Non-commutativity in nonlocal transport

The Lie bracket for measure-dependent vector fields

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Joint work with Amaury Hayat (CERMICS), Benedetto Piccoli (Rutgers–Camden) & Emmanuel Trélat (LJLL)

Large populations of interacting agents

A common problem in applied mathematics is the dynamics of **many interacting agents/particles**

multi-agent control	crowd & traffic models
opinion dynamics	swarming & flocking

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Each agent feels a velocity that depends on *where everyone else is*

The model: nonlocal continuity equation

The population evolves through the **nonlocal continuity equation**

$$\boxed{\partial_t \mu_t + \operatorname{div}\left(b(\cdot, \mu_t)\,\mu_t\right) = 0}$$

where the velocity field

$$b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d, \qquad (x, \mu) \mapsto b(x, \mu)$$

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- Arises as a **mean-field limit** of interacting particle systems
- The framework of **Measure Differential Equations** [Piccoli '19], write $\dot{\mu} = V[\mu]$
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Local case $b(x, \mu) = X(x)$, ordinary transport along a vector field X

The classical picture: bracket & commutation

Warm-up — linear flows Two linear ODEs $\dot{x} = Ax$ and $\dot{x} = Bx$ have flows e^{tA}, e^{sB}, and

$$e^{tA}e^{sB} = e^{sB}e^{tA} \;\; \forall s, t \quad \Longleftrightarrow \quad [A, B] := AB - BA = 0$$

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General fields For $X, Y \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ with flows Φ^X_t, Φ^Y_s, the **Lie bracket**

$$[X, Y](x) \;=\; DY(x)\,X(x) - DX(x)\,Y(x)$$

measures the defect of commutation

Classical theorem

$$\Phi^X_t \circ \Phi^Y_s = \Phi^Y_s \circ \Phi^X_t \quad \forall s, t \quad \iff \quad [X, Y] \equiv 0$$

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↪ cornerstone of **geometric control**, for the driftless system $\dot{x} = u\,X(x) + v\,Y(x)$, iterated brackets $[X, Y], [X, [X, Y]], \ldots$ **open new directions (Chow–Rashevskii)**

The question

Central question

What is the right notion of **Lie bracket** for a **nonlocal** field $b(x, \mu)$, and how does it control the **commutativity** of the flows of $\partial_t \mu + \mathrm{div}(b(\cdot, \mu)\mu) = 0$?

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The map $x \mapsto b(x,\mu)$ is an honest vector field *for each frozen* μ, so it has a classical bracket, **but the flow itself moves** μ

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We need a bracket that also differentiates *b in the measure variable*
$\Rightarrow$ **Lions differential calculus** on $\mathcal{P}_2(\mathbb{R}^d)$

The tool: Lions L-derivative

Differentiating a functional $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in the **measure variable**

Lift to an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\widetilde{F} : L^2(\Omega; \mathbb{R}^d) \to \mathbb{R}, \qquad \widetilde{F}(X) := F(\mathrm{Law} X)$$

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F is **L-differentiable** at $\mu = \mathrm{Law}(X_0)$ if $\widetilde{F}$ is Fréchet-differentiable at X_0, and then by law-invariance the gradient is represented by a function

$$D\widetilde{F}(X_0)(\omega) \;=\; \partial_\mu F(\mu)\big(X_0(\omega)\big), \qquad \partial_\mu F(\mu) \in L^2(\mu; \mathbb{R}^d)$$

$\partial_\mu F(\mu)(y) \cdot v$ → change when moving small mass from y in direction v

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$\partial_\mu F(\mu)(y) \cdot v \to$ change when moving small mass from y in direction v

For the velocity map, $\partial_\mu b_x[\mu](y) \in \mathbb{R}^{d\times d}$ measures the sensitivity of the velocity at x to a perturbation of the mass located at y

Main result: a bracket for nonlocal fields

The Lie bracket for nonlocal fields (B., Hayat, Piccoli, Trélat)

Two nonlocal fields b^1, b^2 admit a bracket $[b^1, b^2]$, again a nonlocal field, with

$$[b^1, b^2]_\mu(x) = \underbrace{D_x b_x^2[\mu]\, b_x^1[\mu] - D_x b_x^1[\mu]\, b_x^2[\mu]}_{\text{frozen-field classical bracket}} + \underbrace{N^{12}[\mu](x)}_{\text{nonlocal bracket}}$$

with the genuinely new nonlocal term

$$N^{12}[\mu](x) = \int_{\mathbb{R}^d} \Big(\partial_\mu b_x^2[\mu](y)\, b_y^1[\mu] - \partial_\mu b_x^1[\mu](y)\, b_y^2[\mu]\Big)\, \mathrm{d}\mu(y)$$

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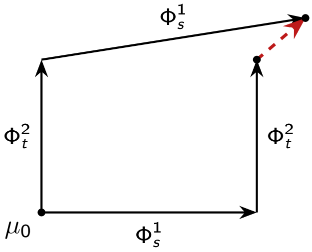
Commutativity of the nonlocal flows

$$\Phi^1_t \circ \Phi^2_s = \Phi^2_s \circ \Phi^1_t \quad \forall s, t \iff [b^1, b^2] \equiv 0$$

Why a nonlocal term appears

Flow first by b^1 during s, then by b^2 during t (and compare with the reverse order)



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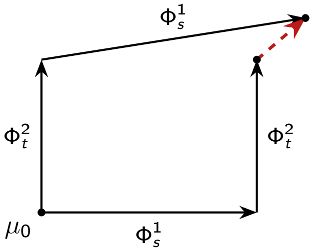

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Two effects feed the gap

1. the fields vary *in space*
   → **classical bracket**
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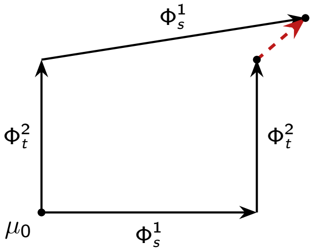

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Nonlocality is exactly what produces the new obstruction to commutativity

Micro commutes, macro doesn't: an example

On $\mathcal{P}_2(\mathbb{R}^2)$, with barycenter $m(\mu)$ and variance $V(\mu)$, take two **nonlocal fields**

$$b^1(x,\mu) = -k\,(x - m(\mu)), \qquad b^2(x,\mu) = \frac{L_0}{1+V(\mu)}\,J\,(x - m(\mu))$$

($J =$ rotation by $\frac{\pi}{2}$): a contraction and a rotation, both about the barycenter

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Frozen μ (microscopic) Homothety & rotation about the *same* center commute

$$\left[b^1, b^2\right]_{\mu}^{\text{frozen}} \equiv 0$$

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Mean-field (macroscopic)

$$N^{12}[\mu](x) = \frac{2kL_0\,V(\mu)}{(1 + V(\mu))^2}\,J\,(x - m(\mu)) \neq 0.$$

Contracting ⇒ faster spin

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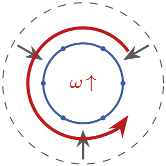

contracting the cloud
speeds up its rotation

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Everything commutes microscopically, yet the mean-field flows do not

The message

Local case ⇒ classical bracket

If the fields are *local* (ODE fields X, Y, independent of μ), then $\partial_\mu b^i = 0$, so $N^{12} = 0$ and

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Non-commutativity is created by the dependence of the velocity on the distribution itself

Behind the scenes & perspectives

Already established (kept implicit today, for simplicity)

A **Lie algebra** S^1_L (Hörmander growth + Lions regularity)

An **exponential map** $\exp : S^1_L \to \mathrm{Aut}(\mathcal{P}_c(\mathbb{R}^d))$

A **Baker–Campbell–Hausdorff formula** for the product of exponentials

$\Rightarrow$ Genuine **Lie theory** for nonlocal fields

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Main message

Nonlocality gives rise to macroscopic non-commutativity

Thank you!!!