VERTICAL MOTION OF A FLOATING CYLINDER IN THE BOUSSINESQ REGIME

CANUM 2026

Ewan Contentin

IRMAR, ENS Lyon

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Joint work with: G. Beck, L. Martaud

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1. **Part I:** Modelisation of the problem

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    - Context & Notations
    - Modelisation & reformulation of the problem

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   - Spectral scheme
   - Comparaison with the first method

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PART I:
Context & Notations

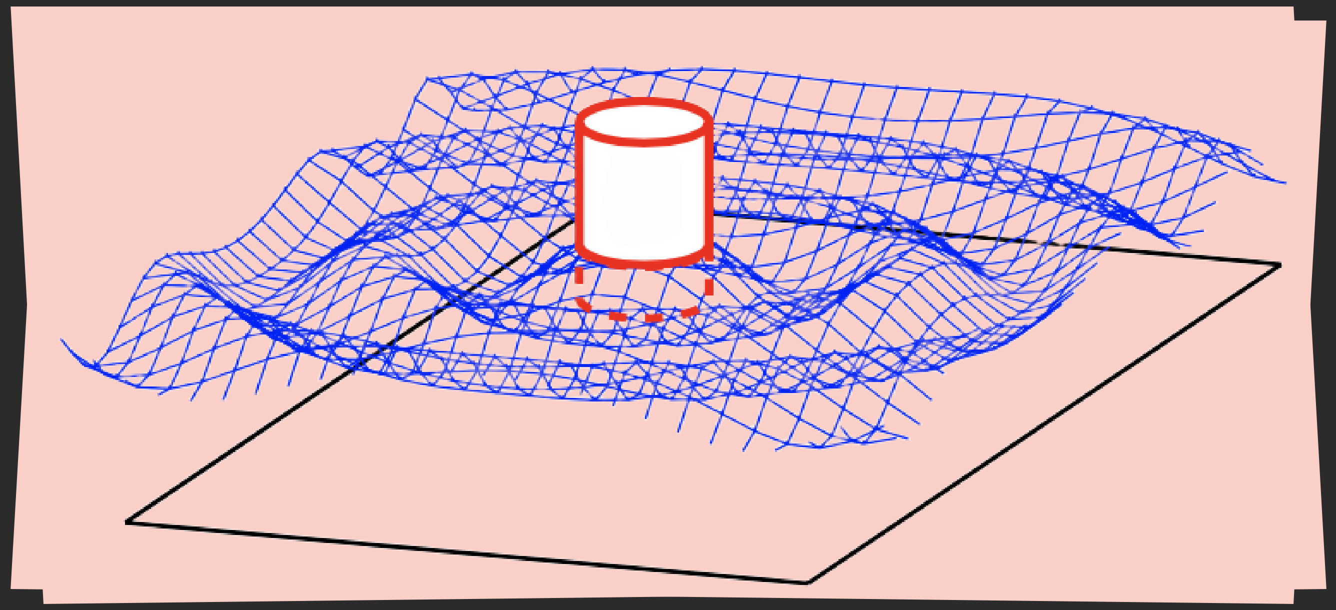

Figure - Freely floating cylinder

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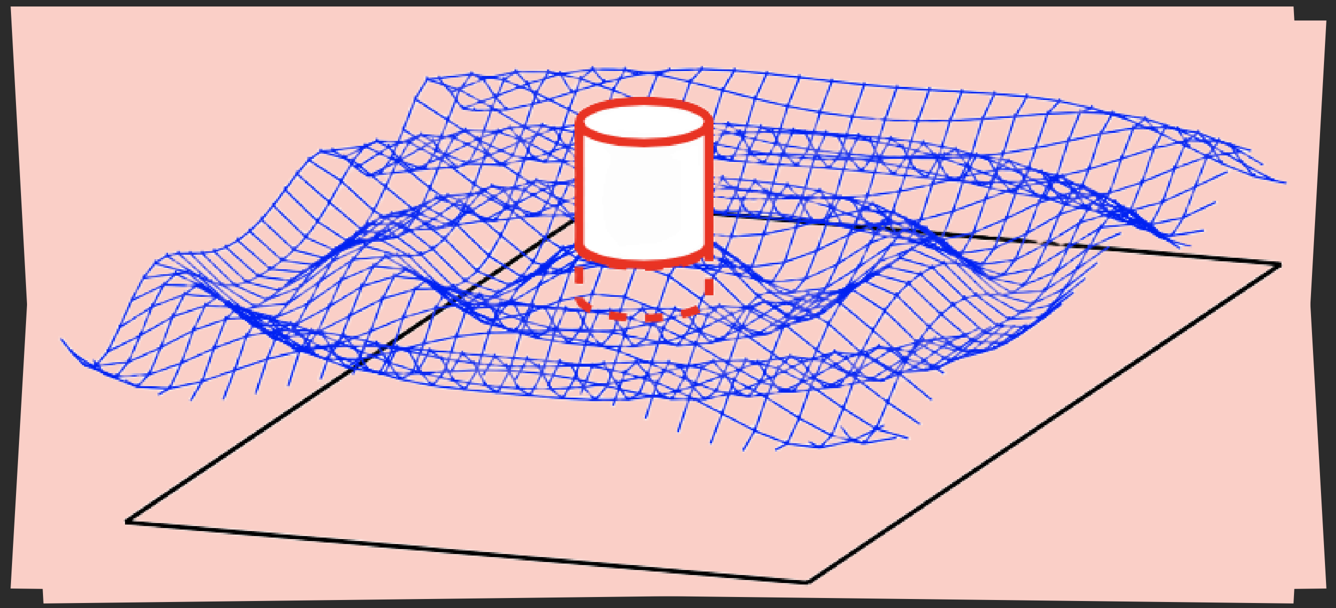

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- Cylinder of radius $R > 0$,

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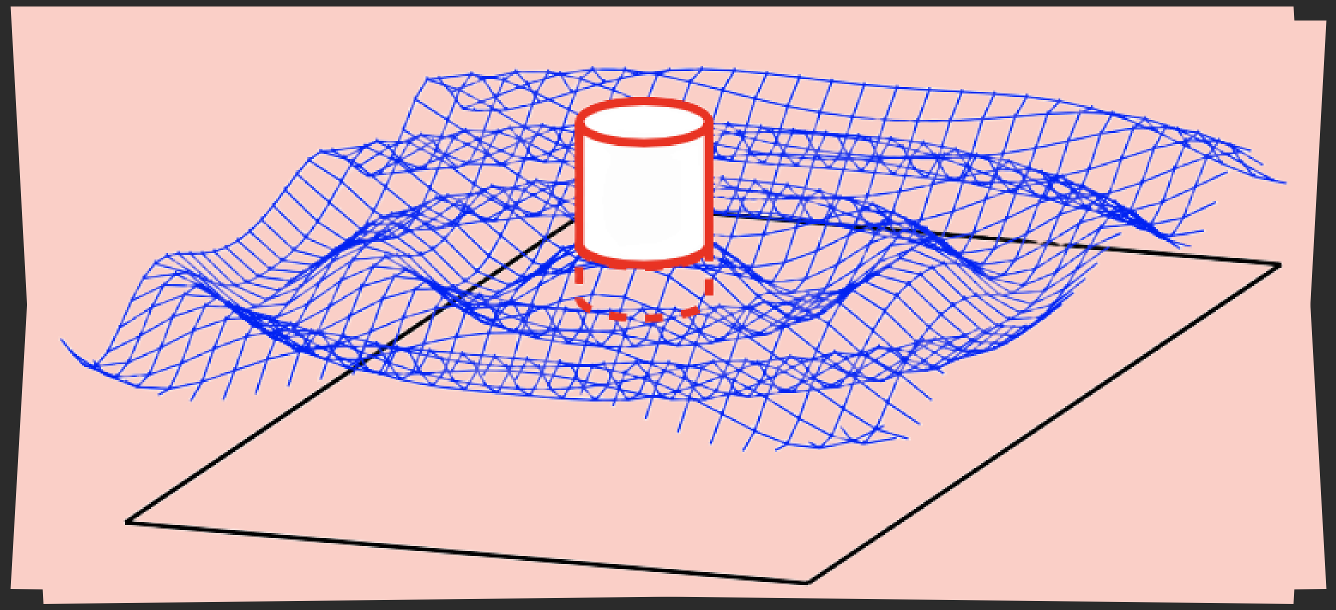

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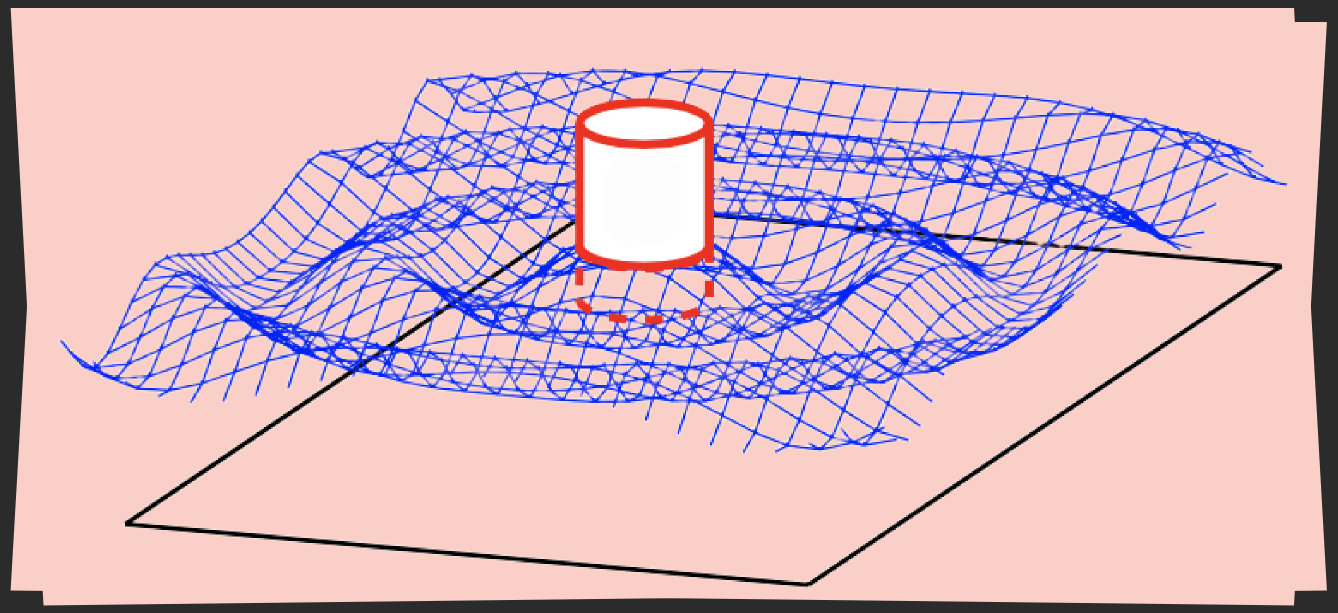

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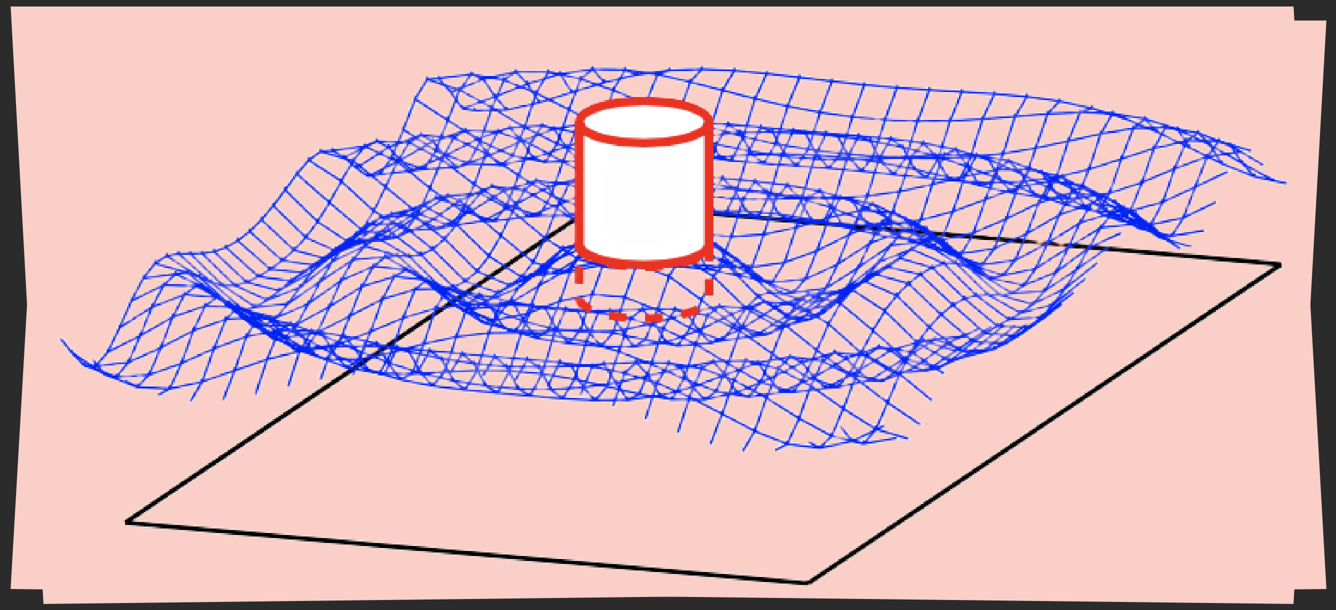

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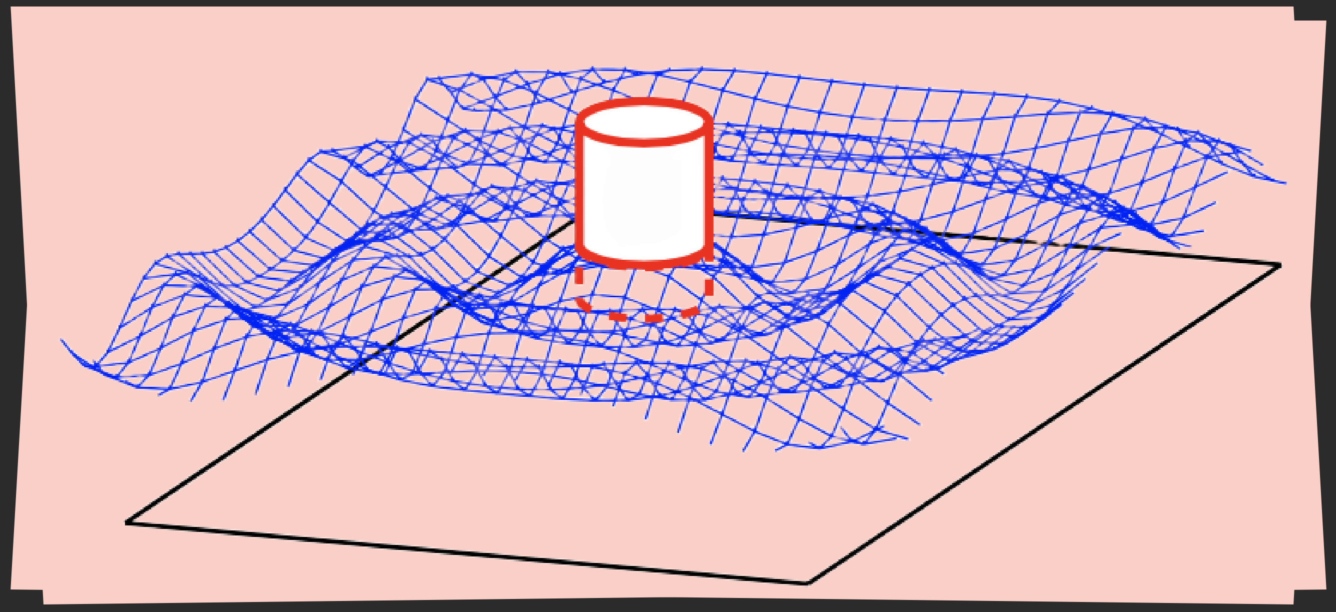

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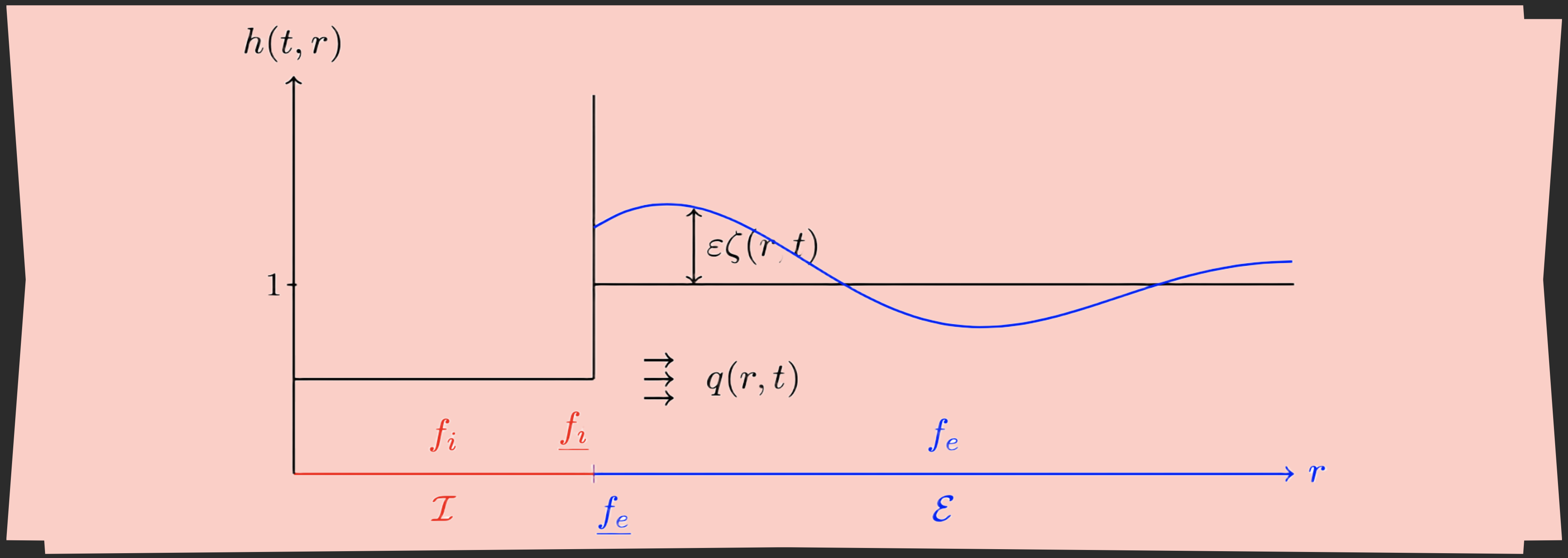

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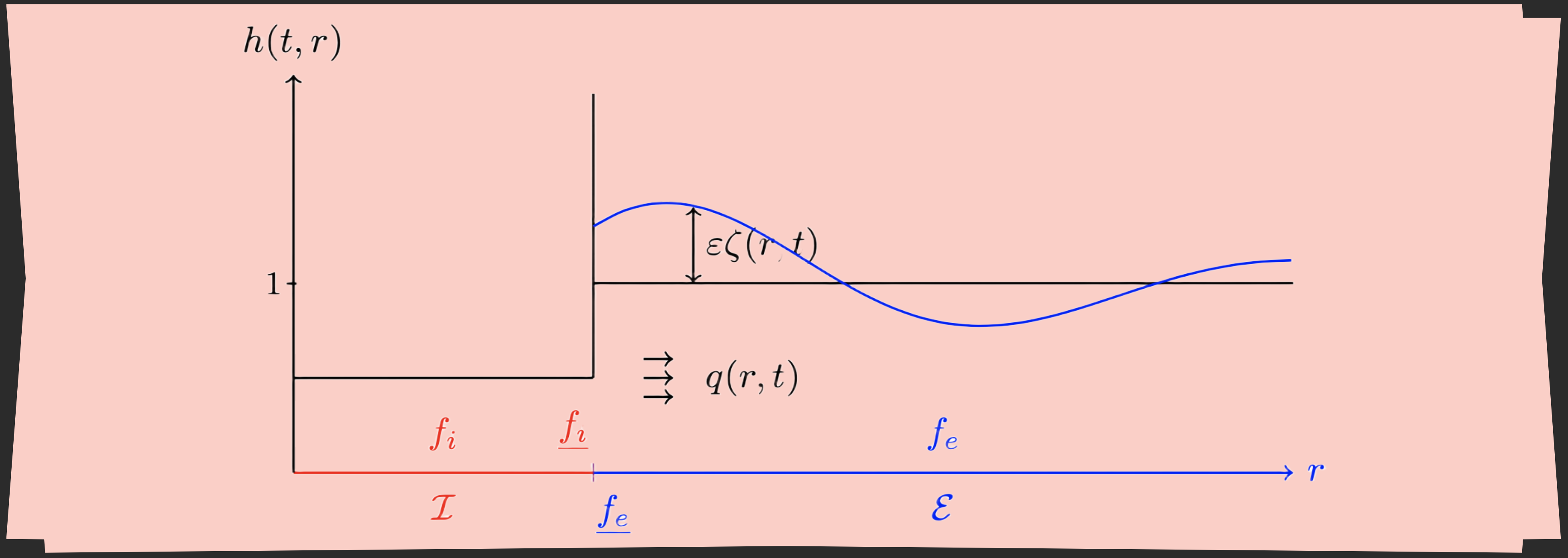

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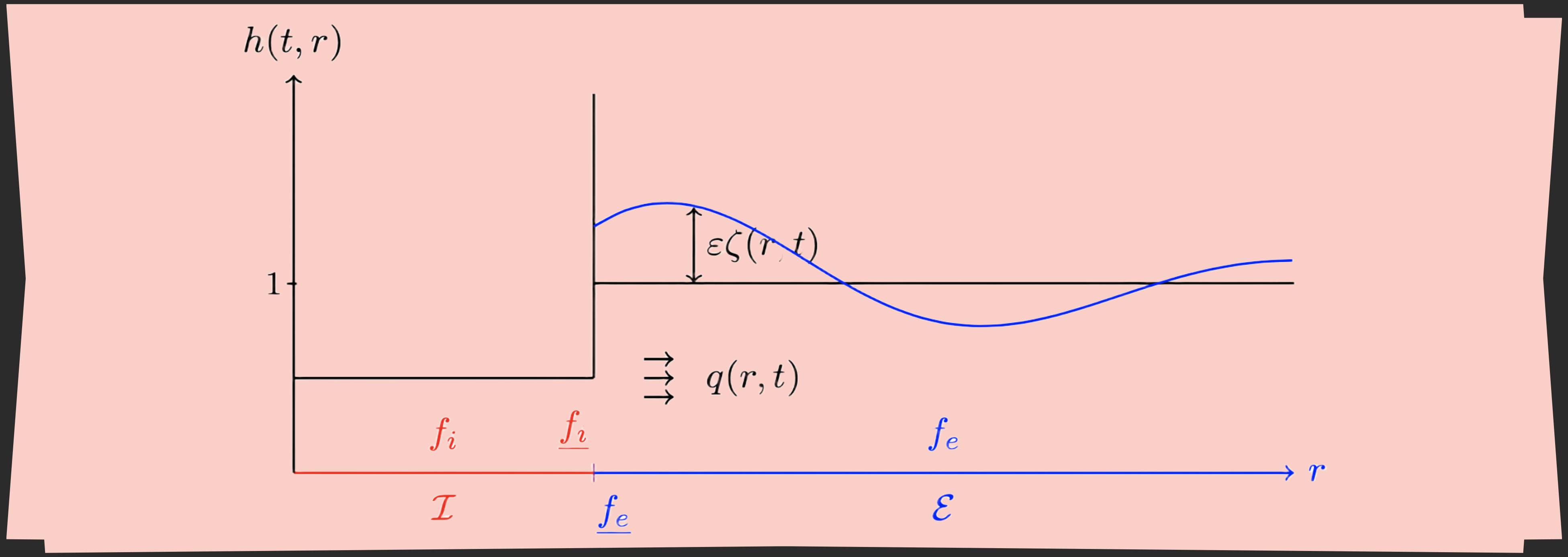
- $\kappa = \dfrac{\text{typ. depth}}{\text{typ. horiz. scale}}$



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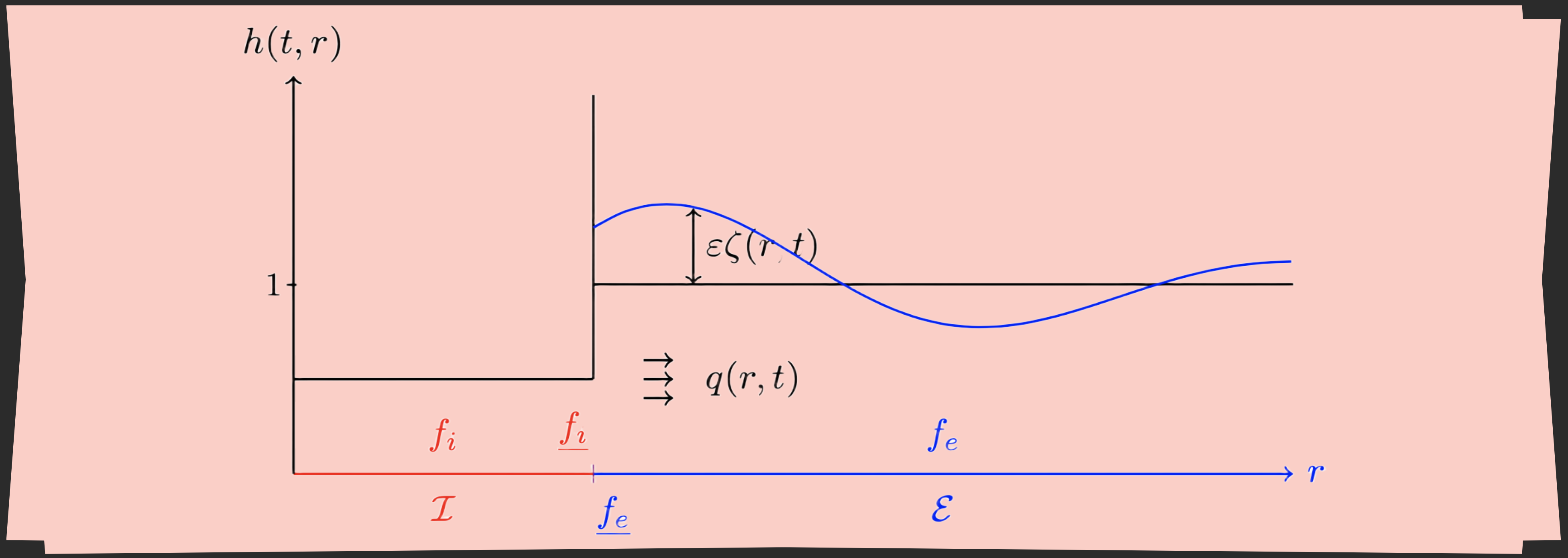

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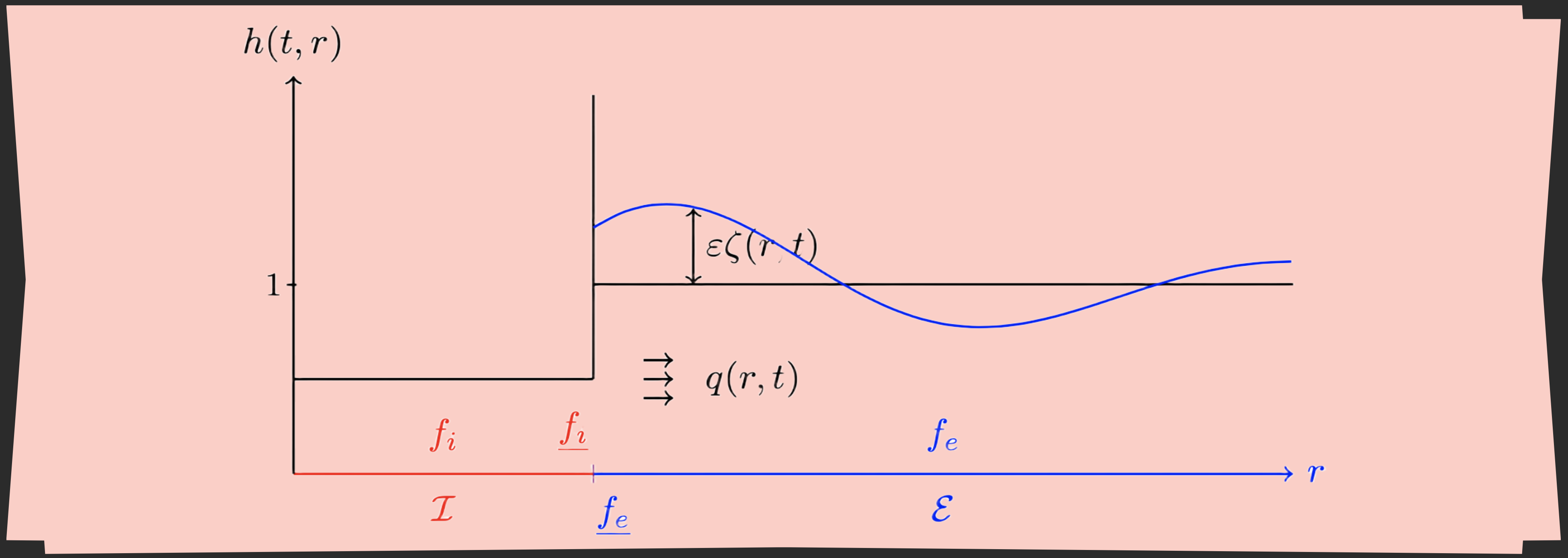

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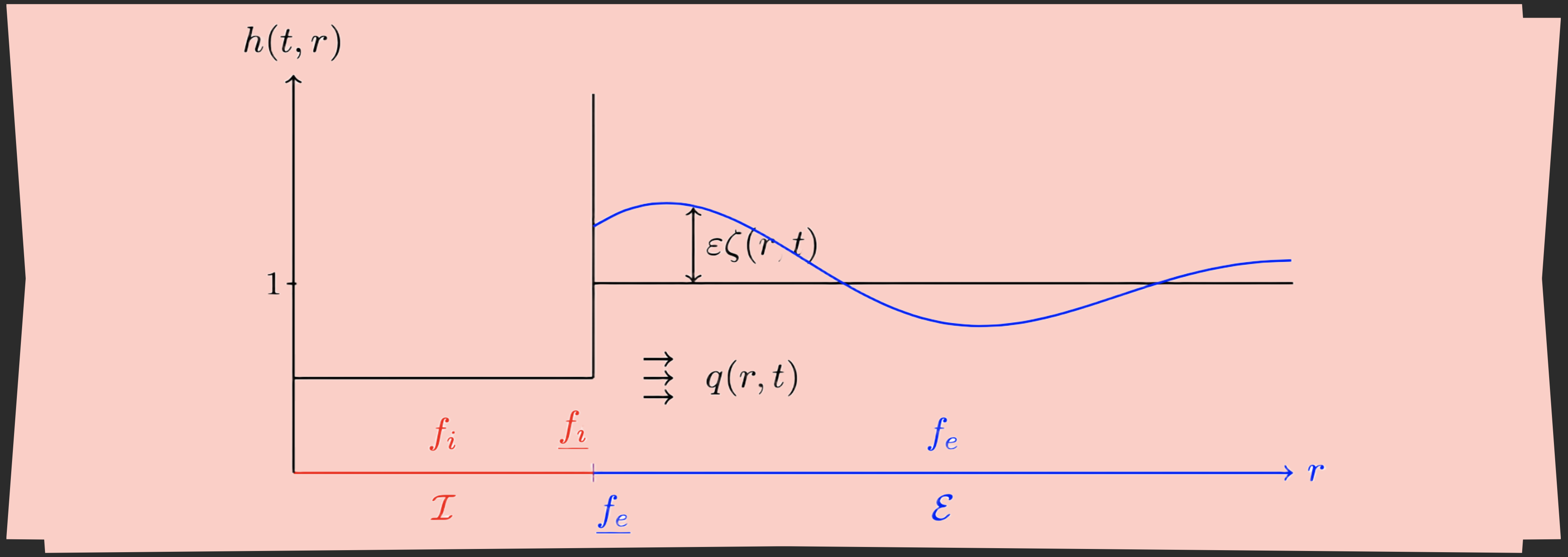

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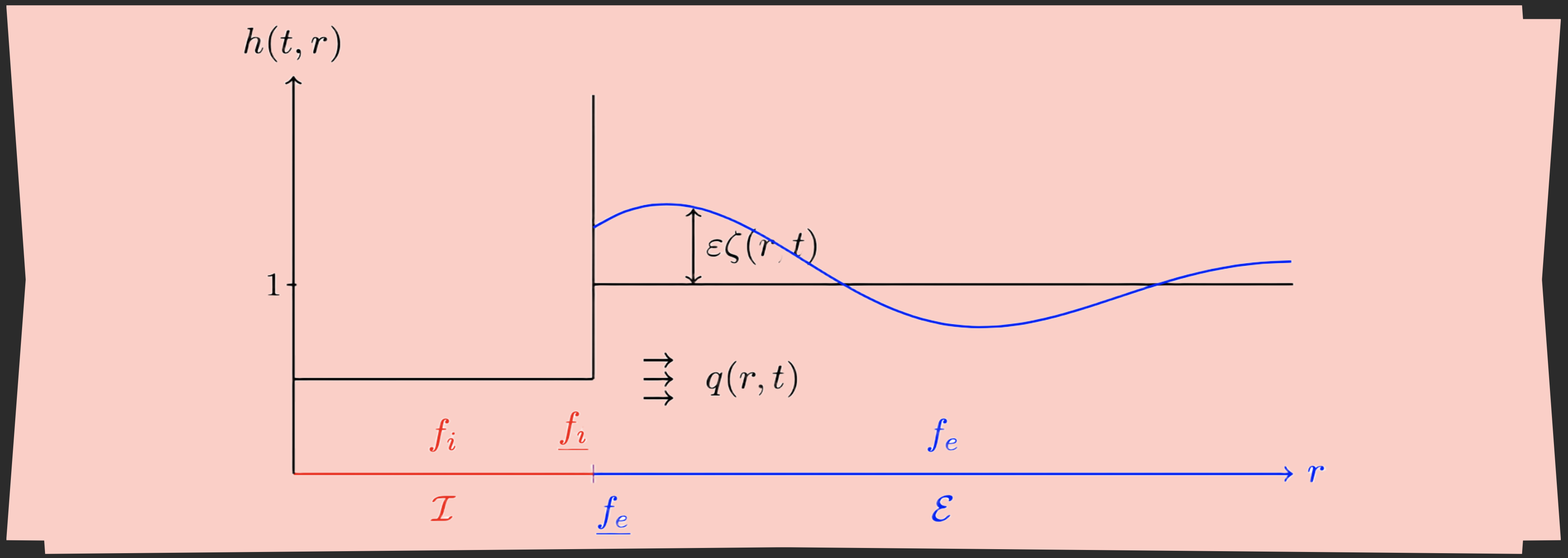

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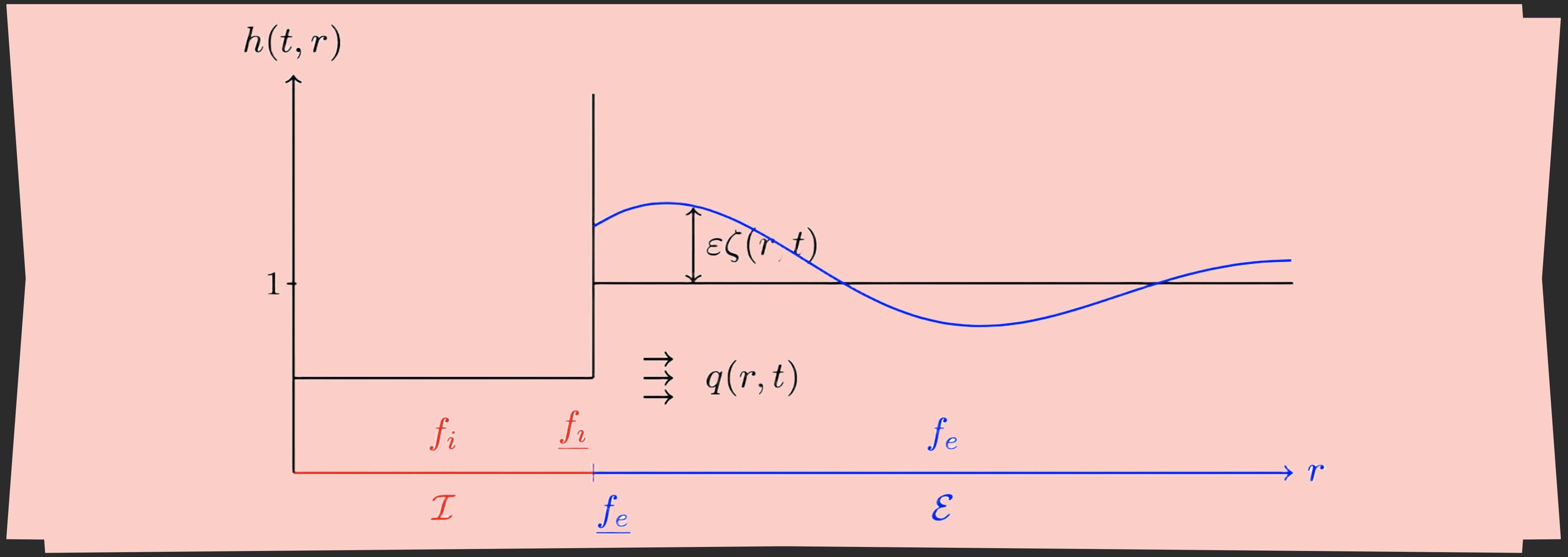

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Remarks:

- In $\mathcal{I}$, $\dot{\zeta}_i = \dot{\delta}$ and P_i is unknown
- In $\mathcal{E}$, $P_e = P_{atm}$ and ζ_e is unknown

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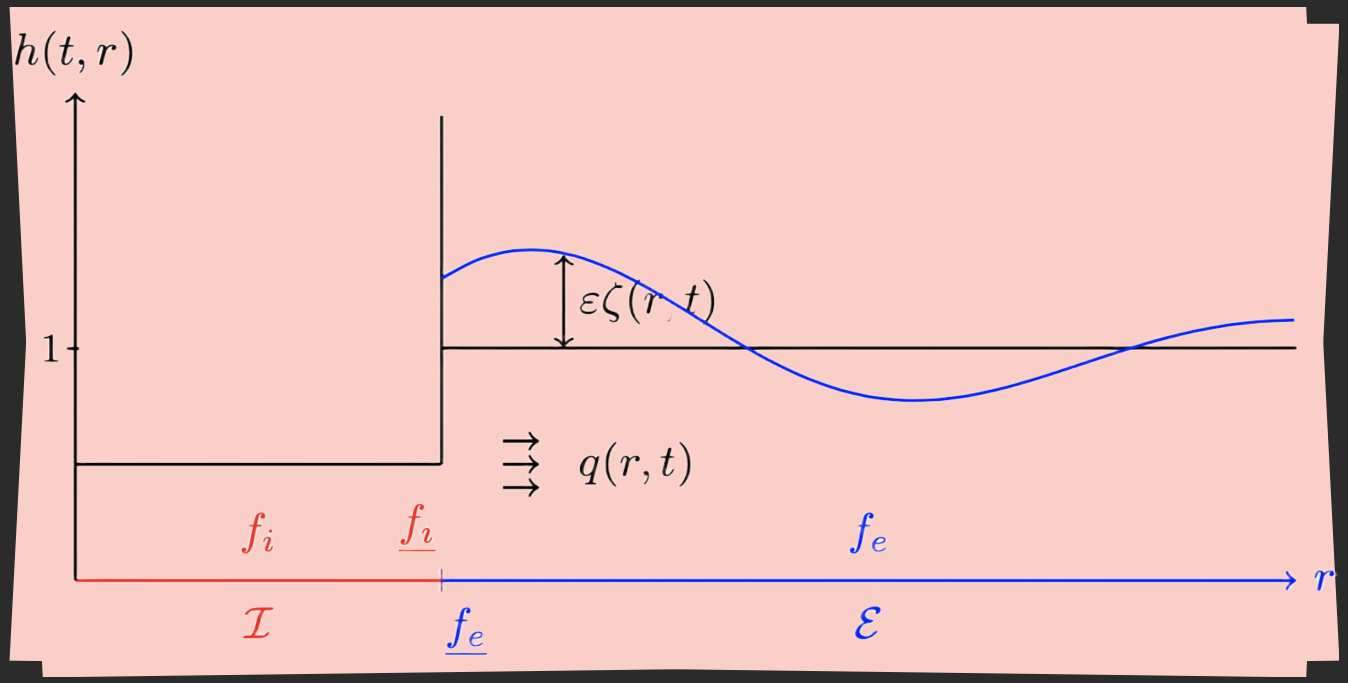
Modelisation



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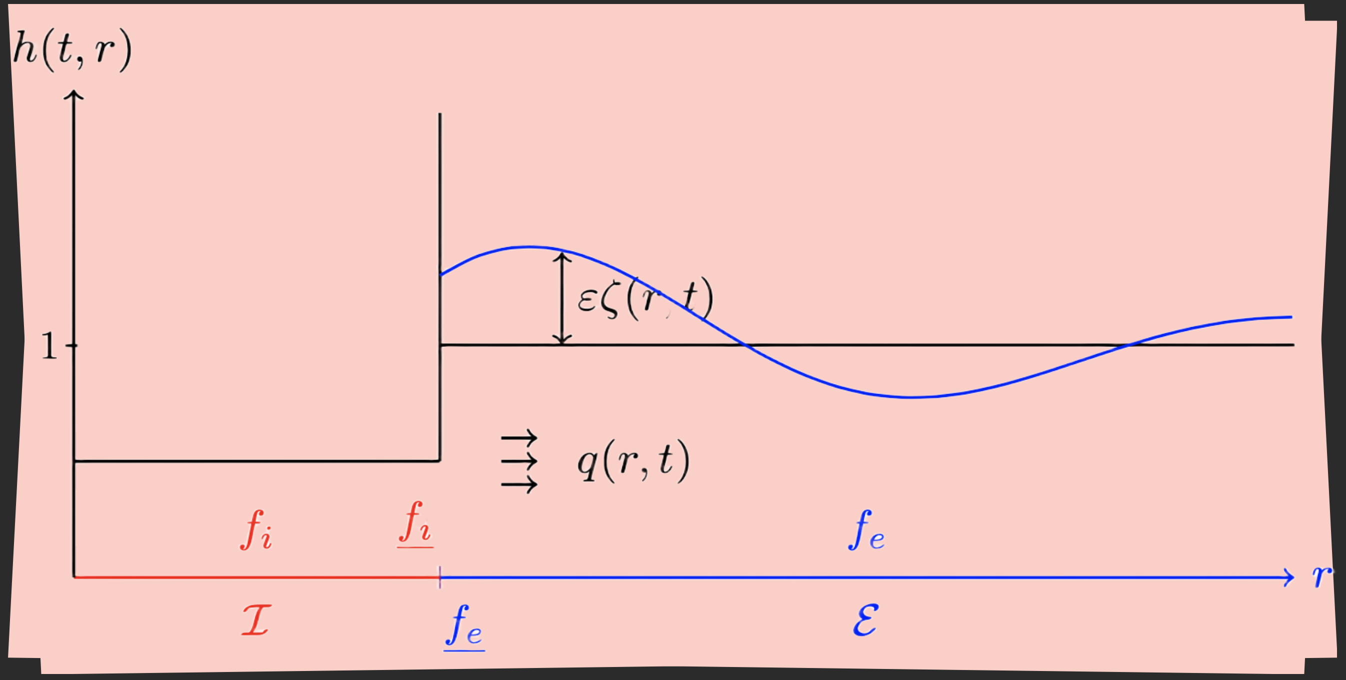

Boussinesq-Abbott equations:

$$\begin{cases} \partial_t \zeta + \mathrm{div} q = 0, \\ (1 - \kappa^2 \partial_r \mathrm{d}_r)\partial_t q + \partial_r \zeta = -\partial_r P, \\ \mathrm{d}_r = \partial_r + \frac{d-1}{r}. \end{cases}$$

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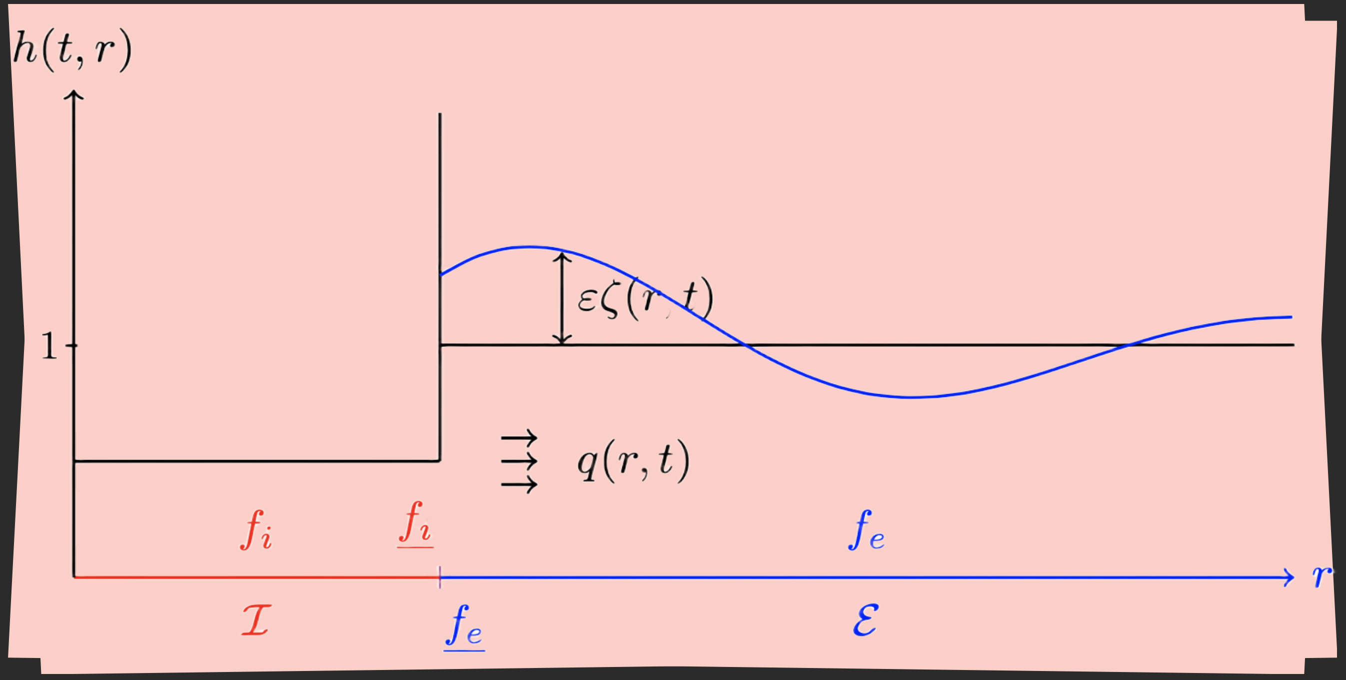

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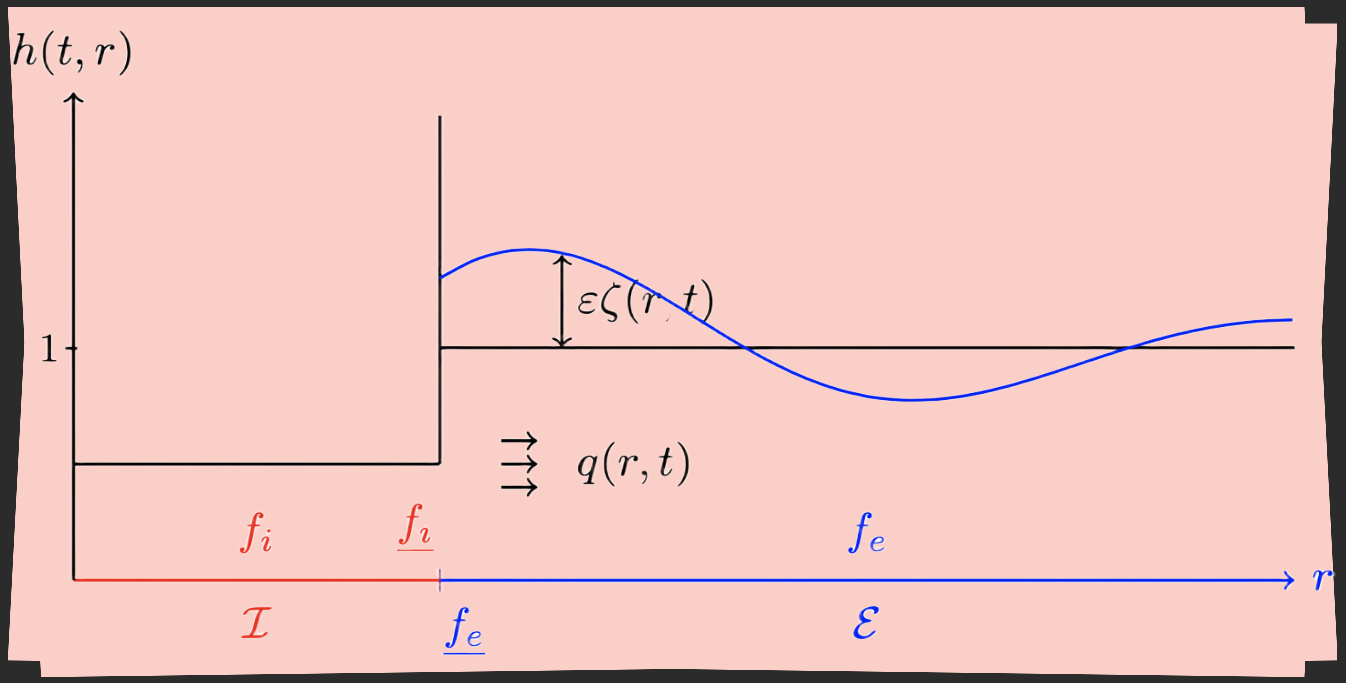

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  - $\underline{f_e} = \lim_{\rho \to R} f(r+\rho)$

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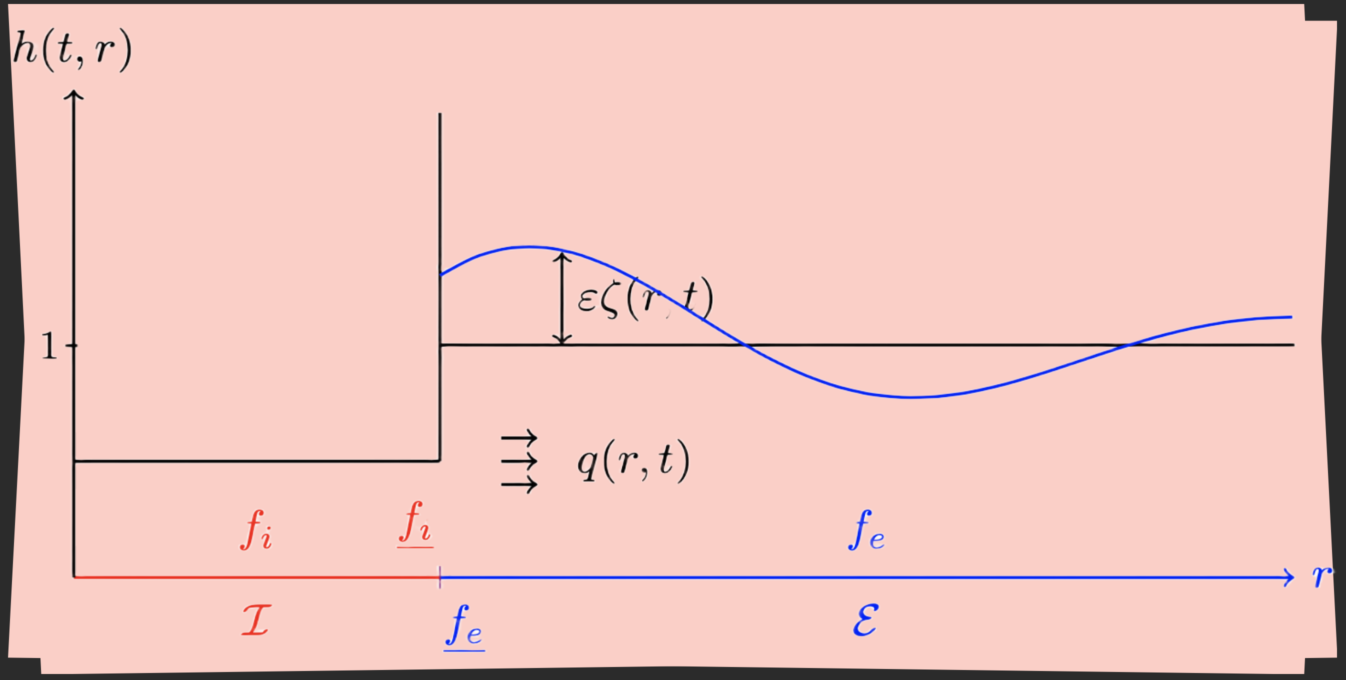

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  - $\underline{P_i} = \mathcal{L}(\zeta, q, \delta)$

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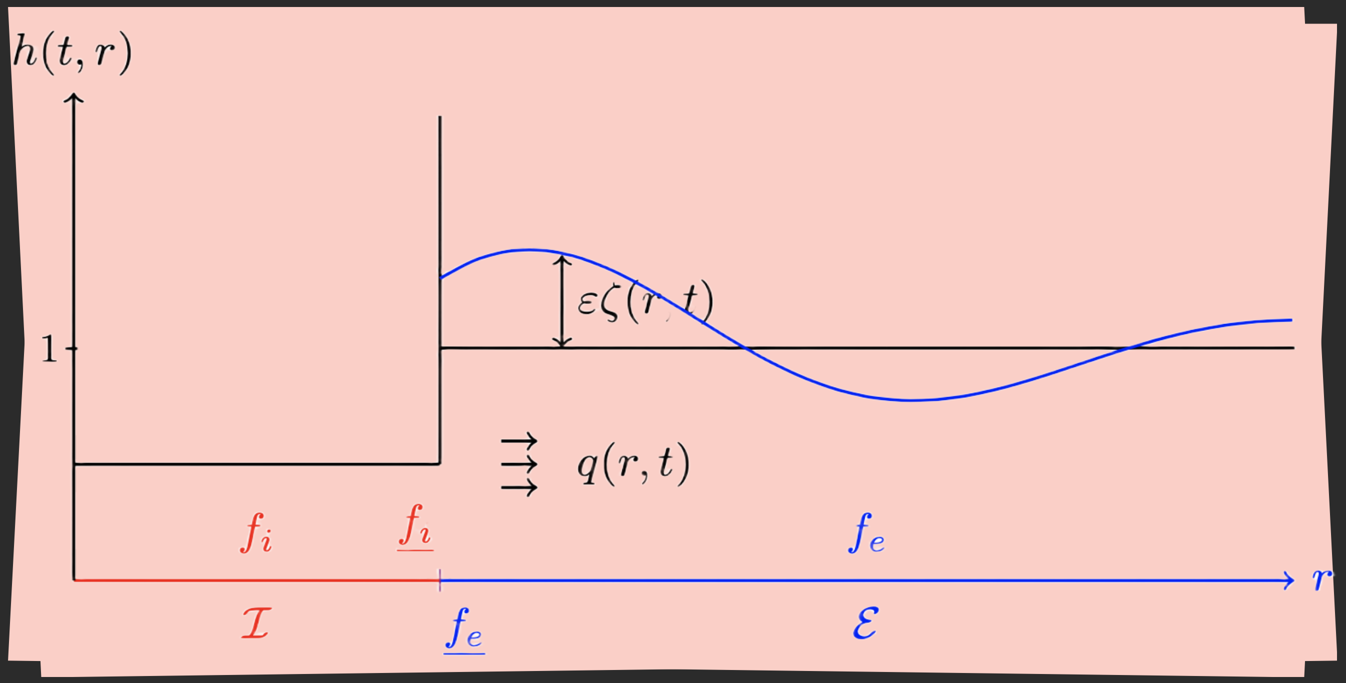

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  - Since $\dot{\zeta_i} = \dot{\delta}$, then $\underline{q_e} = -\frac{R}{2}\dot{\delta}$

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Reformulation of the problem

THEOREM 1 Reformulation

PDE system on $\mathbb{R}^+$ + Transmission conditions

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Augmented formula: PDE system on $(R, +\infty)$ + ODE at the boundary

$$\begin{cases} \partial_t \zeta + \partial_r q = -\frac{q}{r}, \\ \partial_t q + \partial_r \mathcal{R}\zeta = -\ddot{\delta} K(r/\kappa), \quad r > R,\ t > 0 \\ \underline{q} = -R\dot{\delta}, \end{cases} \quad + \quad \begin{cases} \kappa^2 \underline{\ddot{\zeta}} + \underline{\zeta} = \underline{\mathcal{R}\zeta} - G(R/\kappa)\ddot{\delta}, \\ \tilde{m}\ddot{\delta} + \delta = \kappa^2 \underline{\ddot{\zeta}} + \underline{\zeta}, \end{cases} \quad t > 0.$$

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$$\Rightarrow \begin{cases} \partial_t\zeta + \mathrm{div} q = 0, \quad r > R \\ \kappa^2\ddot{\zeta} + \zeta = \mathfrak{R}_1\zeta - \frac{R}{2}\ddot{\delta}G(r/\kappa), \quad r \geq R \\ \tilde{m}\ddot{\delta} + \delta = \kappa^2\underline{\ddot{\zeta}} + \underline{\zeta}, \\ \underline{q} = -\frac{R}{2}\dot{\delta}. \end{cases}$$

THEORICAL RESULTS

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We define the space H_r^1 as the functionnal space on $(R, +\infty)$ equipped with the norm

$$||\cdot||^2_{H_1^r} = ||\cdot||^2_{L_r^2} + ||\text{div}\cdot||^2_{L_r^2} + \kappa^2||\partial_r \text{div}\cdot||^2_{L_r^2}$$

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The augmented formula (PDE on (ζ, q) $(R, +\infty)$ + ODE on $(\delta, \dot{\delta}, \underline{\zeta}, \dot{\underline{\zeta}})$ at $r = R$) is well-posed in $H_r \times H_r^1 \times \mathbb{R}^4$ for time order $T_{\max} = O(\min(\varepsilon^{-1}\kappa, \varepsilon^{-1/2})$.

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THEOREM 3 Decay at ∞

$$\delta \in H^2(\mathbb{R}^+) \quad \text{but} \quad \forall \beta \in (0,2),\ \delta, \dot{\delta}, \ddot{\delta} \notin L^2(t^\beta dt, \mathbb{R}^+).$$

Also, for large time scales:

$$\forall k \in \{0,1,2\},\ \forall 0 < \rho \ll 1, t^{-1/2-\rho} \lesssim |\delta^{(k)}(t)|$$

PART II
Numerical simulations

The augmented formula reads:

$$\begin{cases} \partial_t \zeta + \mathrm{div} q = 0, \quad r > R \\ \kappa^2 \ddot{\zeta} + \zeta = \mathcal{R}\zeta - R\kappa\ddot{\delta} G(r/\kappa), \quad r \geq R \\ \tilde{m}\ddot{\delta} + \delta = \kappa^2 \underline{\ddot{\zeta}} + \underline{\zeta}, \\ \underline{q} = -R\dot{\delta}, \end{cases} \quad \text{with} \quad \begin{cases} (1 - \kappa^2 \mathrm{d}_r \partial_r)\mathcal{R}u = u, \\ \underline{\partial_r \mathcal{R}u} = 0. \end{cases}$$

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- Computation of $\mathcal{R}\zeta$ (depending on the dimension)

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$$\begin{cases} \partial_t \zeta + \mathrm{div} q = 0, \quad r > R \\ \kappa^2 \ddot{\zeta} + \zeta = \mathcal{R}\zeta - R\kappa\ddot{\delta} G(r/\kappa), \quad r \geq R \\ \tilde{m}\ddot{\delta} + \delta = \kappa^2 \underline{\ddot{\zeta}} + \underline{\zeta}, \\ \underline{q} = -R\dot{\delta}, \end{cases} \quad \text{with} \quad \begin{cases} (1 - \kappa^2 \mathrm{d}_r \partial_r)\mathcal{R}u = u, \\ \underline{\partial_r \mathcal{R}u} = 0. \end{cases}$$

Therefore the computation of ζ and q are decoupled:

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- Computation of $\mathcal{R}\zeta$ (depending on the dimension)
- Computation of δ and ζ

PART II
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- Computation of δ and ζ
- Reconstruction of q (Optionnal)

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$$\begin{cases} \kappa^2 \frac{\zeta_i^{n+1} - 2\zeta_i^n + \zeta_i^{n-1}}{\Delta t^2} + (I - \mathfrak{R}_1)\zeta_i^n = M^{-1}((\mathfrak{R}_1 \zeta^n)_0 - \delta^n) G_i, \quad i \in \{1, \dots, N\}, \\ \frac{\delta^{n+1} - 2\delta^n + \delta^{n-1}}{\Delta t^2} = M^{-1}((\mathfrak{R}_1 \zeta^n)_0 - \delta^n), \end{cases}$$

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THEOREM 3 Energy inequality on the scheme

Let $\kappa > 0$ and $\Delta t \lesssim \kappa$, $\Delta x \lesssim \kappa$. If $\mathcal{R}$ satisfies $||\mathcal{R}u||_{h^1_\kappa} < ||u||_{l^2}$ and $|\underline{\mathcal{R}u}| < ||u||_{l^2}$, then the time scheme satisfies the discrete energy inequality:

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with $\mathfrak{E}^{n+1/2} = E_{\text{solid}}^{n+1/2} + E_{\text{fluid}}^{n+1/2}$, $E_{\text{solid}}^{n+1/2} := \frac{(v^{n+1/2})^2}{2} + \frac{M}{2}(\delta^{n+1/2})^2$, where the speed is $v^{n+1/2} := \frac{\delta^{n+1} - \delta^n}{\Delta t}$ and the mean position is $\delta^{n+1/2} := \frac{\delta^{n+1} + \delta^n}{2}$ and $E_{\text{fluid}}^{n+1/2} := \frac{\kappa^2 \|S^n\|_{l^2}^2}{2} + \frac{1}{2}\|\zeta^{n+1} + \zeta^n\|_{l^2}^2$, where $S^n = \frac{\zeta^{n+1} - \zeta^n}{\Delta t}$.

SUMMARY

$$\begin{cases} \kappa^2 \frac{\zeta_i^{n+1} - 2\zeta_i^n + \zeta_i^{n-1}}{\Delta t^2} + (I - \mathcal{R})\zeta_i^n = M^{-1}((\mathcal{R}\zeta^n)_0 - \delta^n)G_i, \quad i \in \{1, \ldots, N\}, \\ \frac{\delta^{n+1} - 2\delta^n + \delta^{n-1}}{\Delta t^2} = M^{-1}((\mathcal{R}\zeta^n)_0 - \delta^n), \end{cases}$$

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Important points:

- Easily adaptable with the dimension,

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$$\Delta t \lesssim \kappa \quad \& \quad \Delta x \lesssim \kappa.$$

NUMERICAL SIMULATIONS

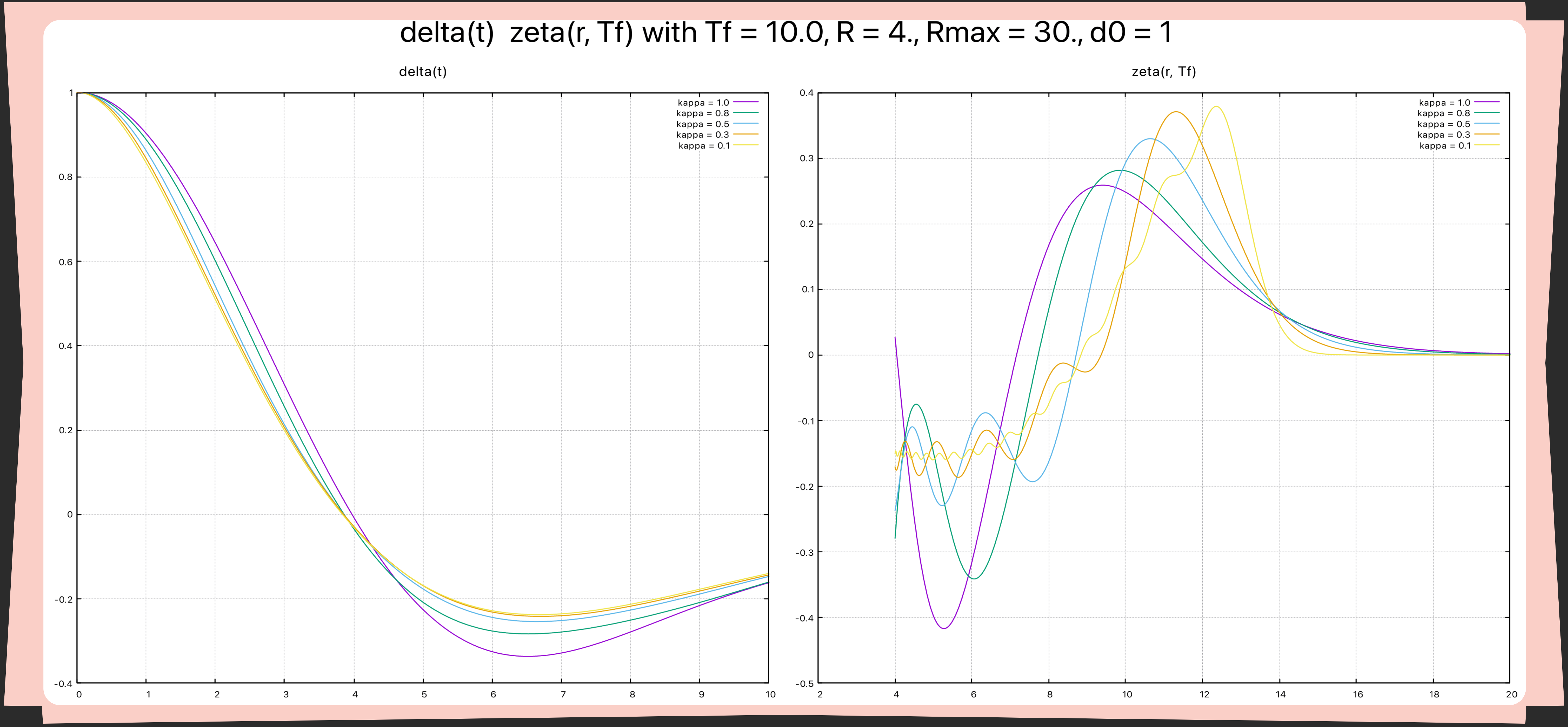

Figure - Simulations with different values of κ

NUMERICAL SIMULATIONS

Small value of κ

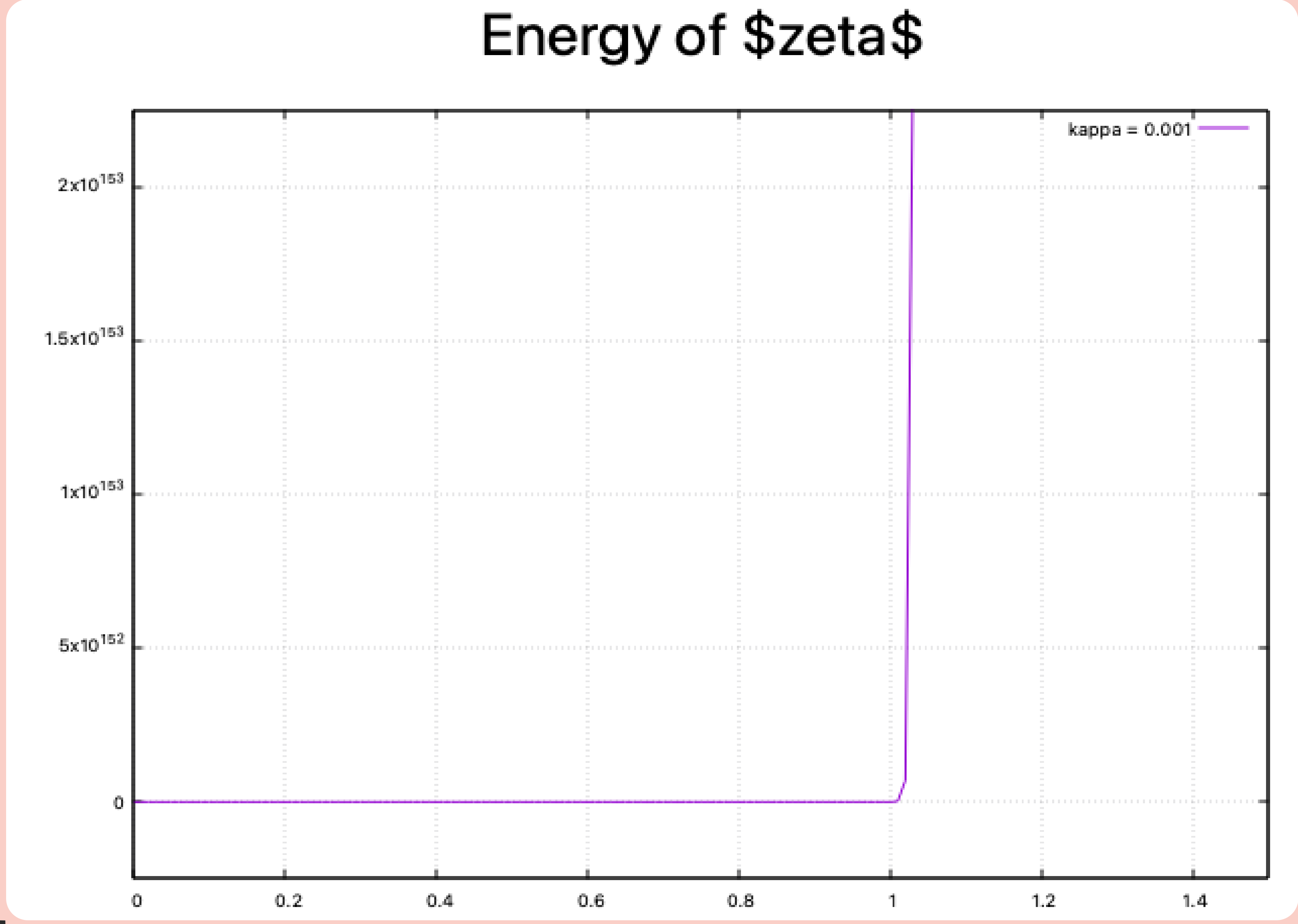

Figure - Simulations with $\kappa = 0.001$

PART III

Overpass the small values of κ in 1D

$$\begin{cases} \kappa^2 \ddot{\zeta} + \zeta = \mathcal{R}\zeta - \frac{R}{2} M^{-1} (\underline{\mathcal{R}\zeta} - \delta) G(r/\kappa), \\ M\ddot{\delta} + \delta = \underline{\mathcal{R}\zeta}. \end{cases}$$

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The Fourier cos transform applied to the equation of ζ:

$$\forall k \in \mathbb{N},\ \kappa^2 \ddot{\hat{\zeta}}_k + \kappa^2 \omega_k^2 \hat{\zeta}_k = c\kappa^2 (\underline{\mathcal{R}\zeta} - \delta) \frac{1}{1 + \kappa^2 k^2}, \quad \omega_k^2 = \frac{k^2}{1 + \kappa^2 k^2}$$

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$$\forall t > 0,\ \underline{\mathcal{R}\zeta} = \underline{\mathcal{R}\zeta}^{\text{homog}} + \int_0^t K(s-t)(\underline{\mathcal{R}\zeta}(s) - \delta(s))ds$$

RESOLUTION

Discretization:

$$\forall k \in \mathbb{N}, \begin{cases} \hat{\zeta}_k^{n+1} = \cos(\omega_k \Delta t)\hat{\zeta}_k^n + \frac{\sin(\omega_k \Delta t)}{\omega_k}\hat{\dot{\zeta}}_k^n + \frac{c}{1+\kappa^2 k^2}(\underline{\mathcal{R}\zeta}^n - \delta^n)\frac{1-\cos(\omega_k \Delta t)}{\omega_k^2}, \\ \hat{\dot{\zeta}}_k^{n+1} = -\omega_k \sin(\omega_k \Delta t)\hat{\zeta}_k^n + \cos(\omega_k \Delta t)\hat{\dot{\zeta}}_k^n - \frac{c}{1+\kappa^2 k^2}(\underline{\mathcal{R}\zeta}^n - \delta^n)\frac{\sin(\omega_k \Delta t)}{\omega_k}. \end{cases}$$

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$$\begin{pmatrix} \delta \\ \dot{\delta} \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & \Delta t \\ -\Delta t M^{-1} & 1 \end{pmatrix} \begin{pmatrix} \delta \\ \dot{\delta} \end{pmatrix}^{n} + \Delta t \begin{pmatrix} 0 \\ M^{-1}\underline{\mathcal{R}\zeta}^n \end{pmatrix}.$$

NUMERICAL SIMULATIONS

Comparaison of the 2 methods

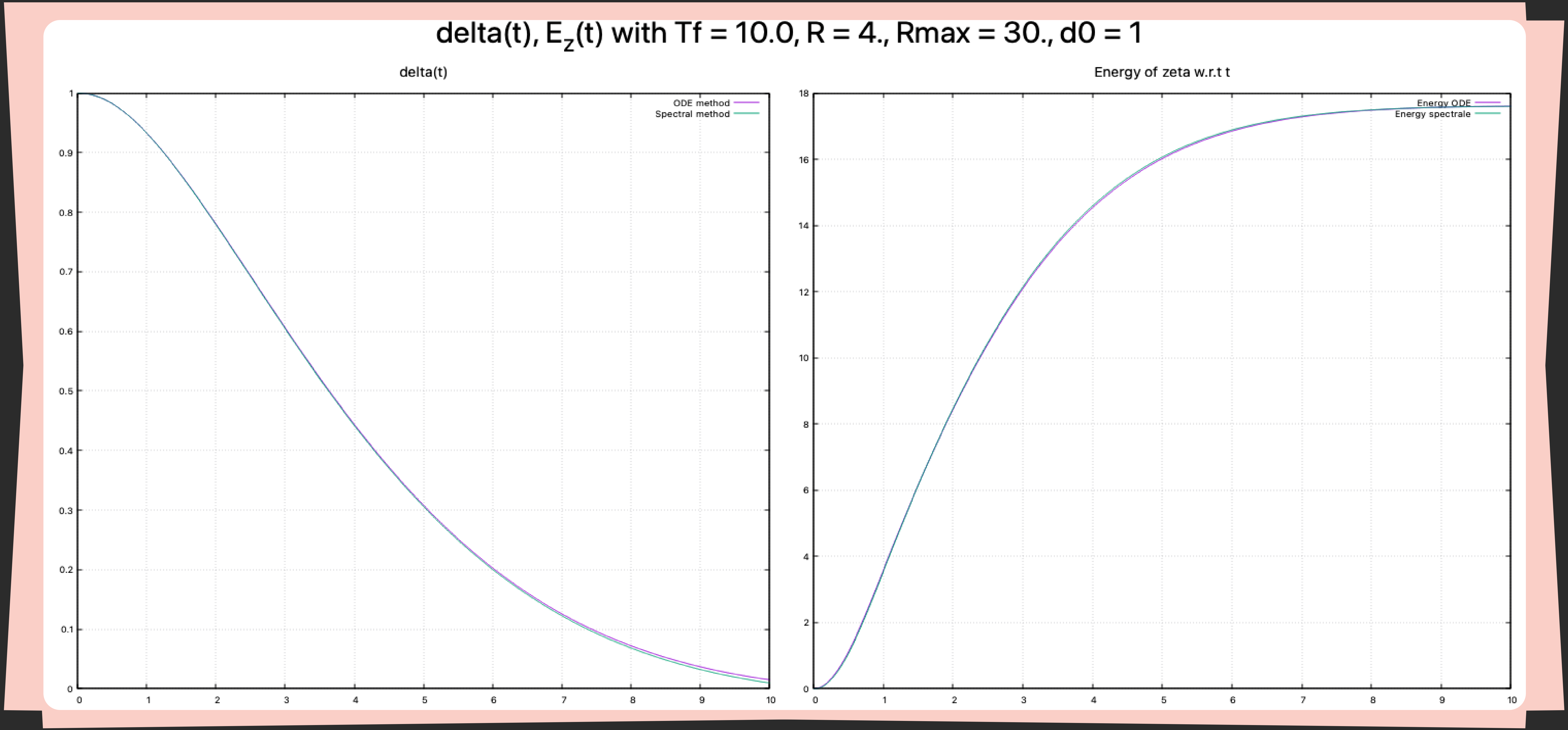

Figure - Comparaison of the methods with $\kappa = 0.3$

NUMERICAL SIMULATIONS

Small value of κ with the spectral method

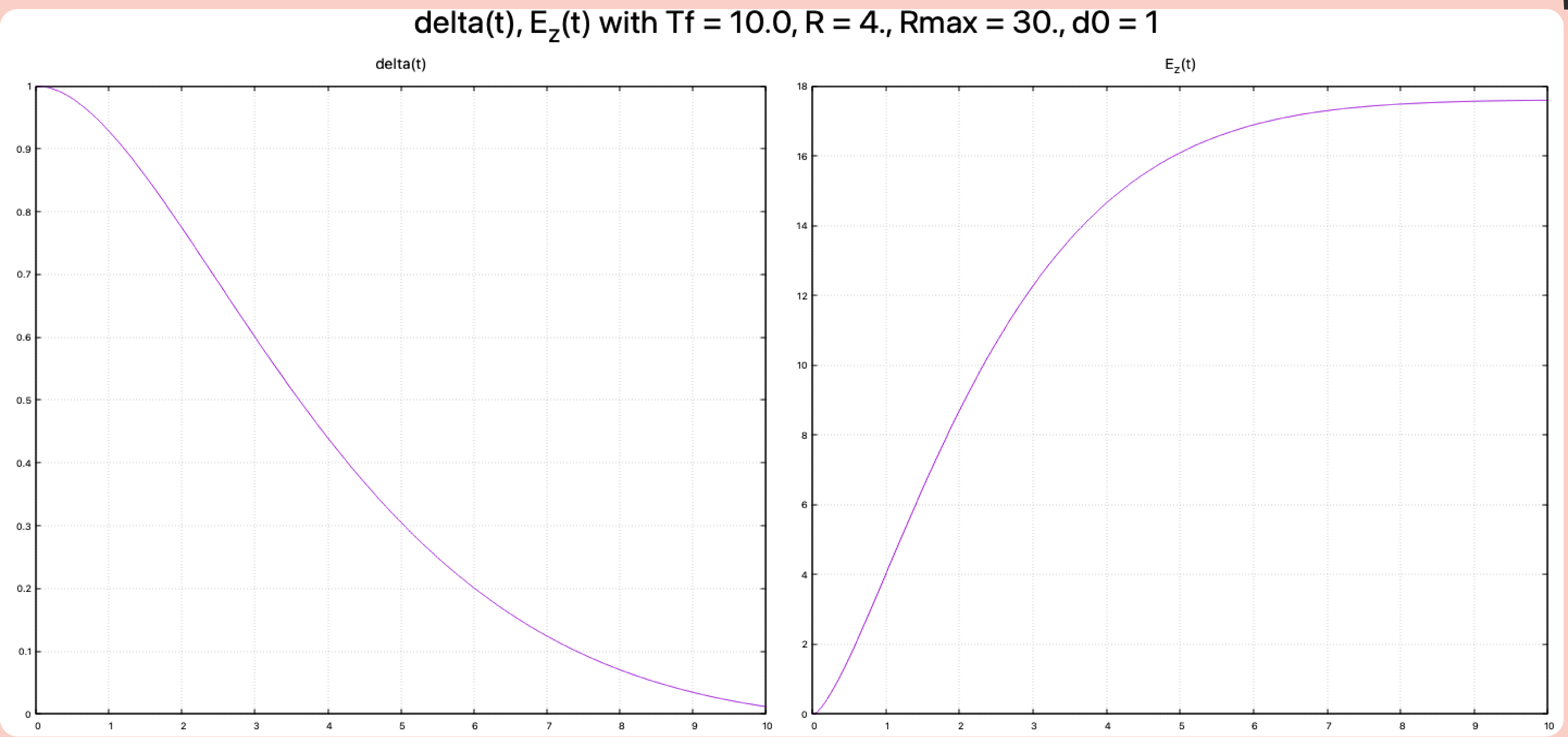

Figure - Spectral method with $\kappa = 10^{-5}$

CONCLUSIONS
& perspectives

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- Non-linear case,

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- Non-linear case,
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REFERENCES

Article: Freely floating cylinder on a 3D fluid governed by the Boussinesq equations in the axisymmetric without swirl case, G.Beck, **E.Contentin**, L.Martaud (2025)

Proceeding: The return to the equilibrium of a floating cylinder in the Boussinesq regime, G.Beck, **E.Contentin**, L.Martaud (2026)

Thank you!

Slides created using reveal.js

Template

Alan Riquier, postdoctoral at Fields Institute.