Diffraction d’ondes par une jonction de guides ouverts. Etude et résolution par la méthode HSM.

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En collaboration avec Anne-Sophie Bonnet-Ben Dhia et Sonia Fliss.

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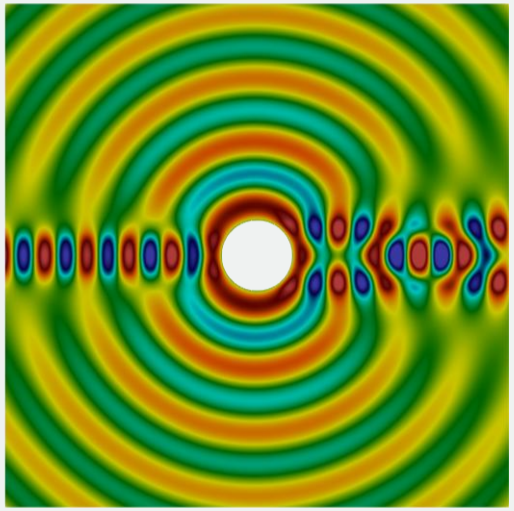
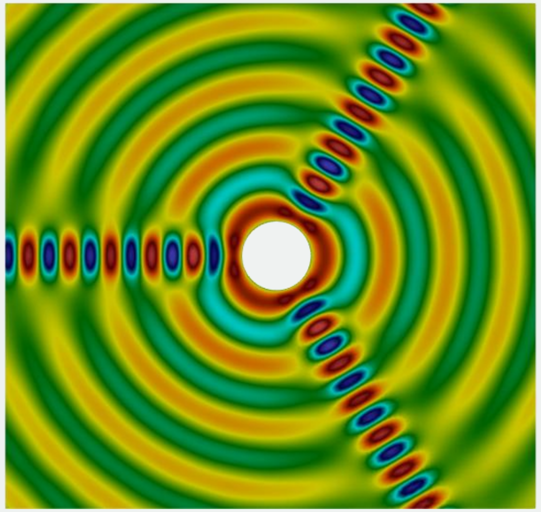
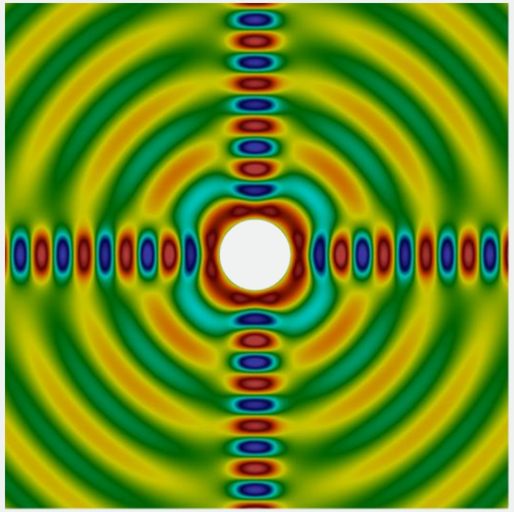
Motivations

We are interested in simulating wave propagation in 2D **unbounded media** which can be described as junctions of stratified media.

More precisely, we are interested in junctions of so-called open waveguides.







Simulations done with XLiFE++

E. Lunéville and N. Kielbasiewicz, xlifepp.pages.math.cnrs.fr

$$\frac{\partial^2 E}{\partial t^2} - c(\mathbf{x})^2 \Delta \vec{E} = f \quad \xrightarrow[f(x,t) = \widehat{f}(x) e^{-i\omega t}]{k(\mathbf{x}) = \frac{\omega}{c(\mathbf{x})}} \quad -\Delta \widehat{E} - k(\mathbf{x})^2 \widehat{E} = \widehat{f}$$

$$E(x,t) = \widehat{E}(x) e^{-i\omega t}$$

The Helmholtz equation in unbounded domains

For some $k \in L^{\infty}(\Omega; \mathbb{R})$, find u such that

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If Ω is *unbounded*, there is in general no solution in $L^2(\Omega)$ (no solution of finite energy in the harmonic regime), but there is infinitely many solutions in $L^2_{loc}(\Omega)$. One needs to prescribe the behaviour of u at infinity to recover the uniqueness of the solution.

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With $\Omega = \mathbb{R}^2$:

- If k is constant, the physical solution satisfies the Sommerfeld radiation condition.

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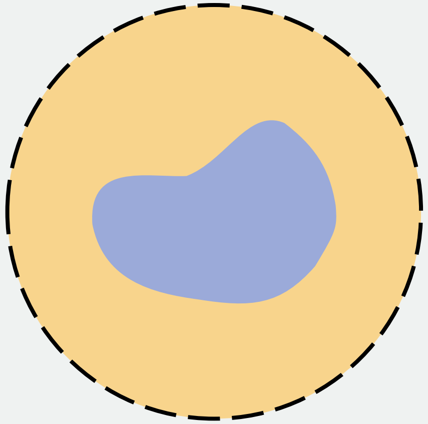
- If k describes a junction of open waveguides, the behaviour at infinity is much more complex and we cannot expect to characterize the solution by similar conditions.

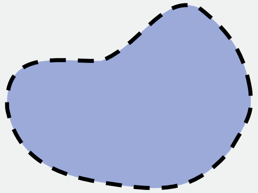
Existing methods

Transparent boundary conditions (DtN…)



Cannot be extended due to the same difficulties as for the radiation conditions.

Integral equations



Green's function cannot be computed directly.

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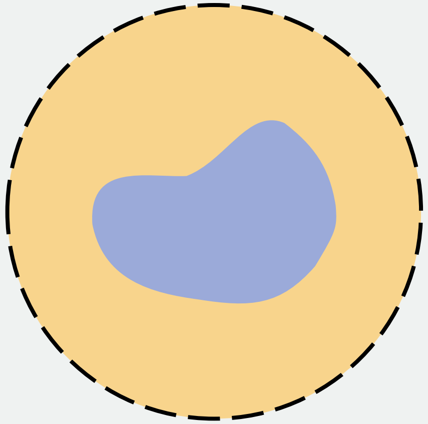

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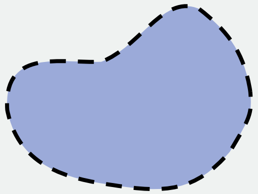

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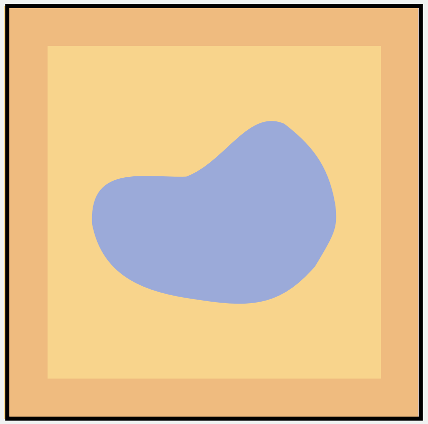
Perfectly Matched Layers



Seem to work with no rigorous justification.

Alternative approach: limiting absorption principle

Consider the Helmholtz equation with dissipation $\varepsilon > 0$

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It has a unique solution in $H^1(\mathbb{R}^2)$.

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Then, the aim is threefold:

Propose a formulation of the problem in presence of dissipation **adapted** to these complex media which **passes to the limit** and leads to a **numerical method**.

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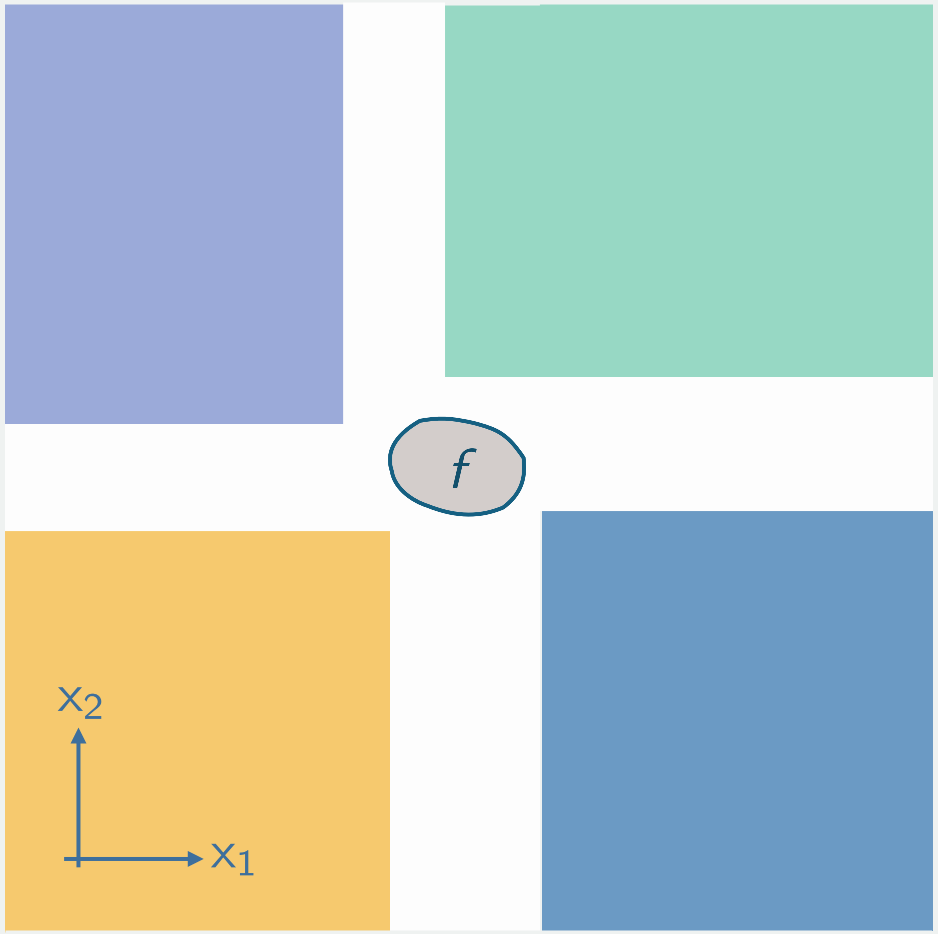

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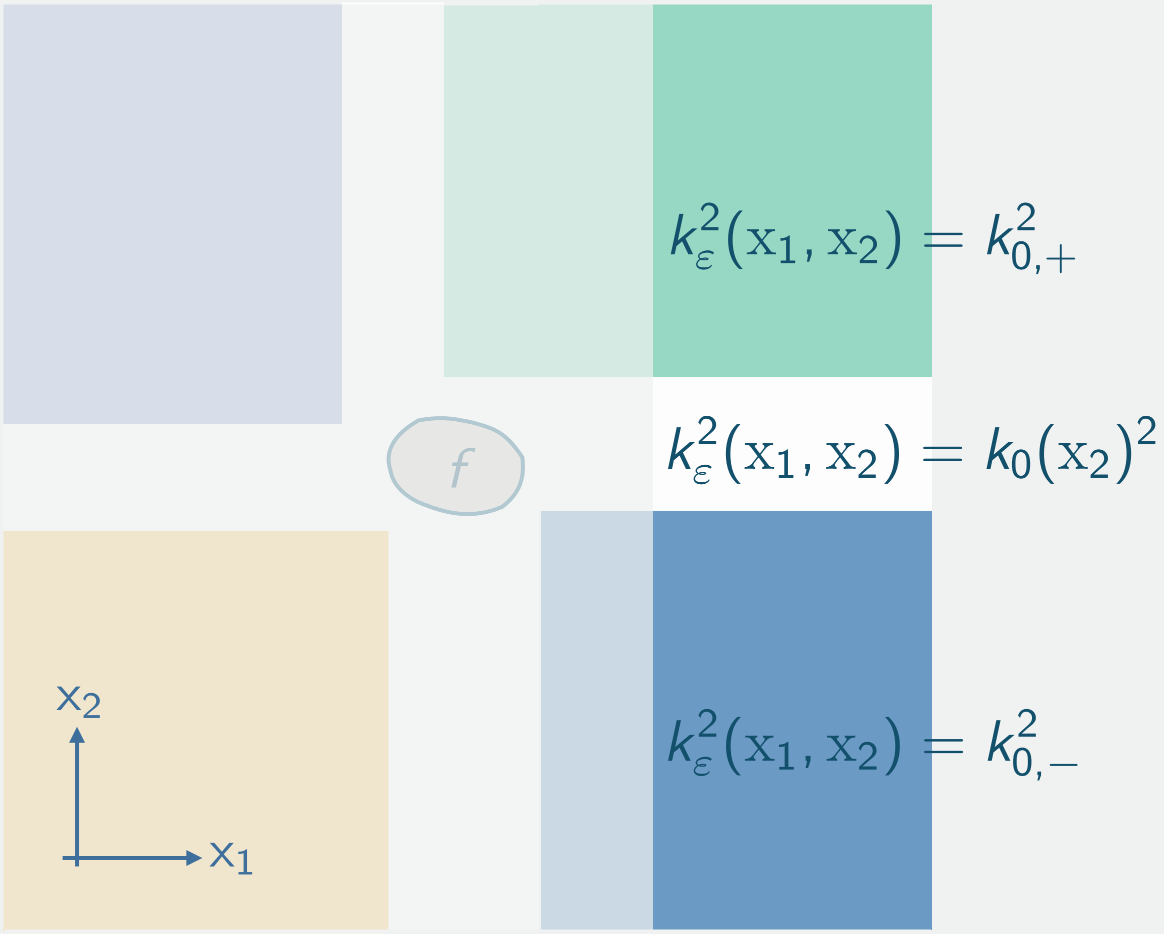

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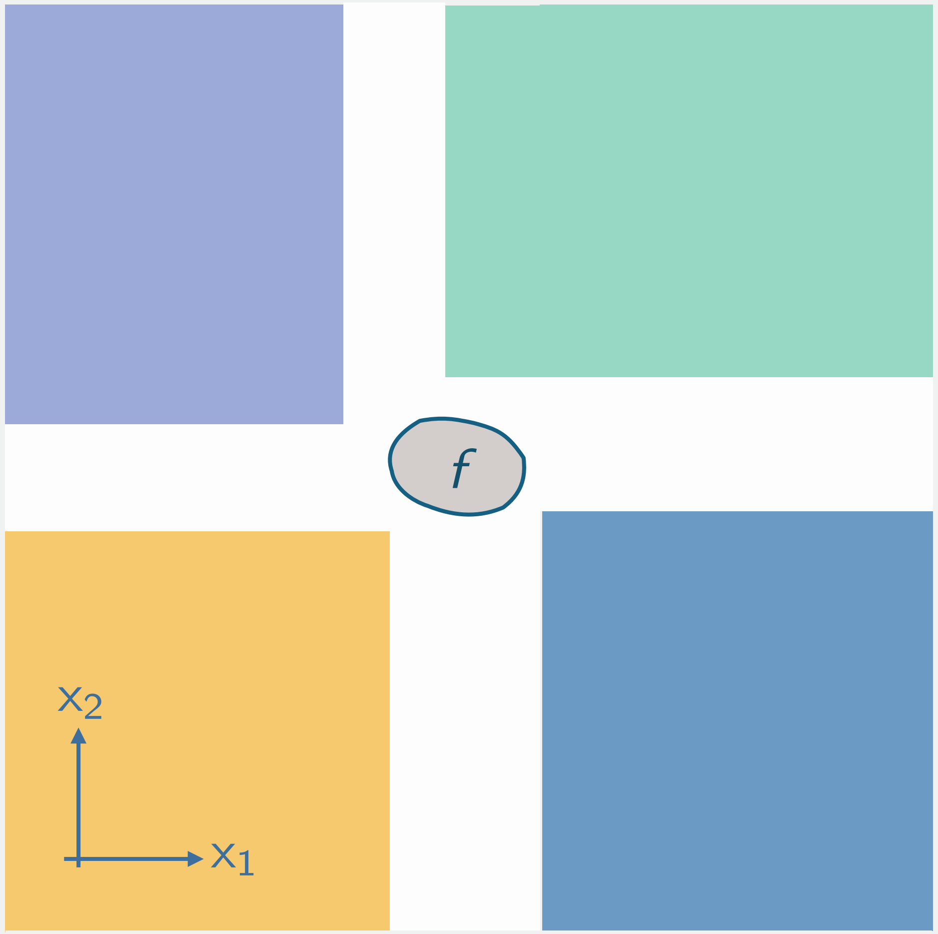

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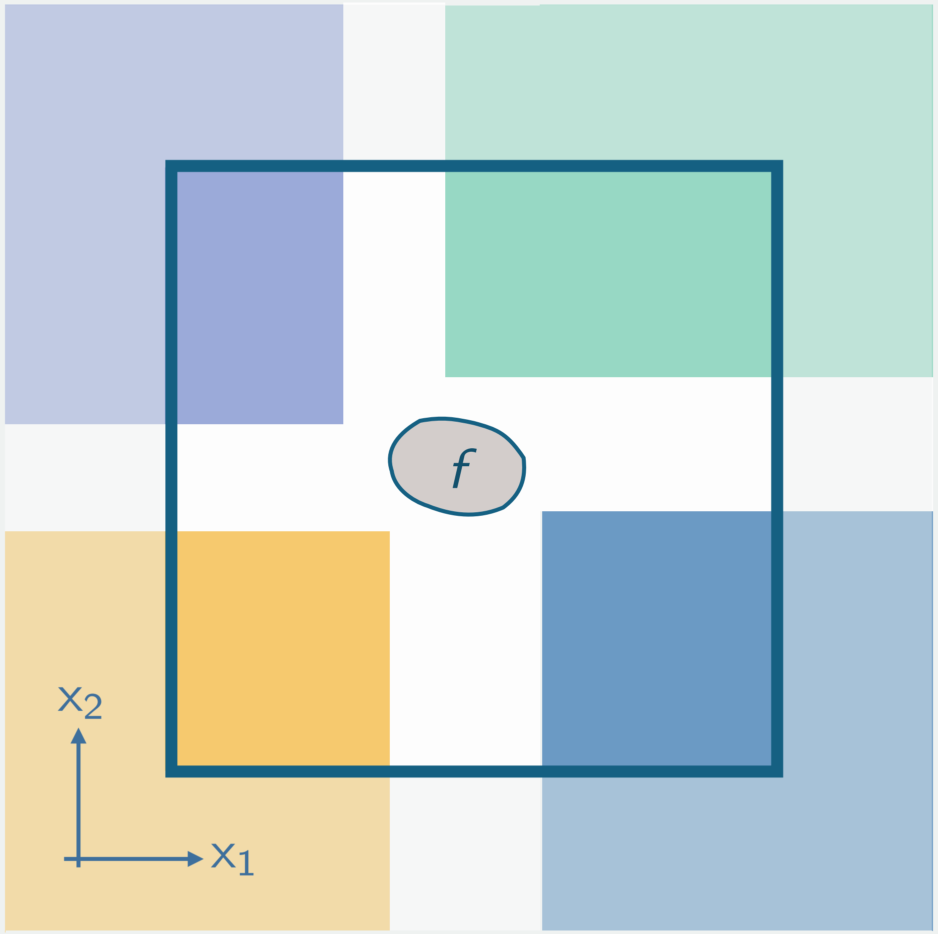

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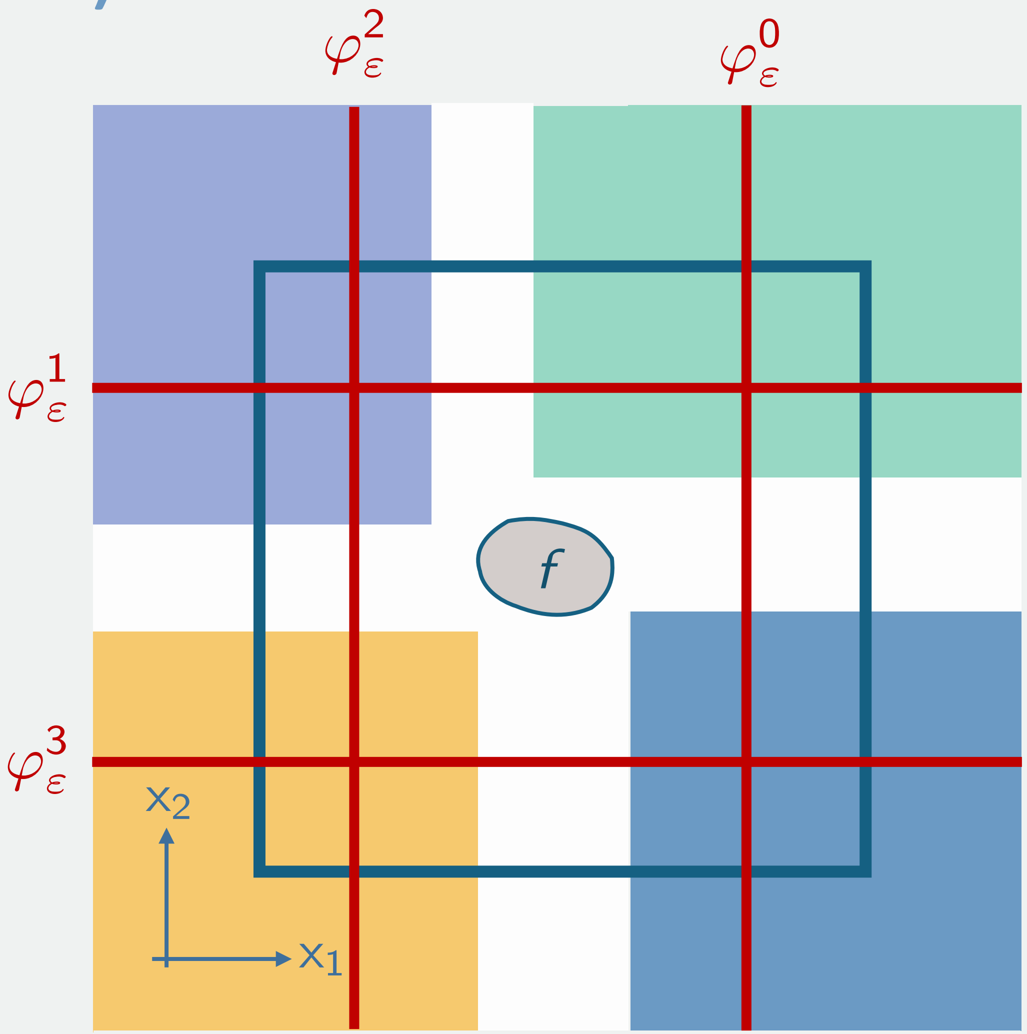

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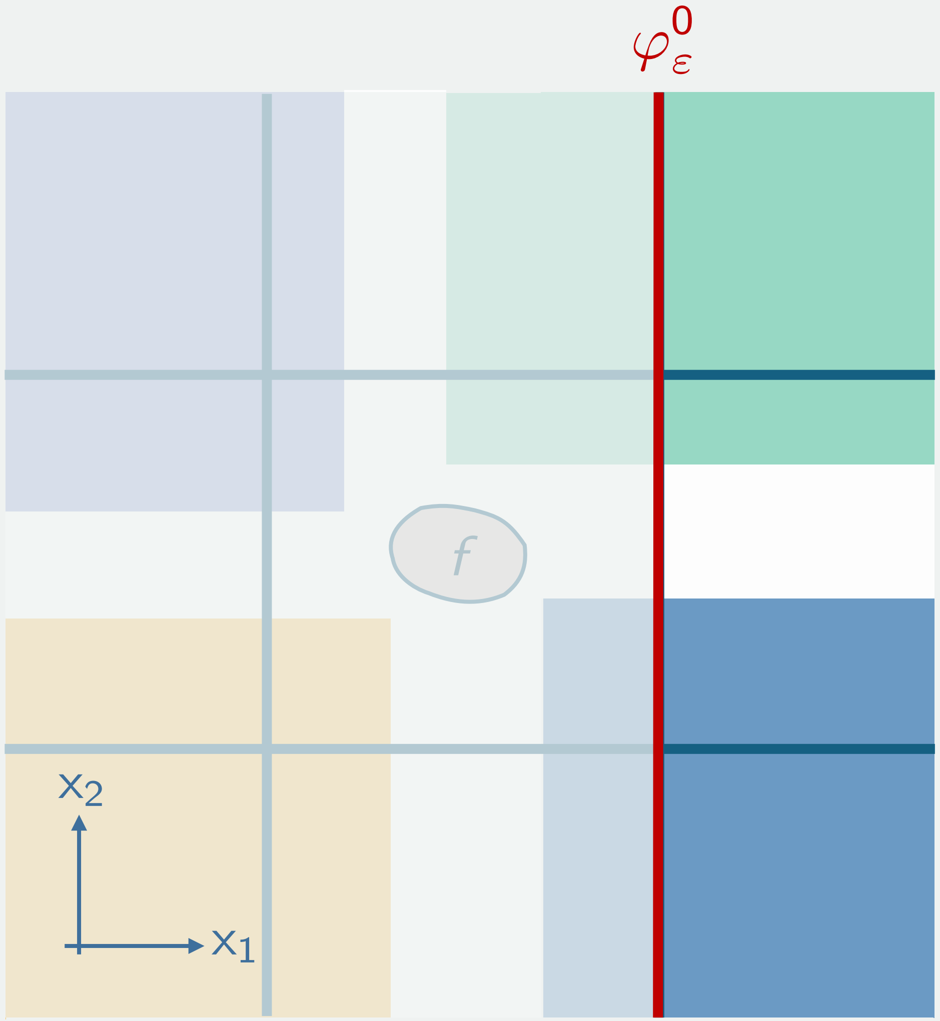

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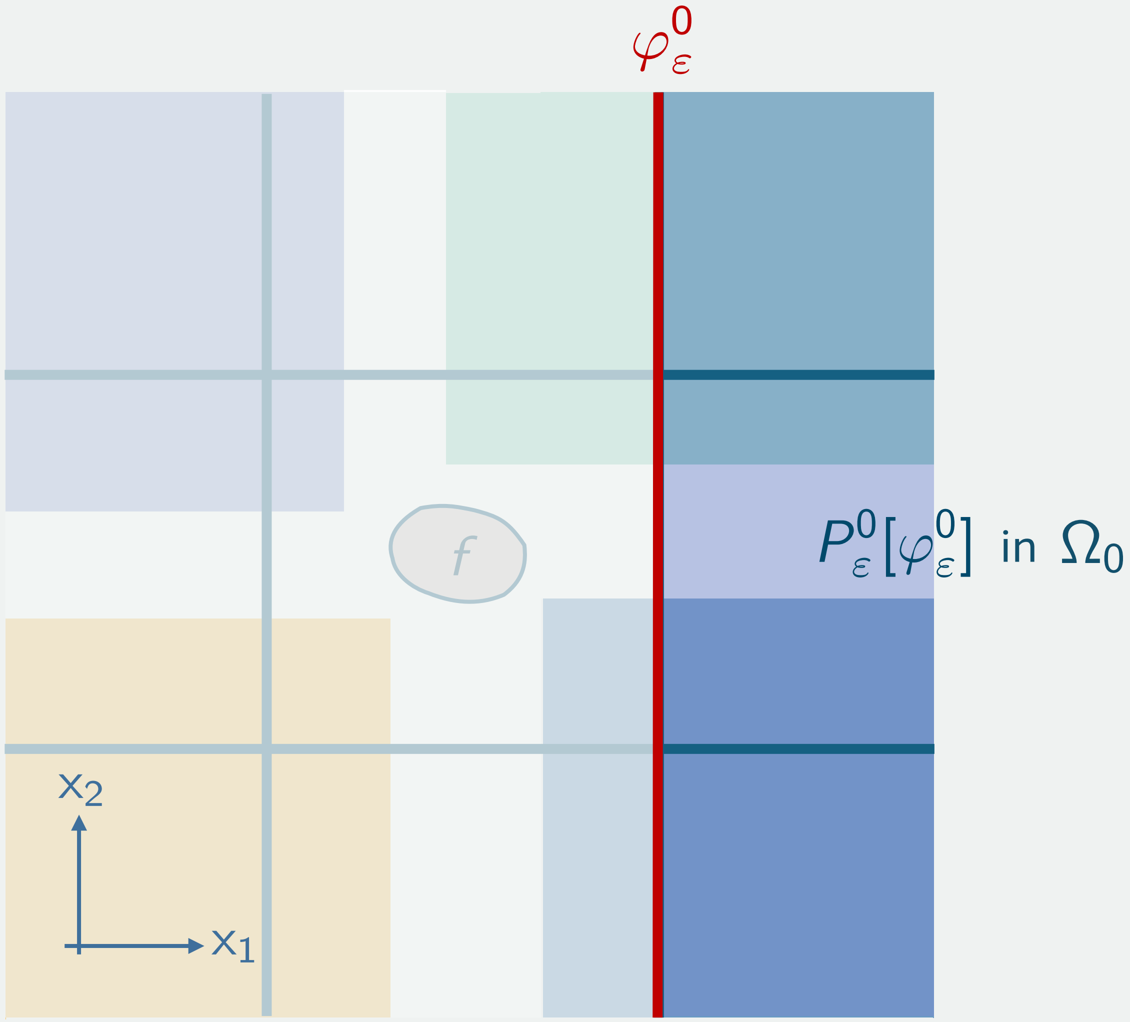

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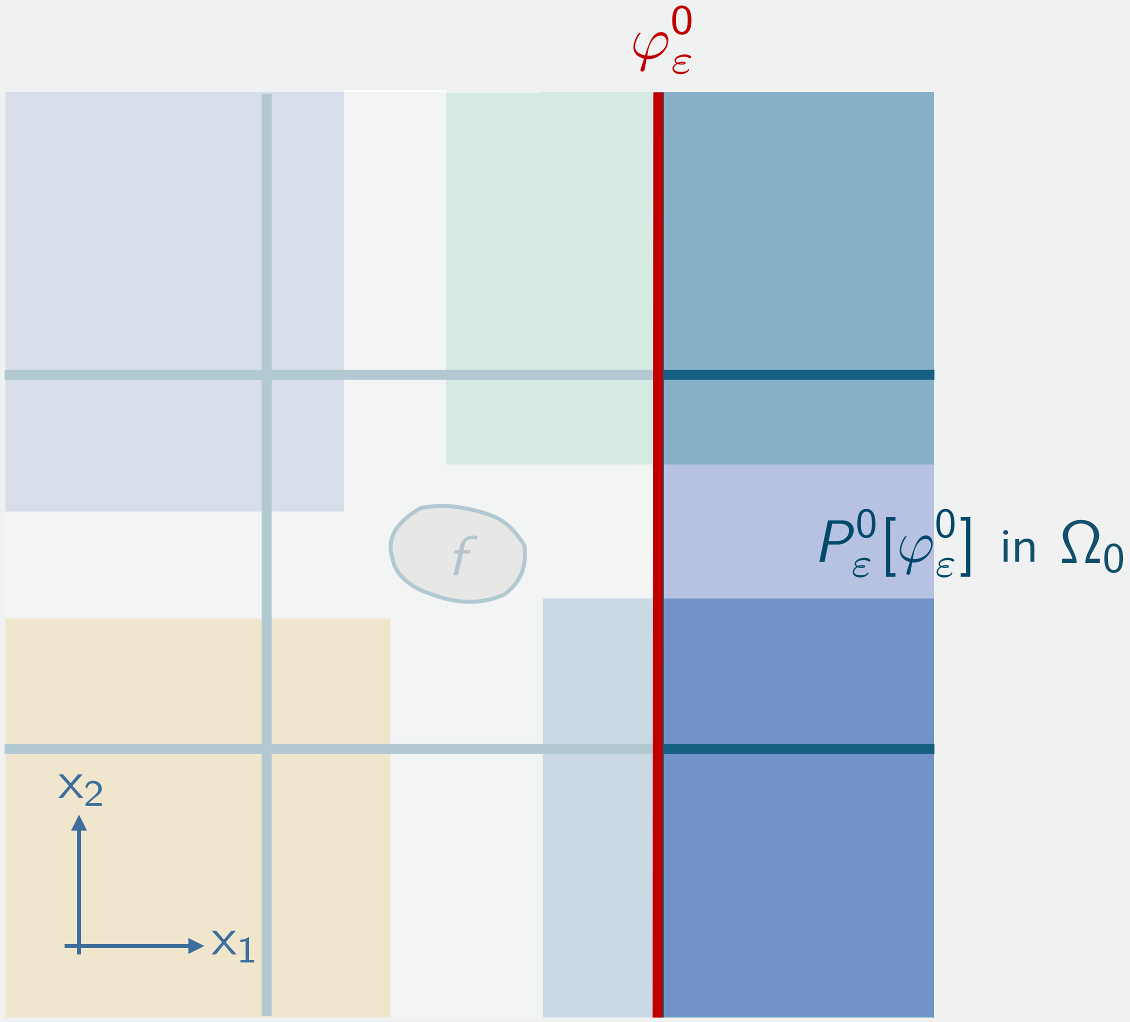

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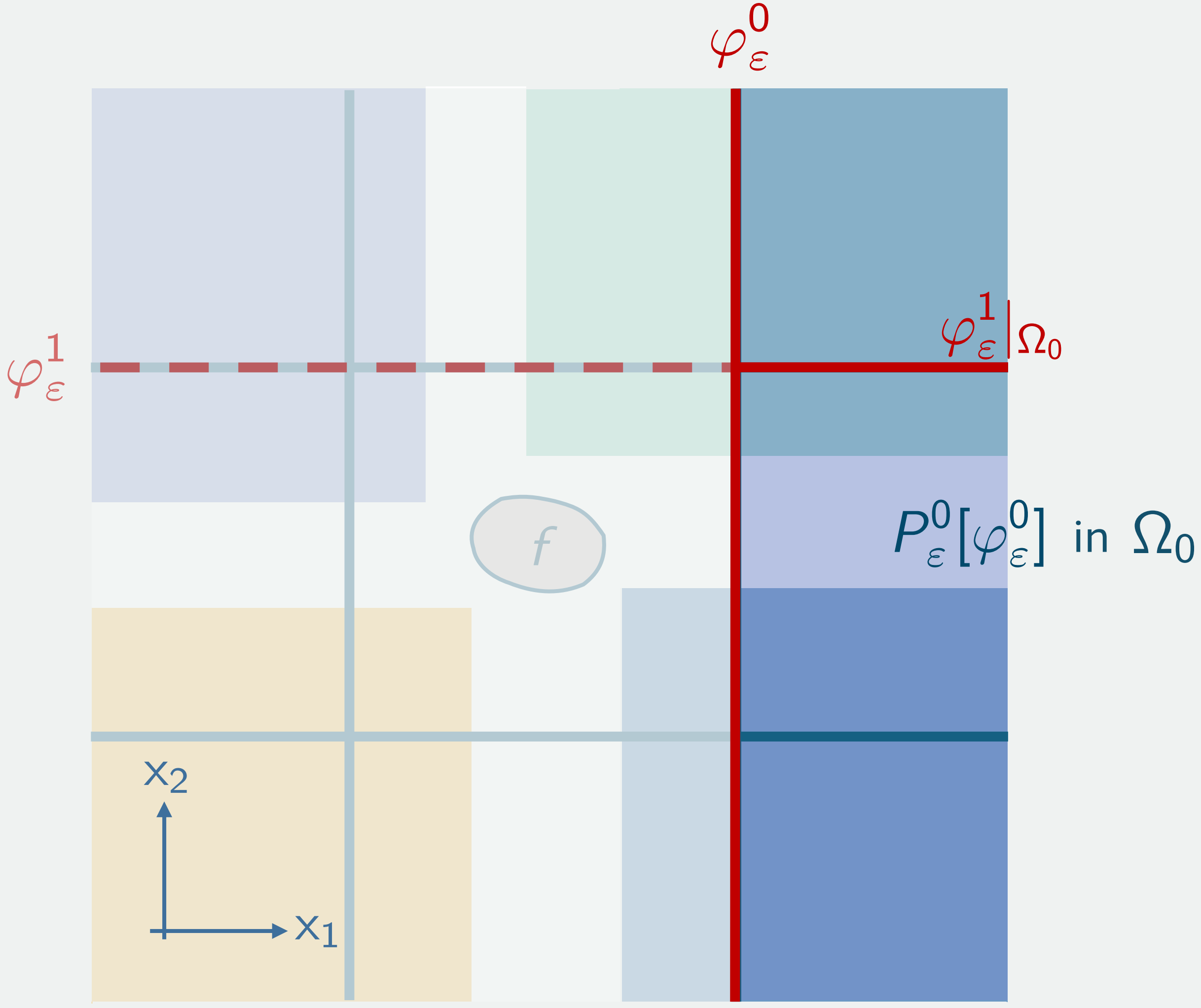

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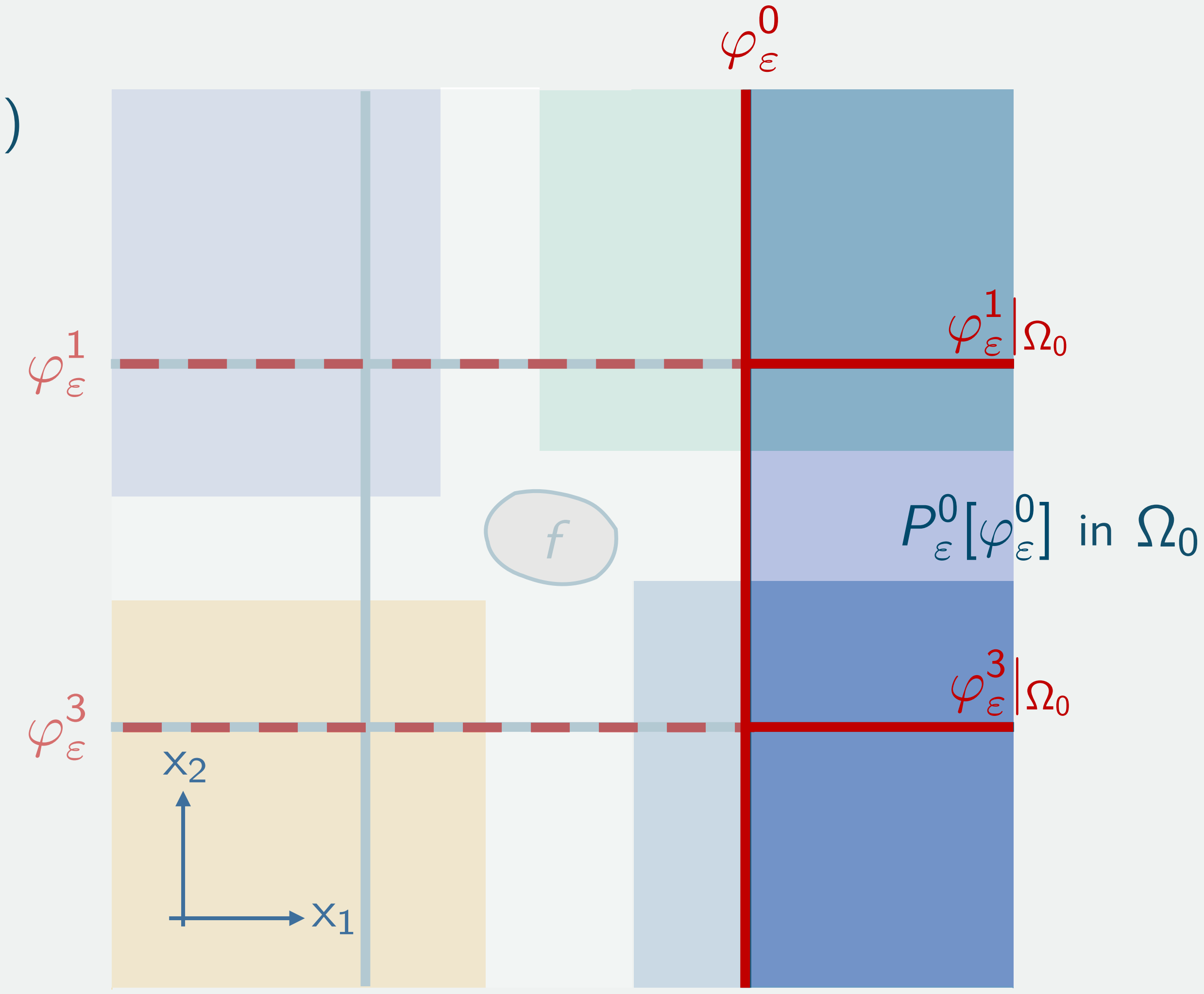

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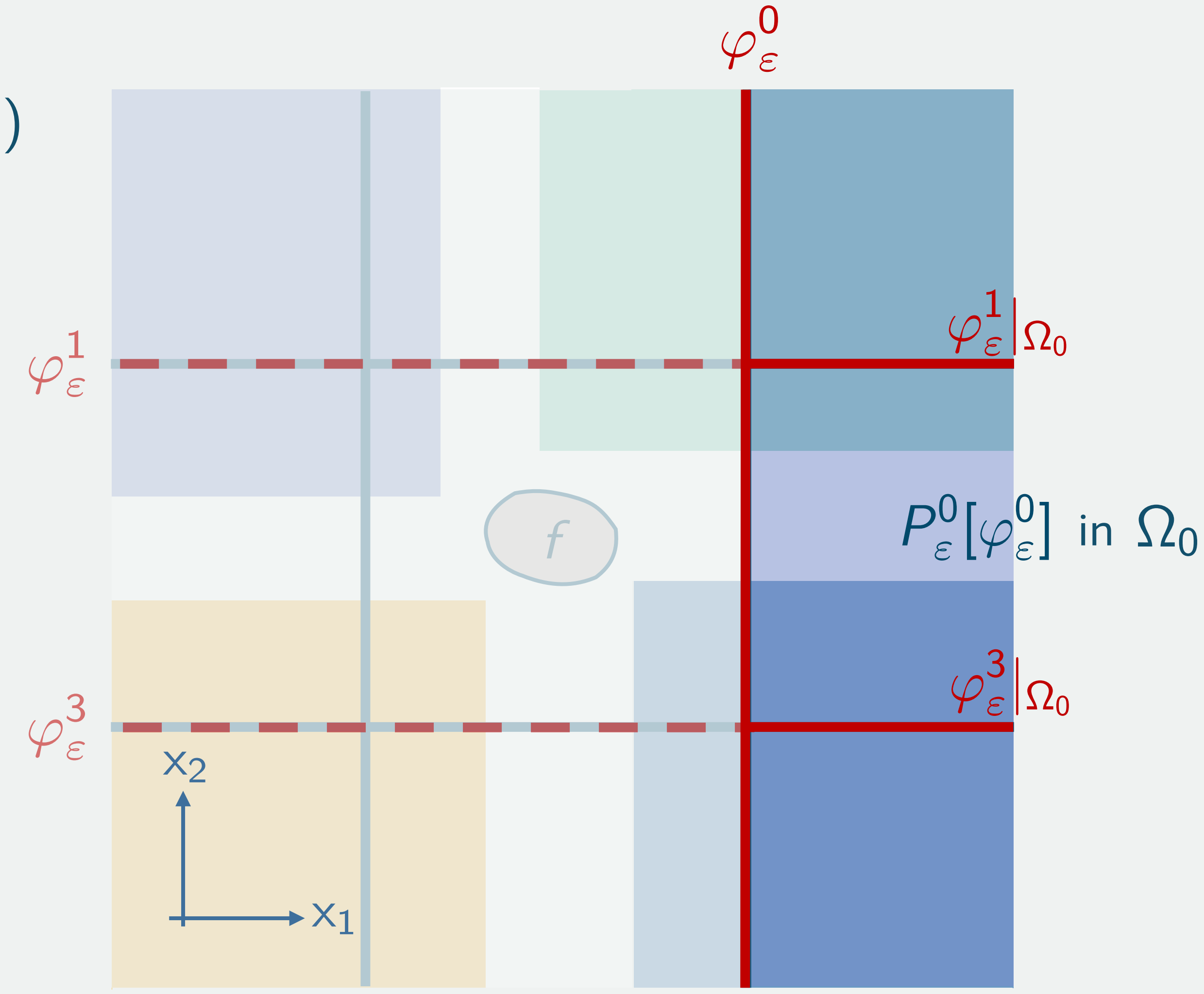

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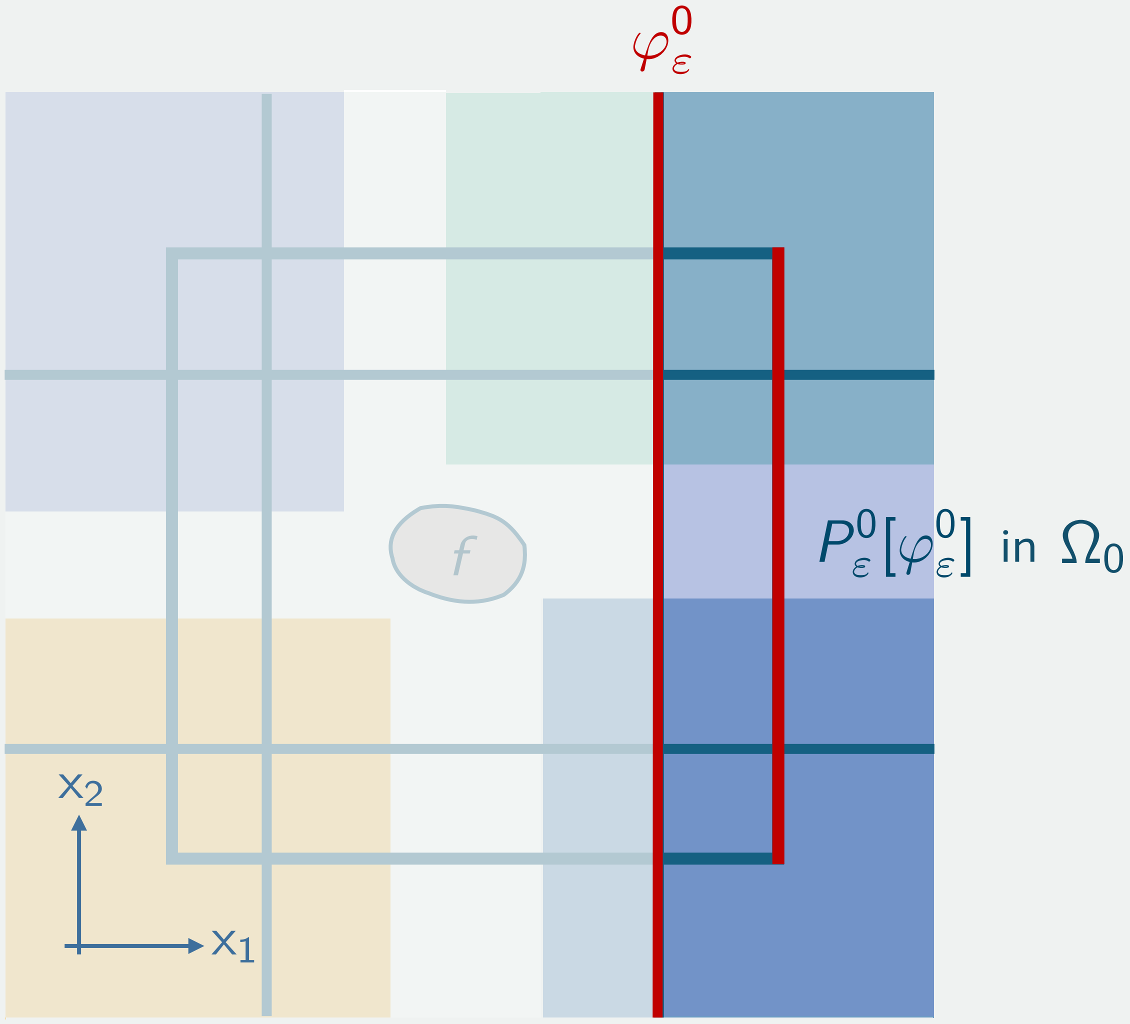

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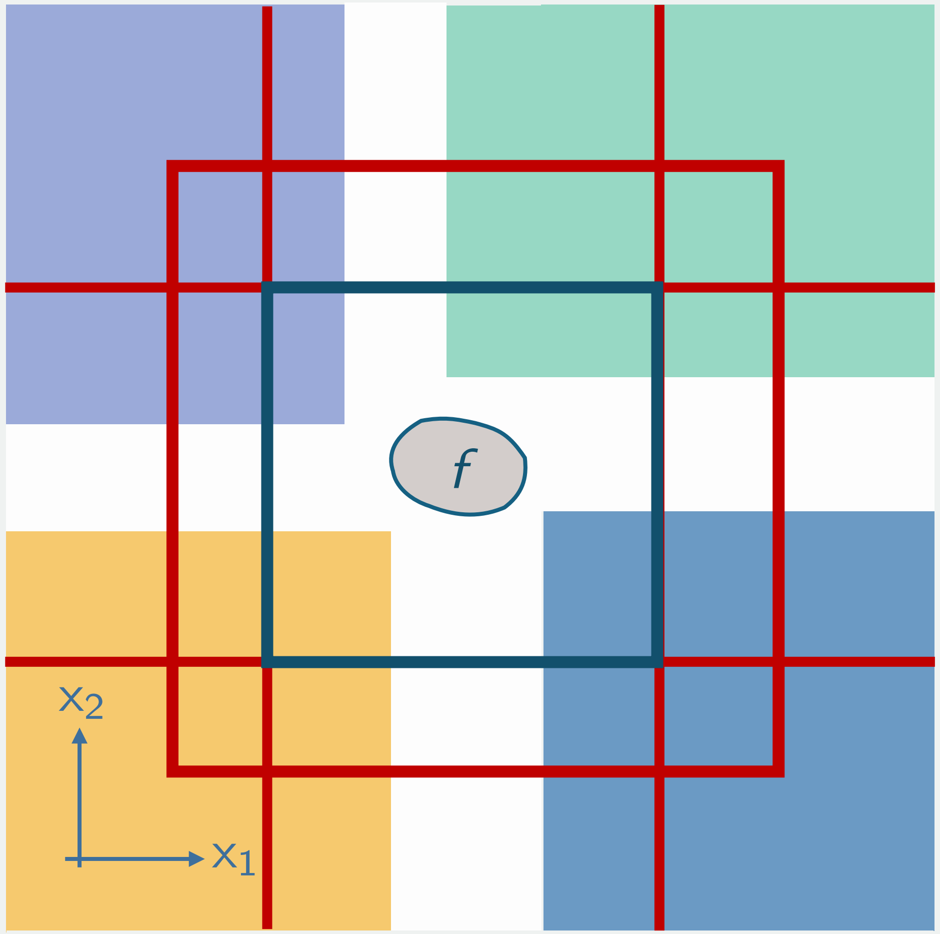

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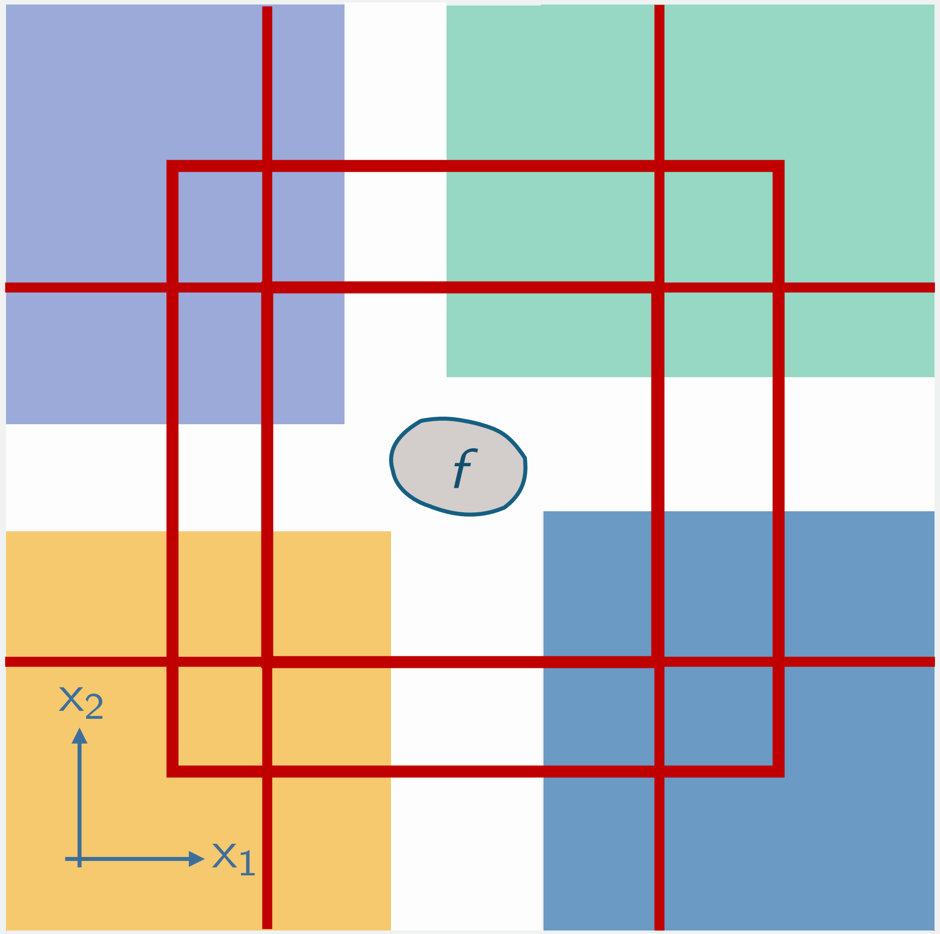

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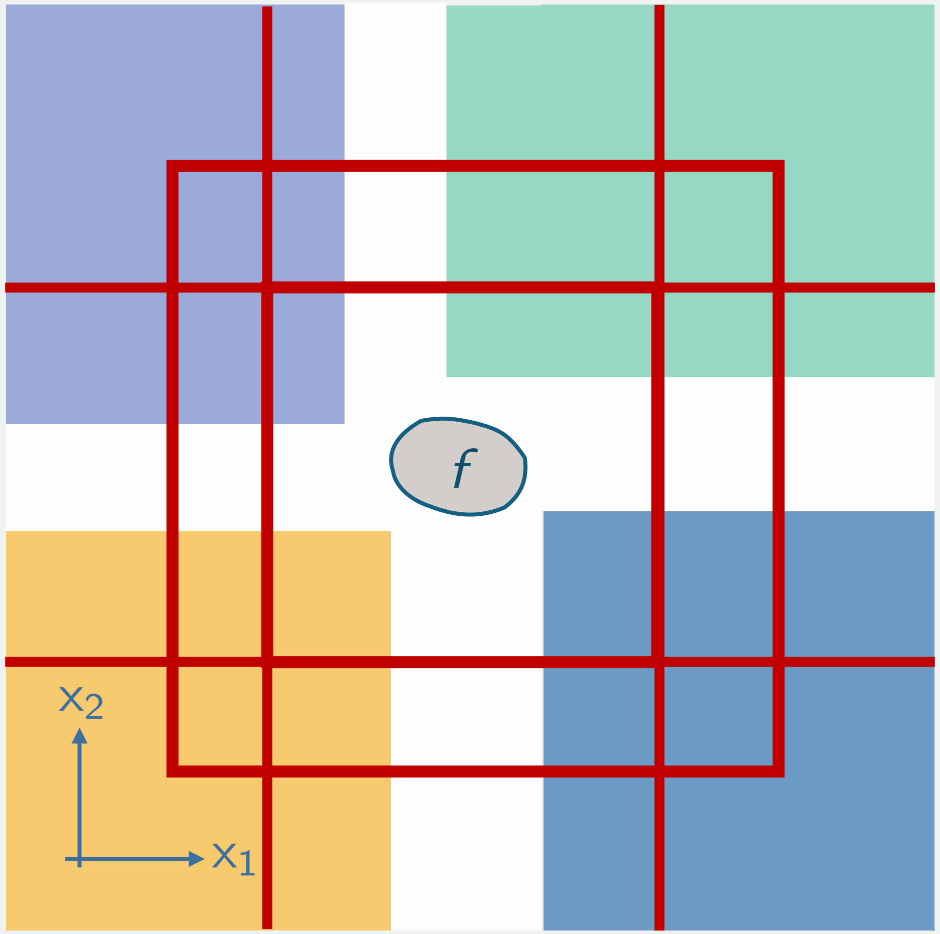

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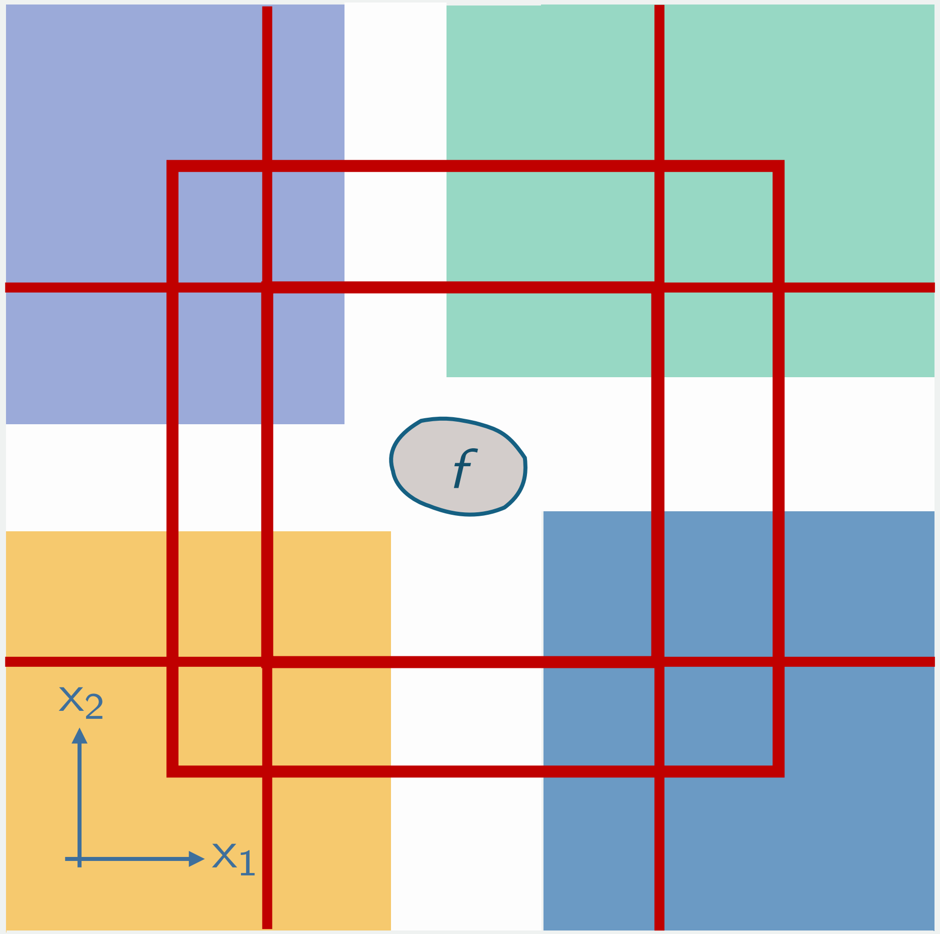

The Halfspace Matching Method : a new method to solve scattering problem in infinite media, *Anne-Sophie Bonnet-Ben Dhia, Sonia Fliss, Antoine Tonnoir, 2018*

Halfspace Matching: a Domain Decomposition Method for Scattering by 2D Open Waveguides, *Julian Ott, 2017*

Some contributions to the analysis of the Half-Space Matching Method for scattering problems and extension to 3D elastic plates, *Yohanes Tjandrawidjaja, 2019*

Properties of the method

Find $\Psi := (u_{\varepsilon,b}, \varphi_\varepsilon^0, \varphi_\varepsilon^1, \varphi_\varepsilon^2, \varphi_\varepsilon^3) \in H^1(\Omega_b) \times \prod_{\mathrm{J}=0}^{3} L^2(\Sigma^{\mathrm{J}})$,

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The framework for the traces is not adapted!

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The idea is to use that the $\varphi_\varepsilon^{\mathrm{J}}$'s have analytic extensions and we can choose a path τ_θ in the complex plane such that the $\varphi_{\varepsilon,\theta}^{\mathrm{J}} := \varphi_\varepsilon^{\mathrm{J}} \circ \tau_\theta$ are in $L^2(\mathbb{R})$. The Complex-Scaled HSM consists in writing equations on these so-called complexified traces.

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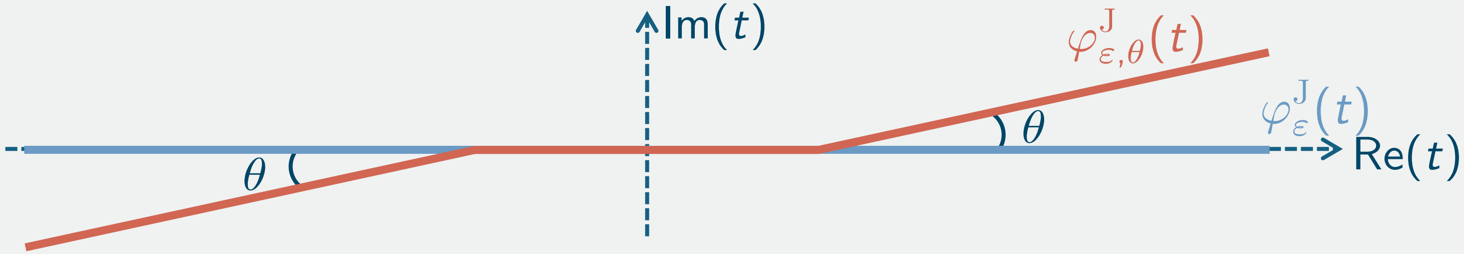

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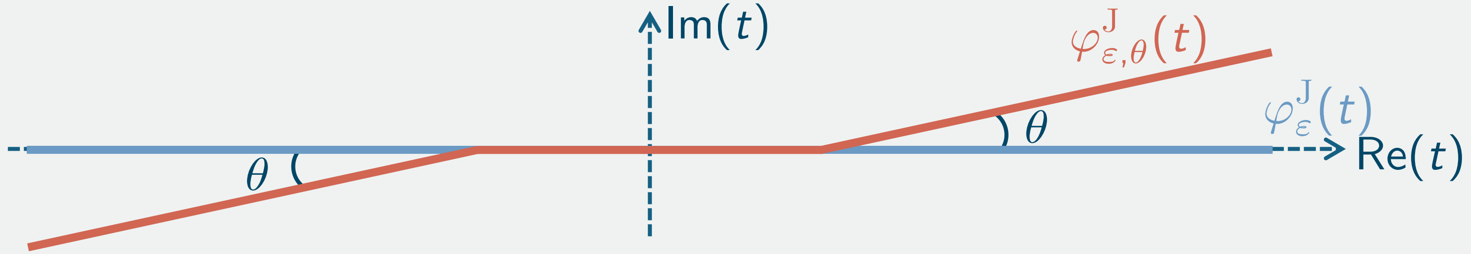

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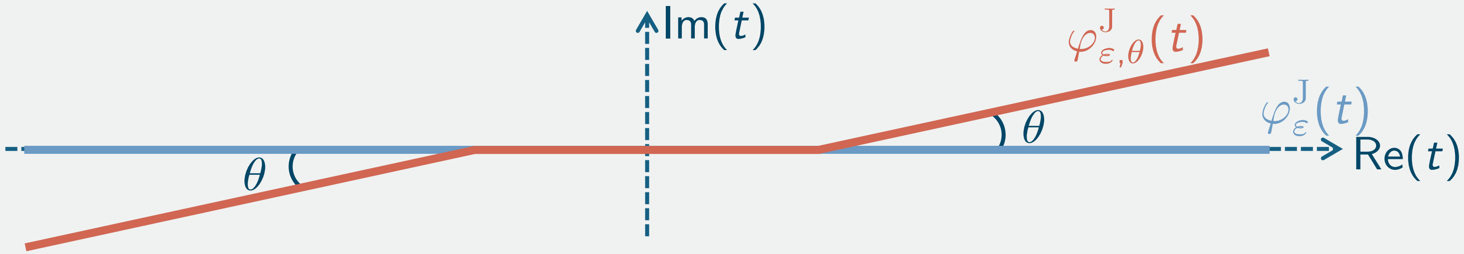

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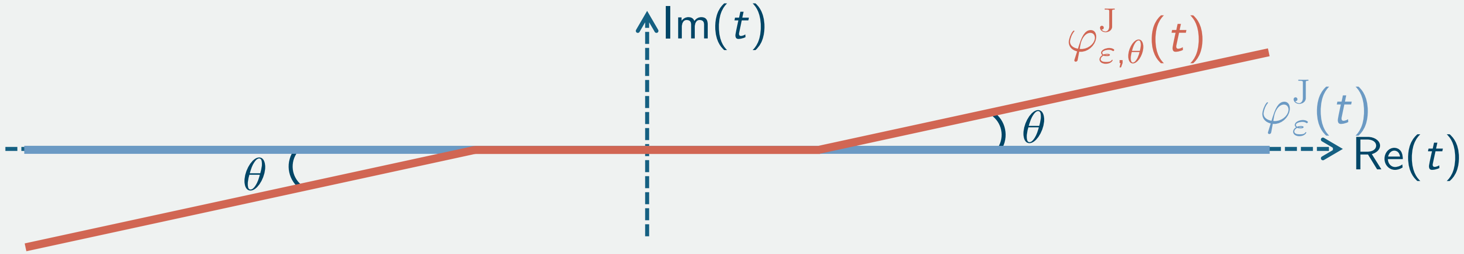

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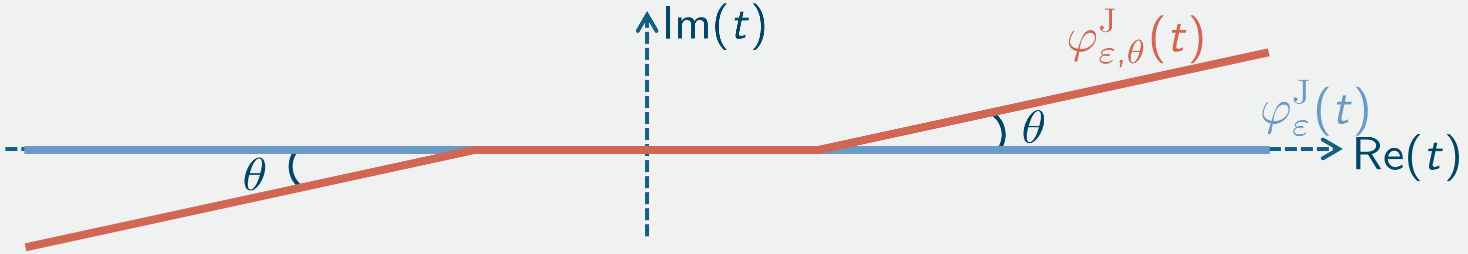

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The right-hand side is analytic in x_1: φ_ε^1 has an analytic extension.

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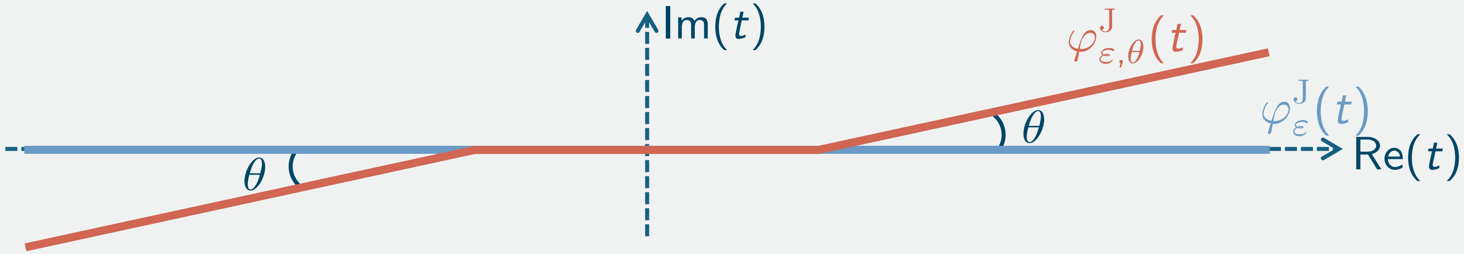
1. 

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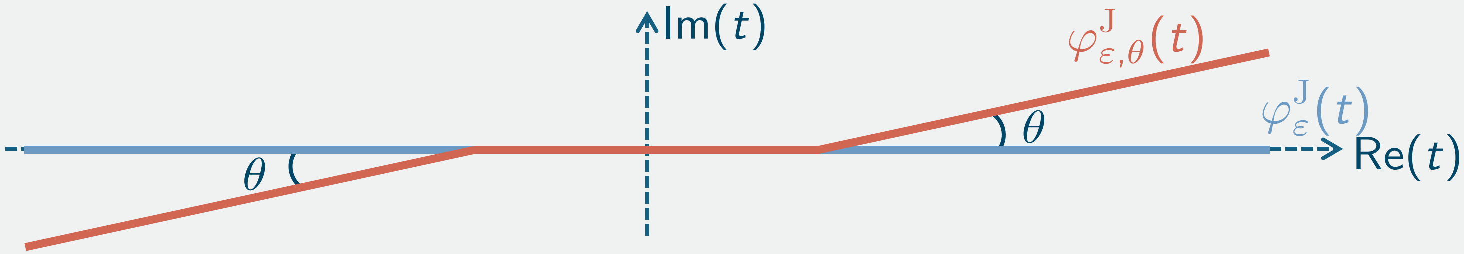
1. 

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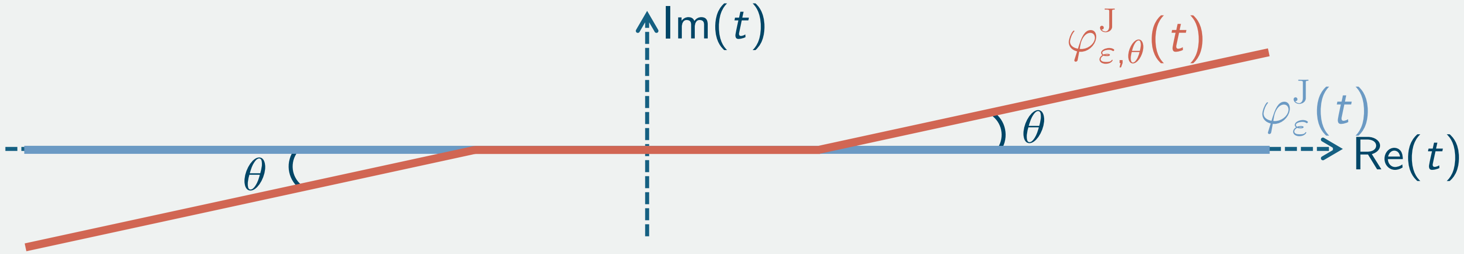

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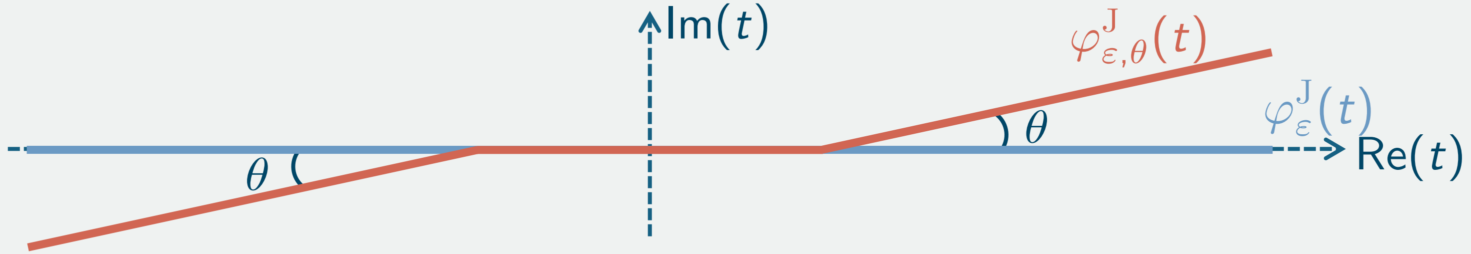
1. 

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Doing this for all the coupling equations between traces and for the equations involving the $\Lambda_\varepsilon^{\mathrm{J}}$ operators yields the CS-HSM system which is of Fredholm type.

$$\left(\mathbb{I} - \mathbb{D}_\varepsilon^\theta + \mathbb{K}_\varepsilon^\theta\right)\Psi_\varepsilon^\theta = \mathbb{G}_\varepsilon^\theta$$

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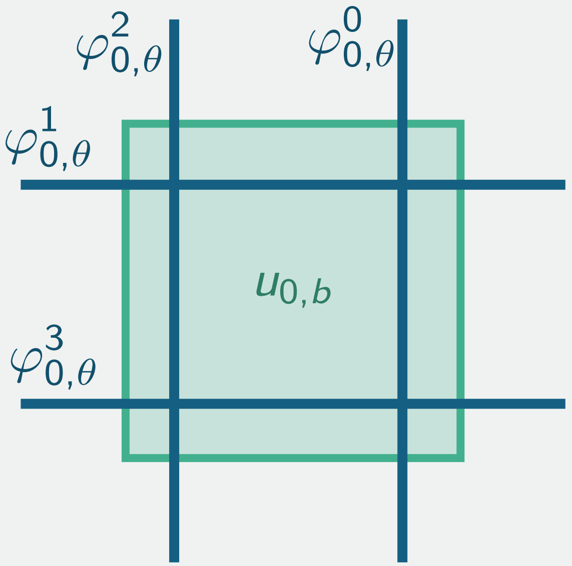

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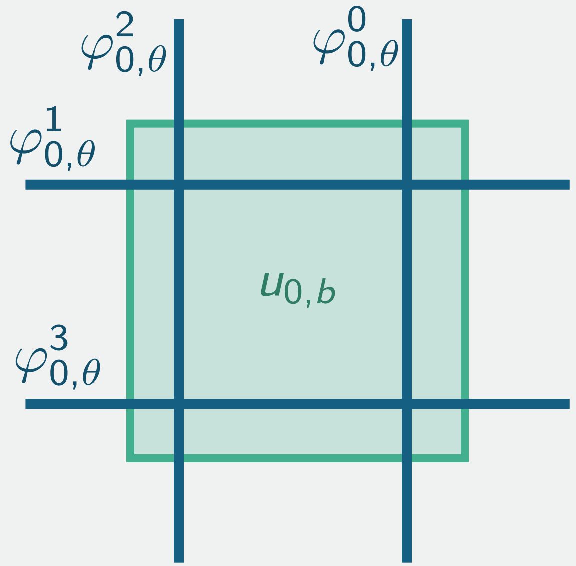

We can compute a representation of the solution based on the complexified traces.

$$P_{0,\theta}^{\mathrm{J}}[\varphi_{0,\theta}^{\mathrm{J}}](\mathbf{x}) := \int_{\mathbb{R}} \varphi_{0,\theta}^{\mathrm{J}}(s) N_{0,\theta}^{\mathrm{J}}(\mathbf{x}; s)\mathrm{d}s$$

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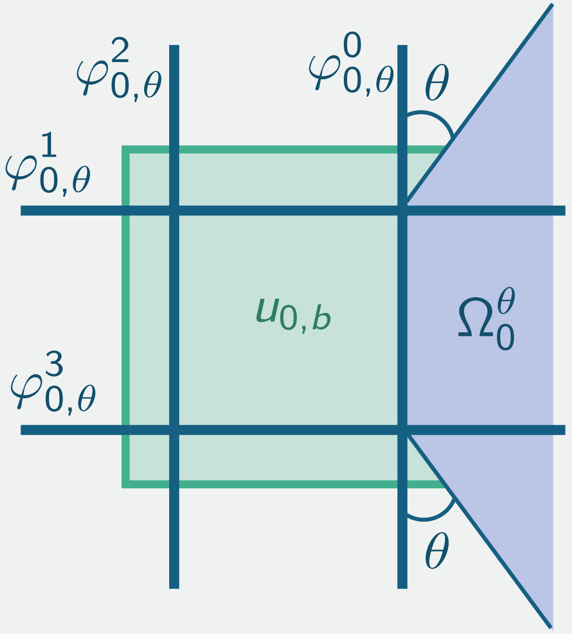

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It is valid in a subdomain.

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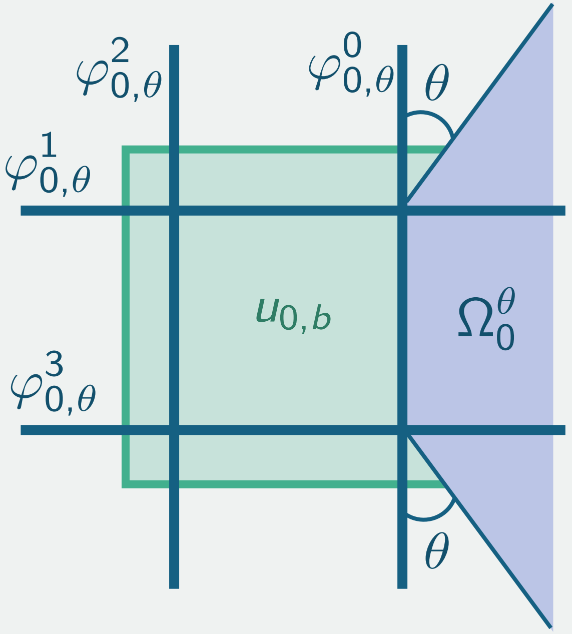

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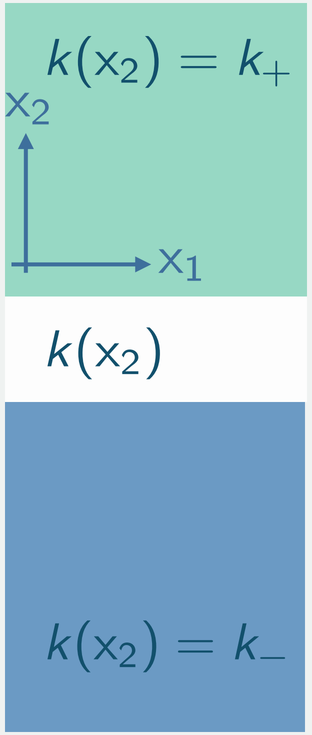
From this, we can recover the physical solution everywhere.

Representation in open waveguides

While the approach is the same as in homogeneous media, the stratification poses additional challenges: deriving a half-space representation adapted to the complex scaling.

$$\forall \mathbf{x} \in \Omega^{\theta}_{\mathrm{J}}, \quad P^{\mathrm{J}}_{\varepsilon}[\varphi^{\mathrm{J}}_{\varepsilon}](\mathbf{x}) = \int_{\mathbb{R}} \varphi^{\mathrm{J}}_{\varepsilon}(s) N^{\mathrm{J}}_{\varepsilon}(\mathbf{x}; s) \mathrm{d}s.$$



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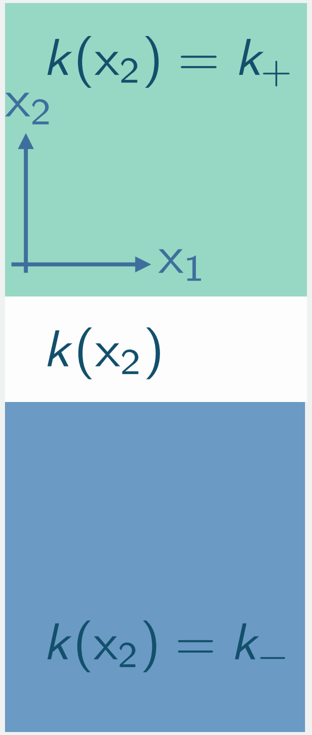

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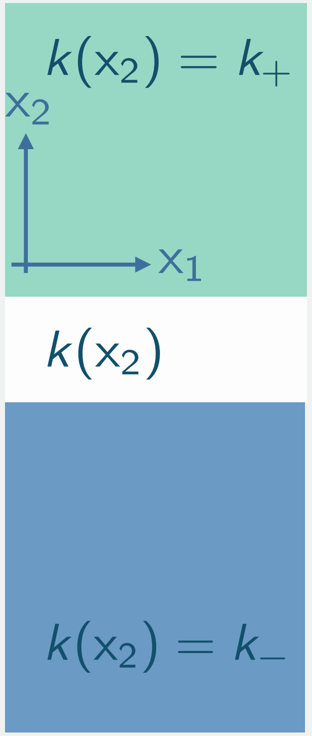

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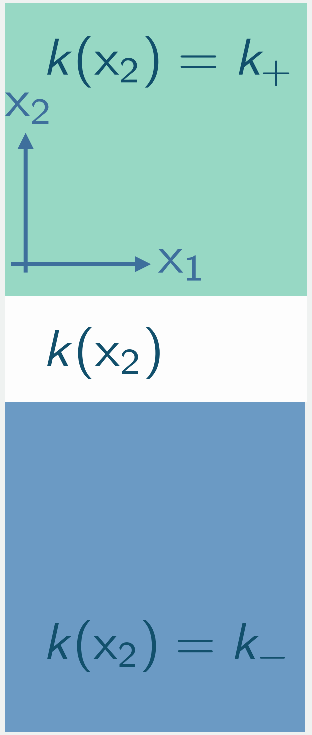

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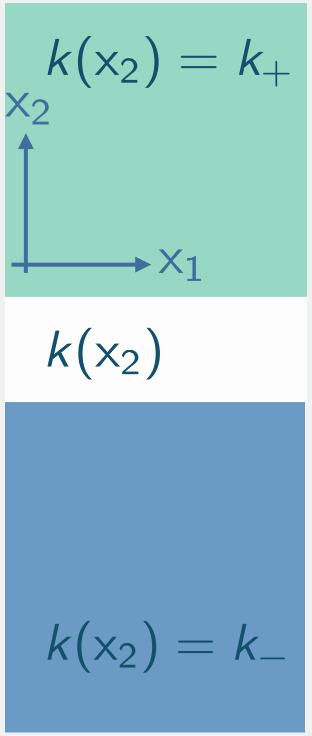
?

Is $N_{\varepsilon}^{\mathrm{J}}$ analytic in x_1 and s?



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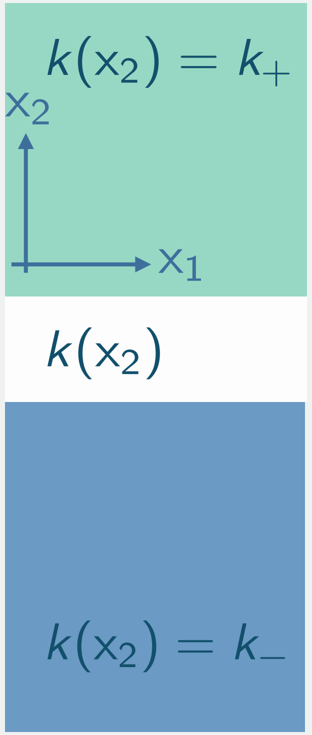

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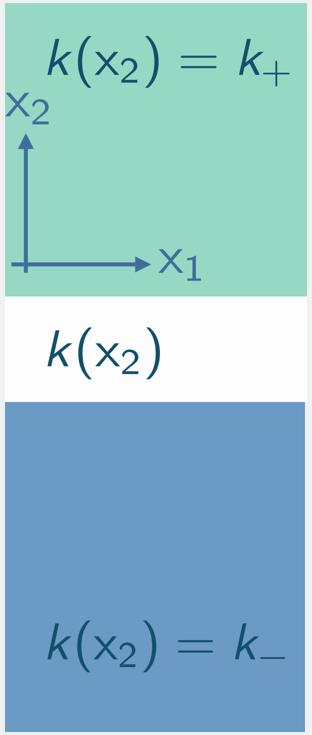

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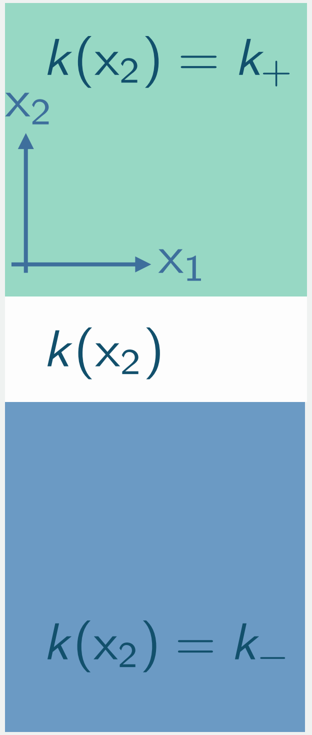

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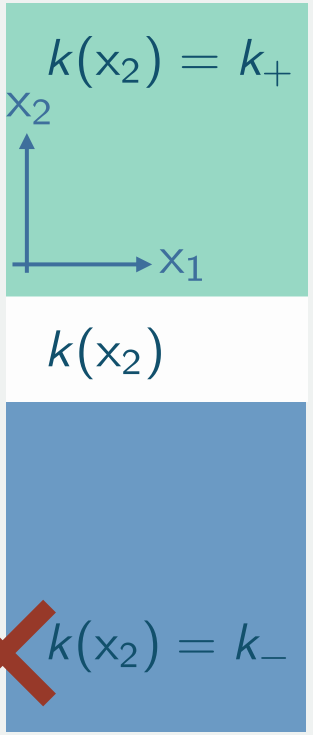

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Not adapted to the complexification as the $\Psi^{\mathrm{J},\pm}$ are meromorphic with an infinity of poles.

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✗

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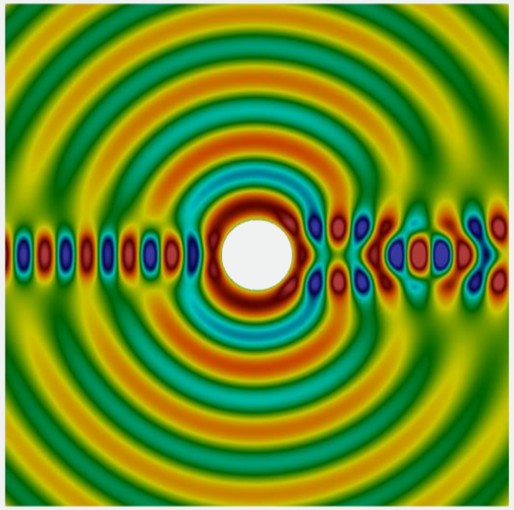
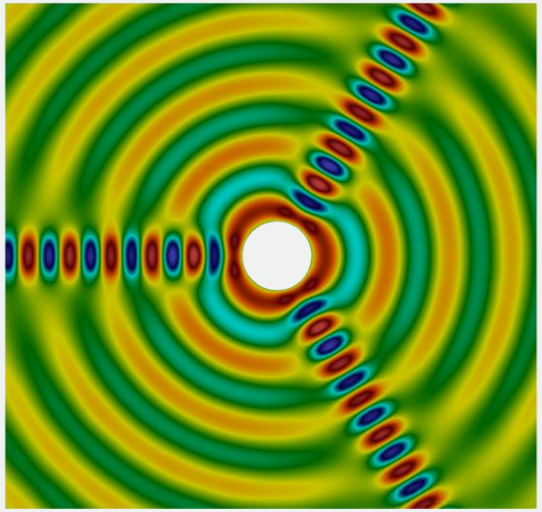
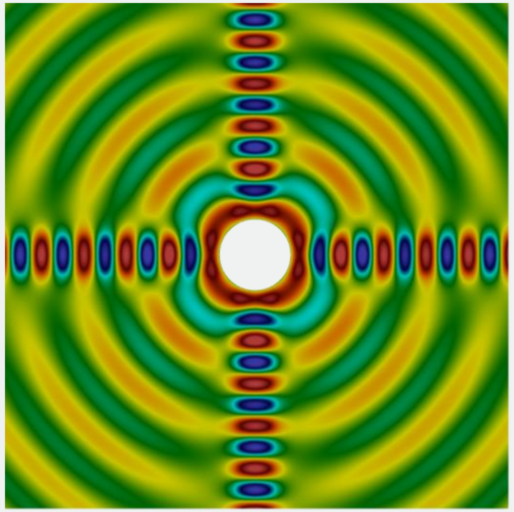
✓

The only poles are zeros of $w_{\varepsilon}^{\mathrm{J}}$ and correspond to the eigenvalues of the transverse operator $-\mathrm{d}_{x_2}^2 - k_{\mathrm{J}}(\cdot)^2$: there is at most a finite number of poles. They can be computed numerically as well as the corresponding residues.

Perspectives and open questions







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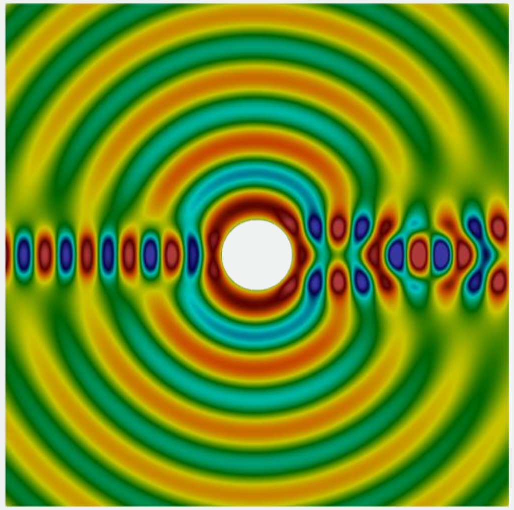

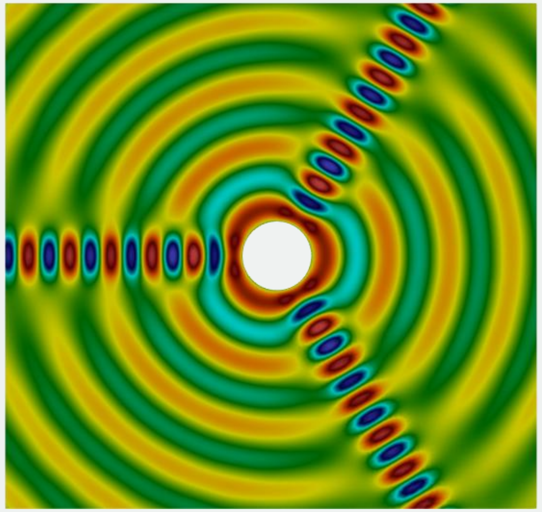

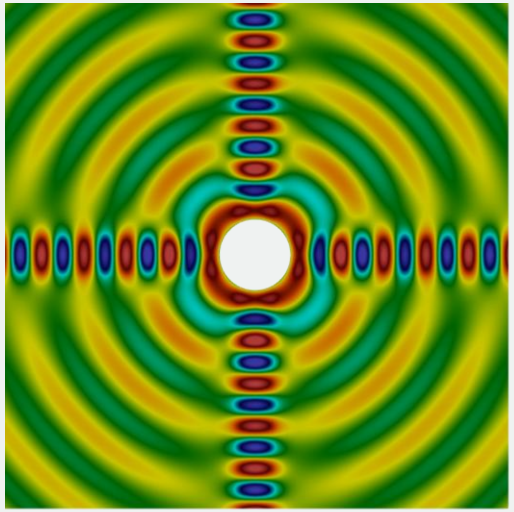

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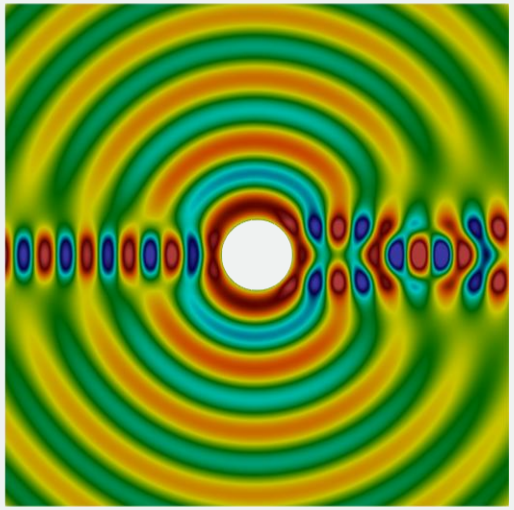

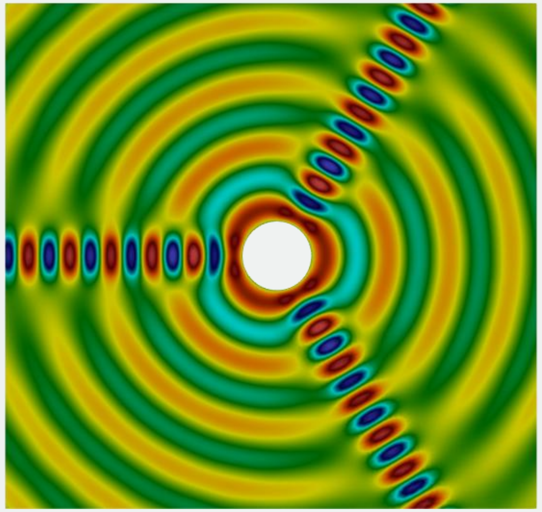

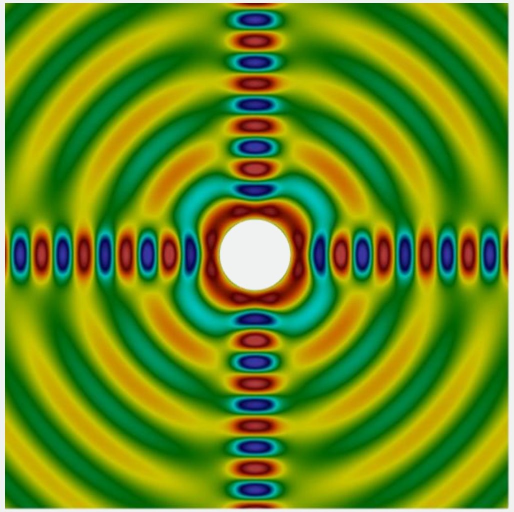

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How to handle waveguides separated by small angles?

Thank you for your attention!