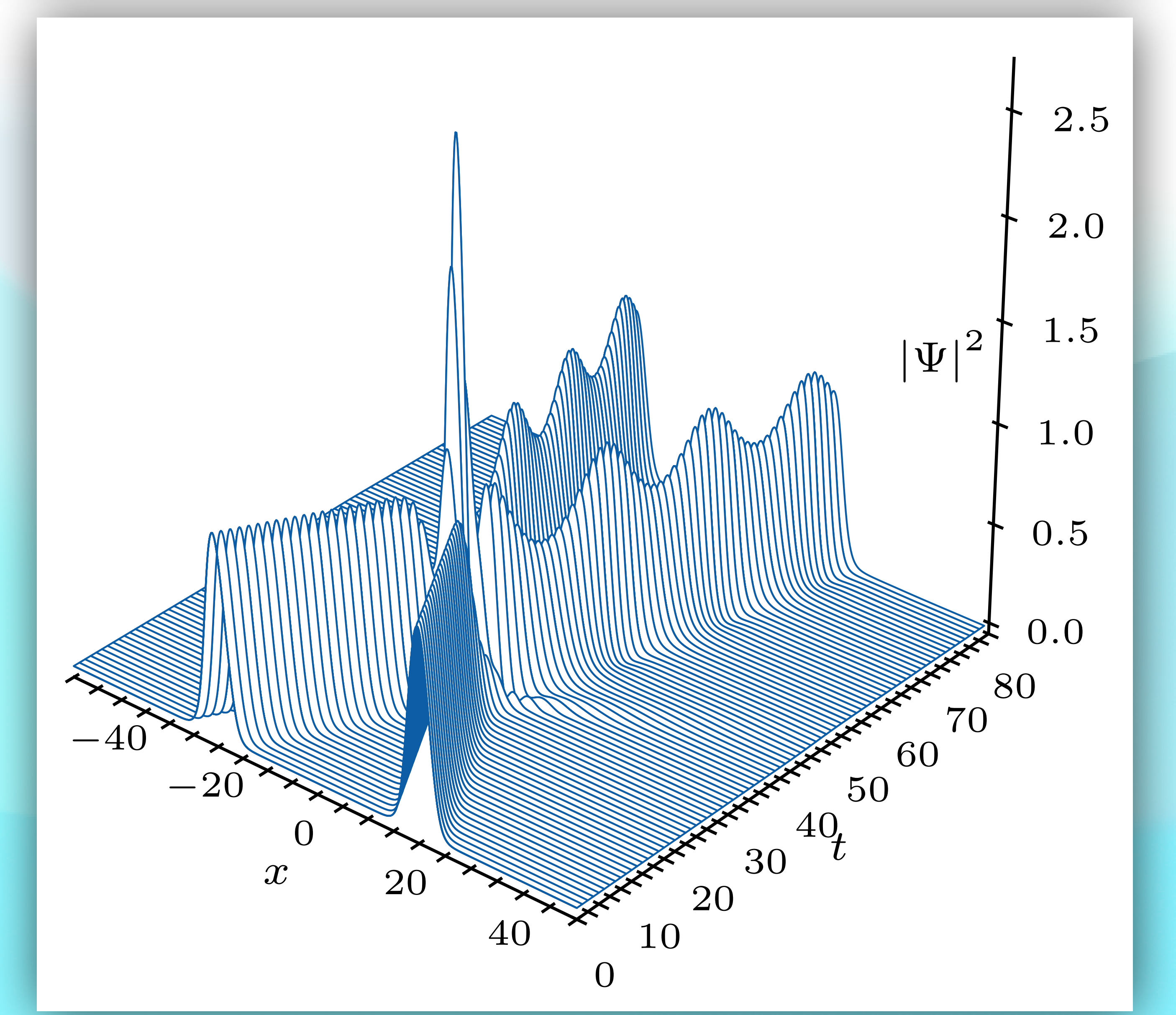
$\kappa = 1$





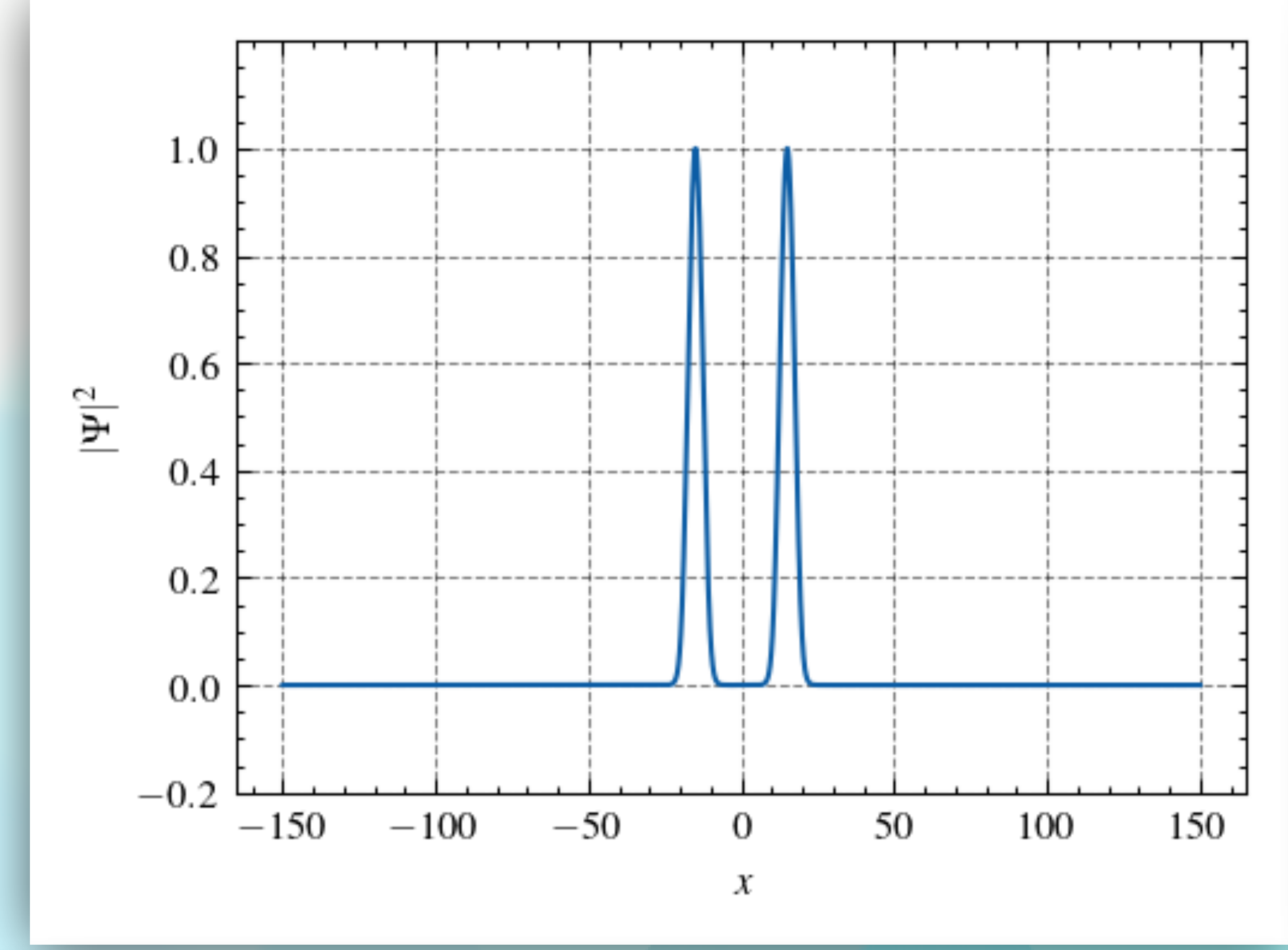
$\kappa = 1.5$

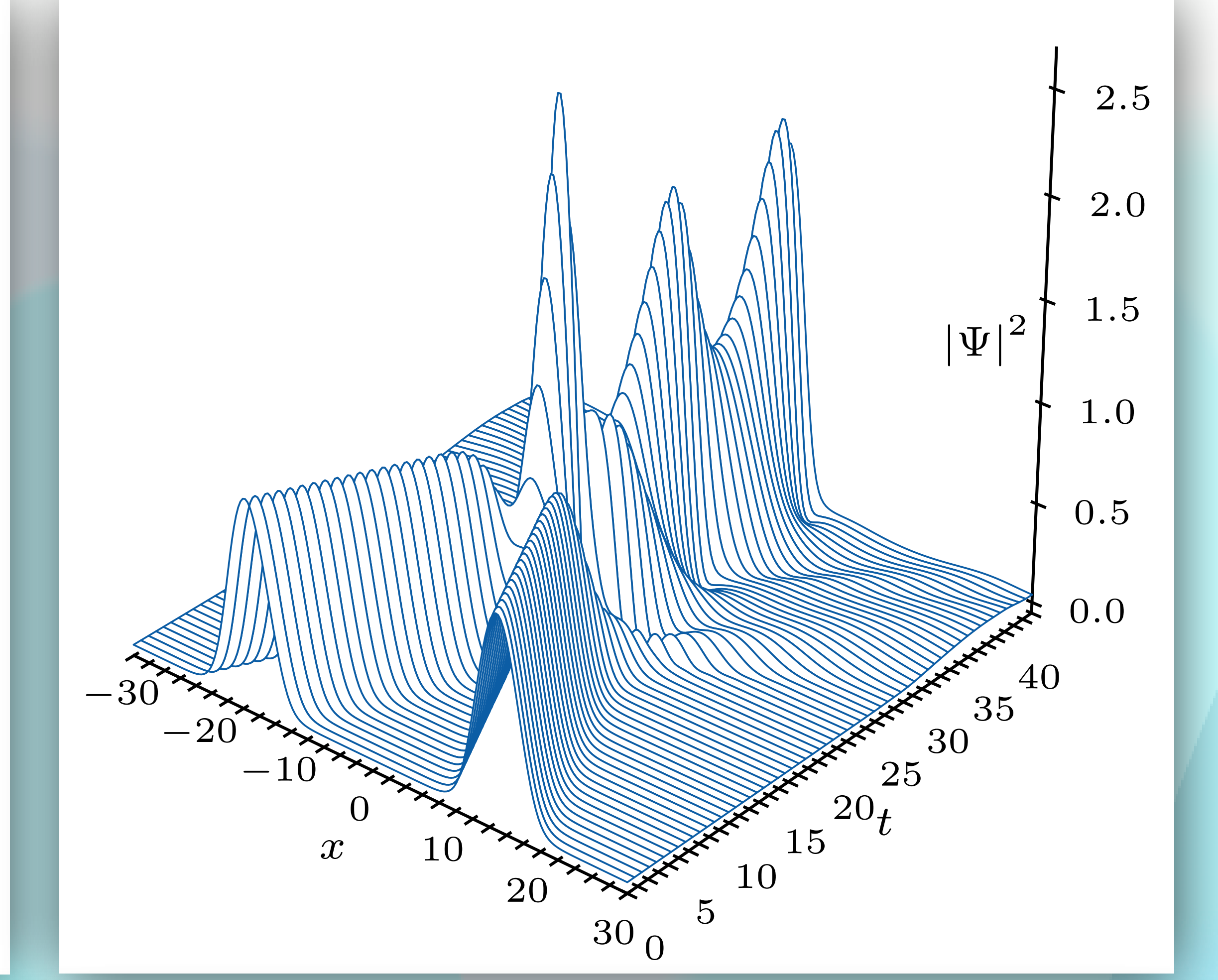




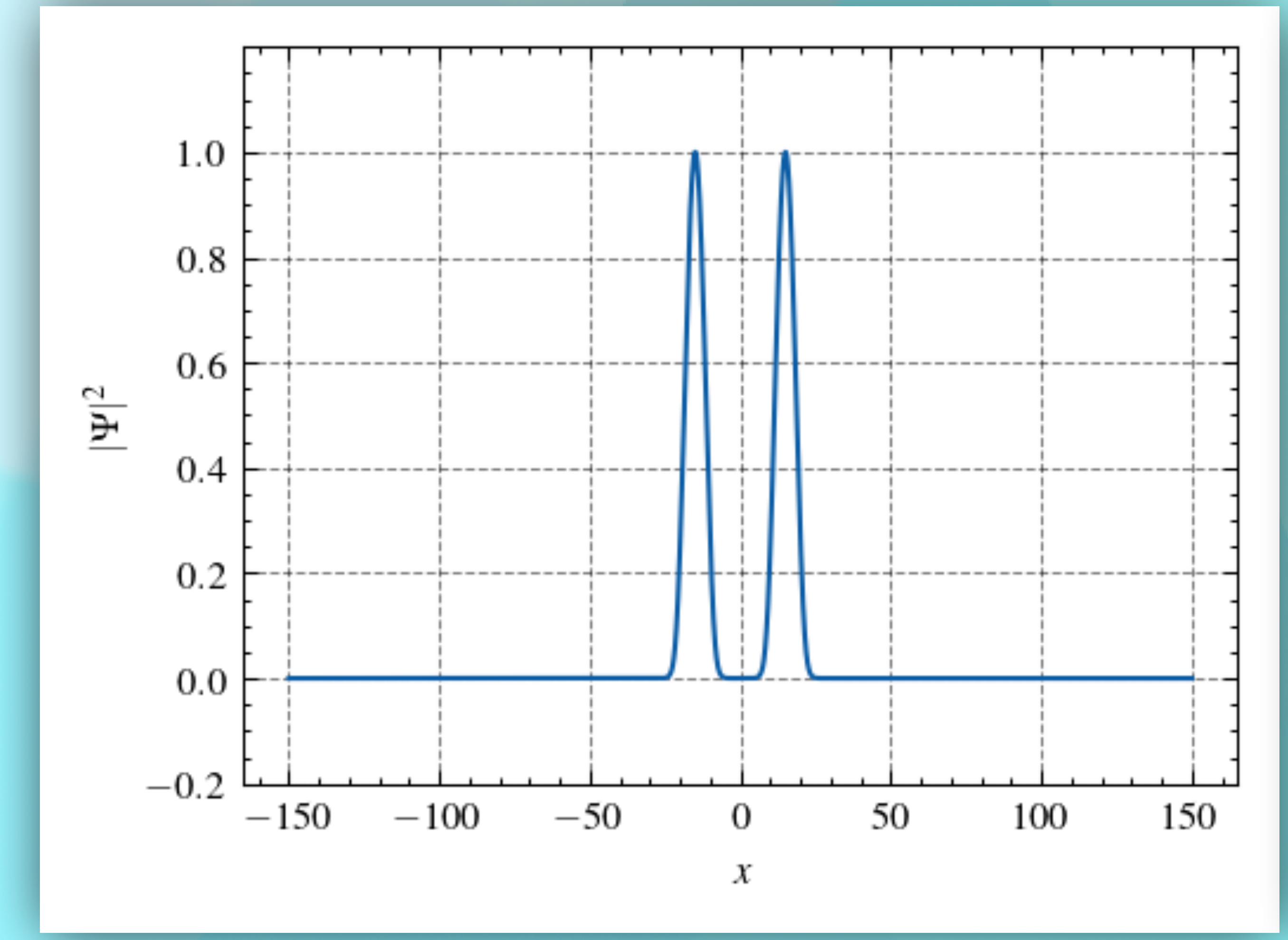
Greater κ in bright soliton collisions

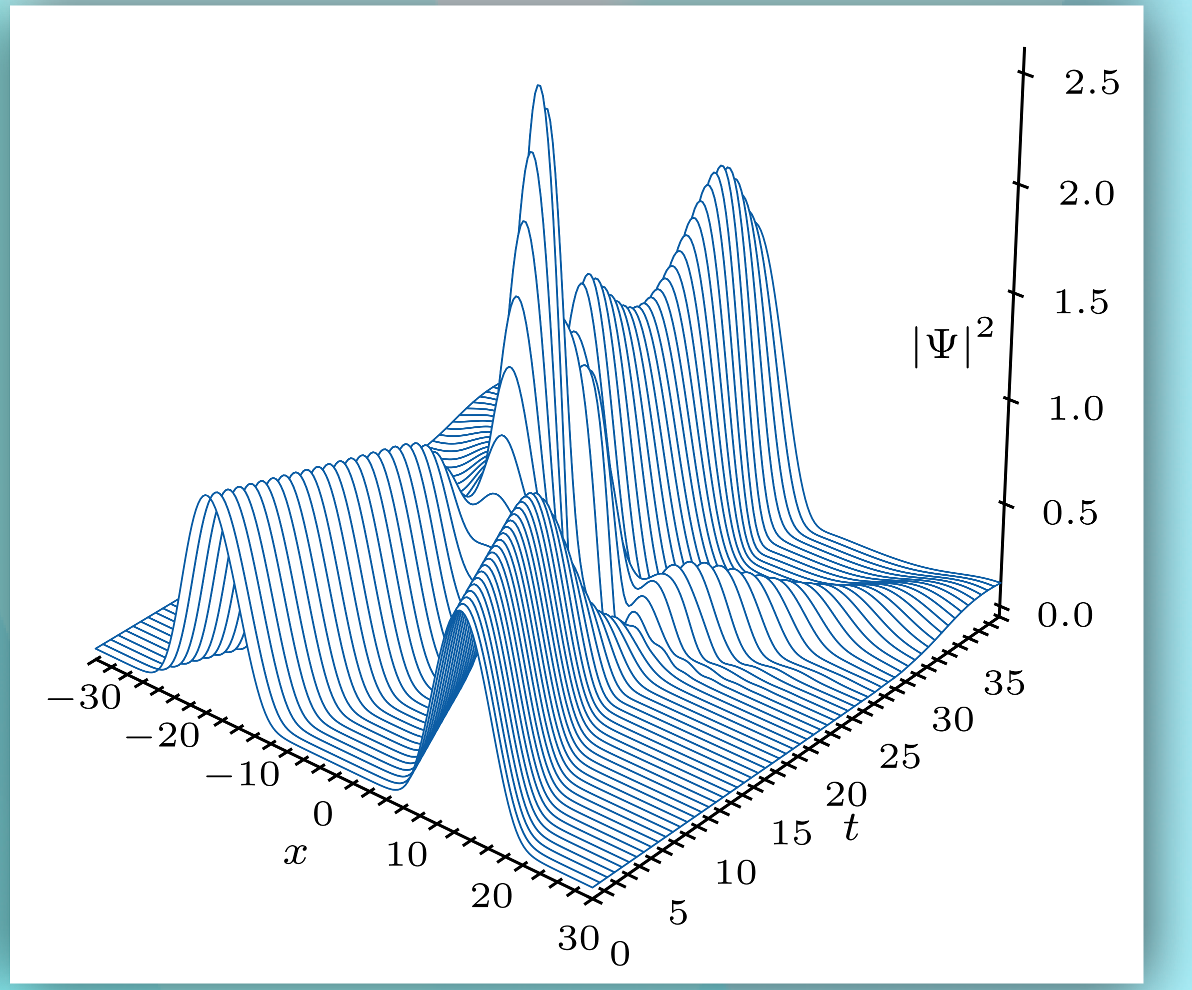
$\kappa = 2.5$



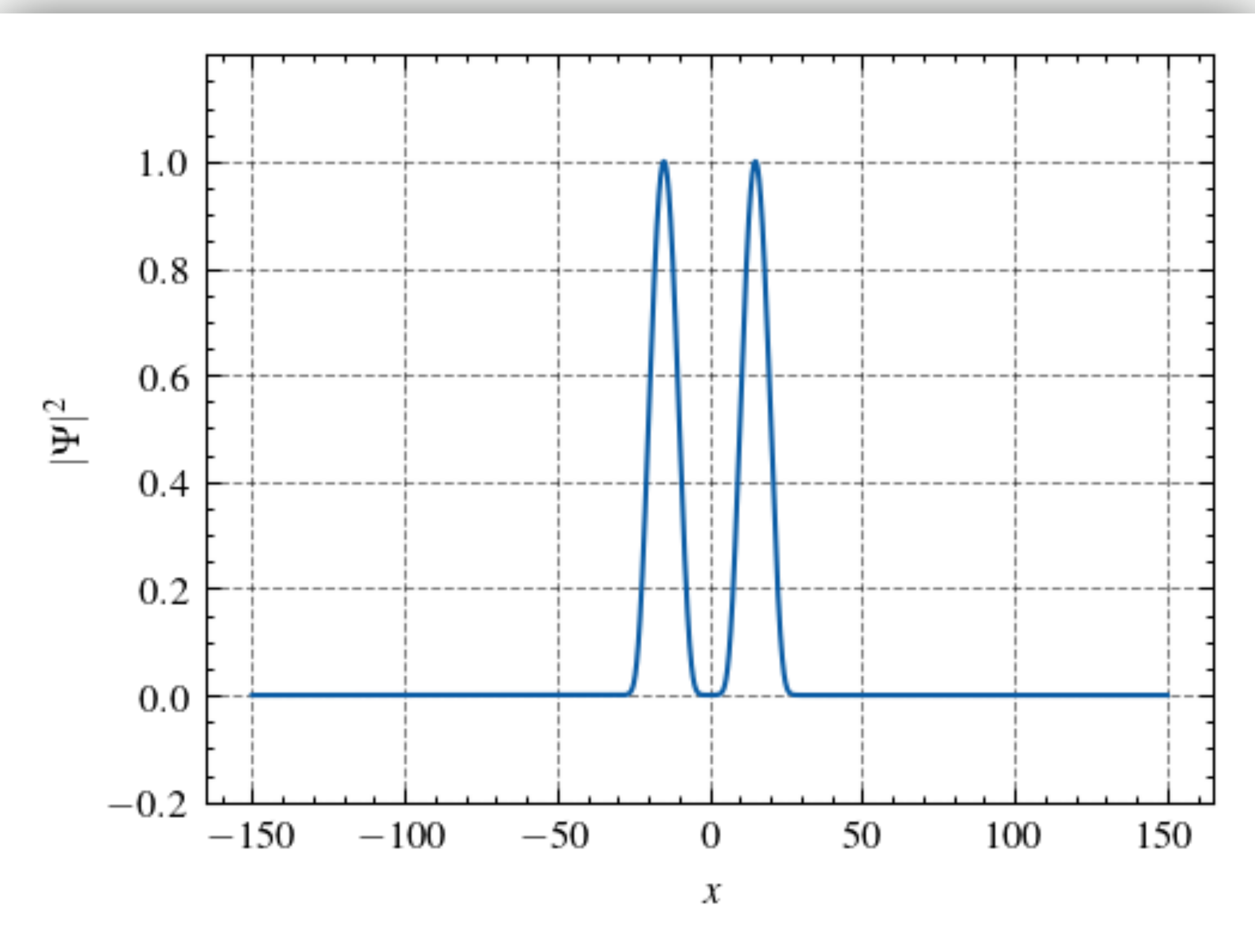


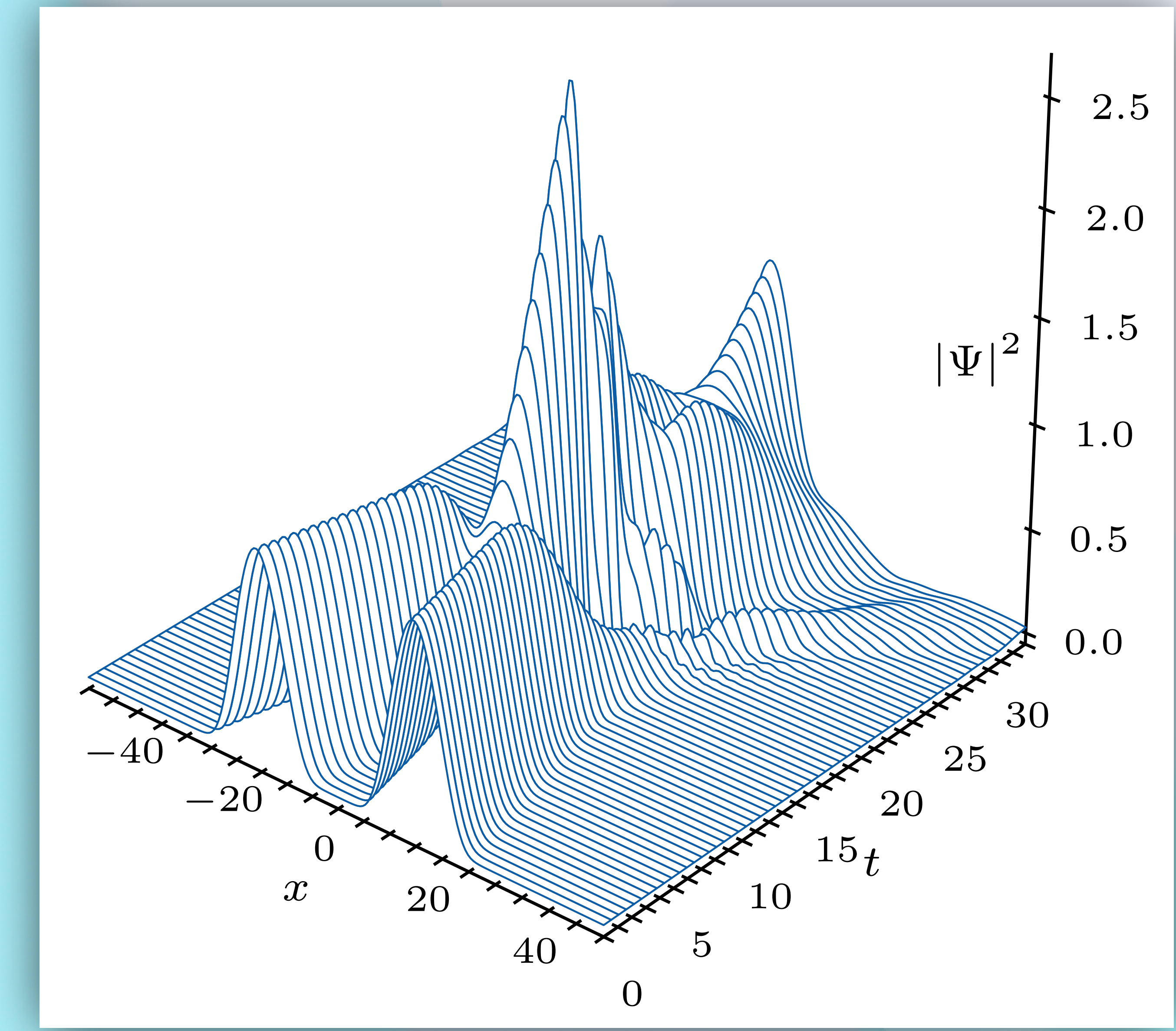
$\kappa = 5$





$\kappa = 10$



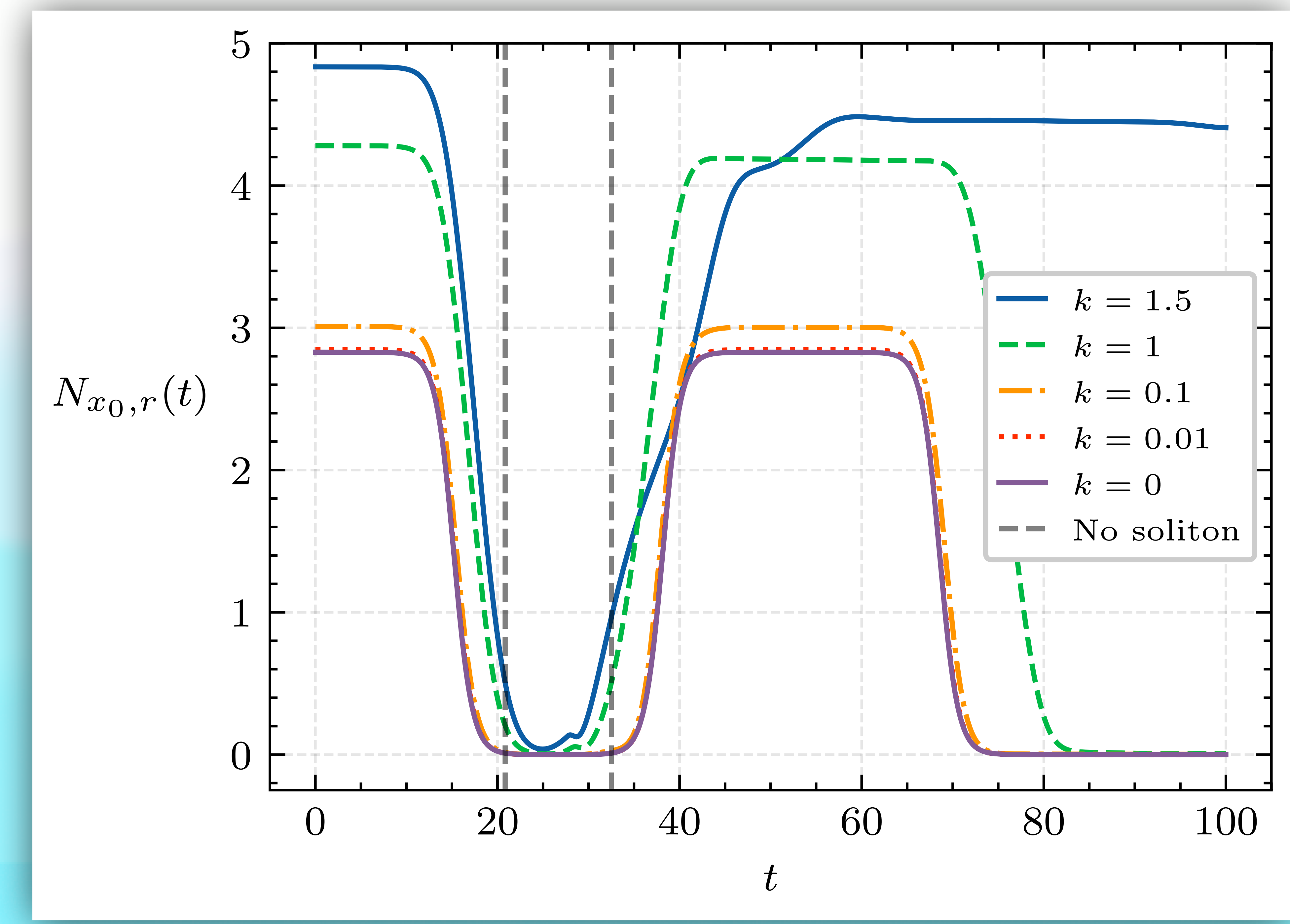


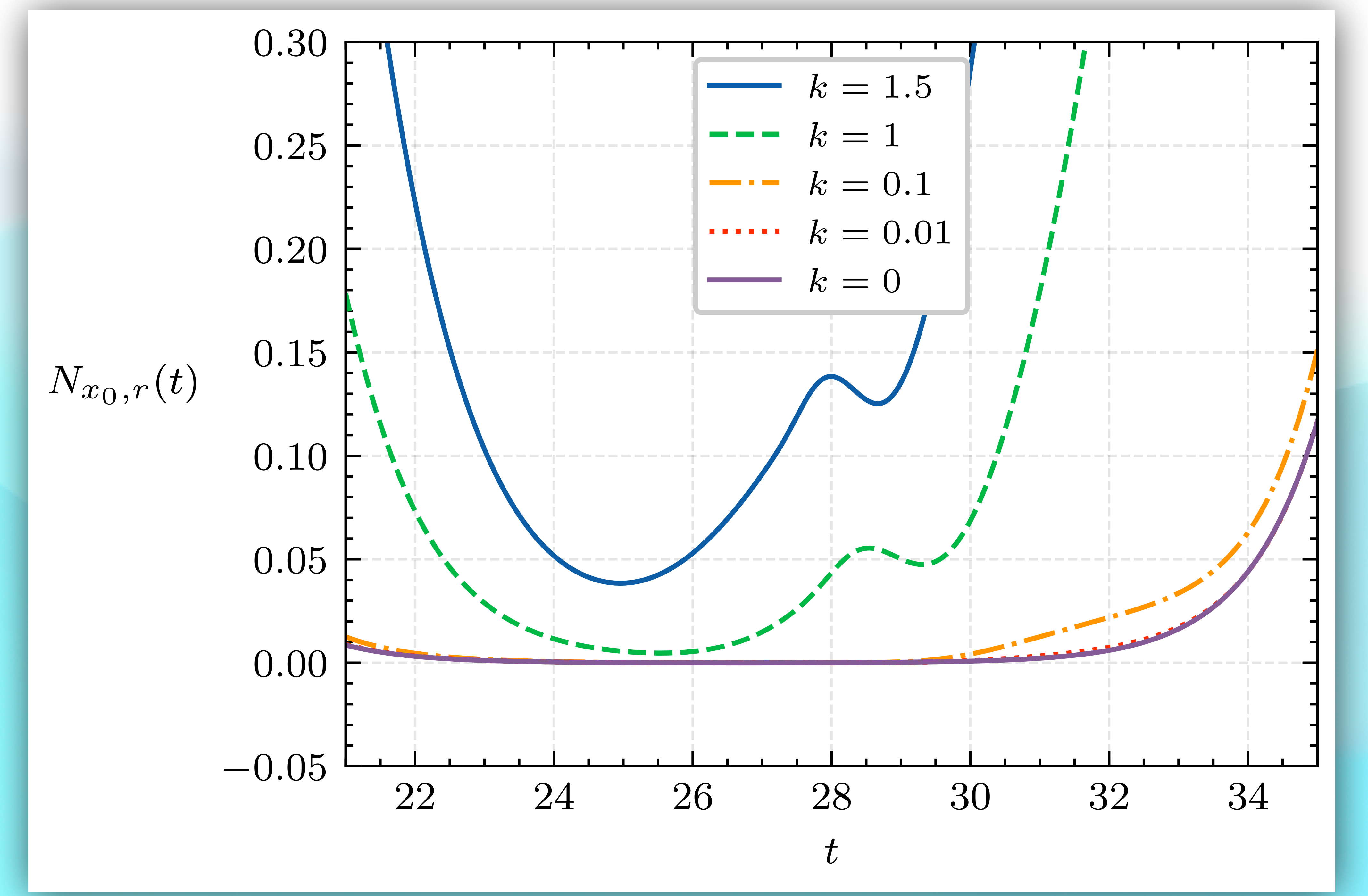
Appendix 5: Dispersion

we notice that in the case $\kappa = 1$ and $\kappa = 1.5$, a little dispersion appears from the moment of collision. To better visualize this dispersion, we compute the local mass:

$$N^n_{x_0,r} = \int_{x_0-r}^{x_0+r} |\Psi^n|^2$$

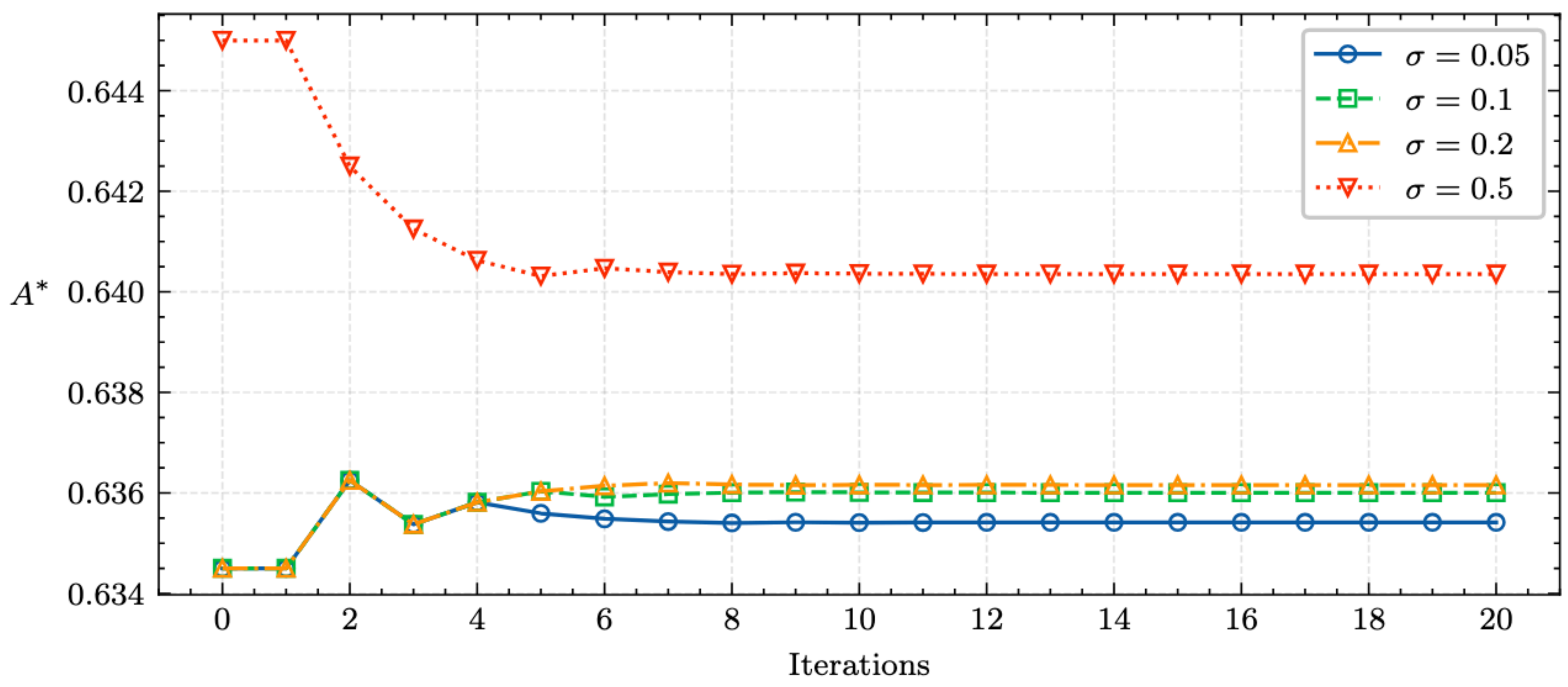
in a (static) window surrounding the right soliton at time $t = 0$, with center x_0 and radius r.





Appendix 6: Blow up - Critial amplitude

From simulations, we can conjecture the existence of a critical amplitude value A^*. We approximate this value by doing a bisection method for different values of σ's.



σ	A^*
0.05	0.63541
0.1	0.63600
0.2	0.63616
0.5	0.64035

Moreover, we can conjecture that A^* depends on σ.

References

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