Multi-indices Butcher Series

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Outline

B-series

The Ordinary Differential Equation (ODE)

$$dy = f(y_t)dt, \quad y(0) = y \in \mathbb{R}^d.$$

"An algebraic theory of integration methods" J.C.Butcher (1972) [2]

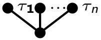
$$\begin{aligned} y_t - y_s &= \int_s^t f(y_{r_1})dr_1 \\ &= \int_s^t \sum_k \frac{1}{k!} f^{(k)}(y_s)(y_{r_1} - y_s)^k dr_1 \\ &= f(y_s)(t-s) + \sum_{k\in\mathbb{N}_+} \int_s^t \frac{1}{k!} f^{(k)}(y_s)(y_{r_1} - y_s)^k dr_1 \\ &= f(y_s)(t-s) \\ &+ \sum_{k\in\mathbb{N}_+} \int_s^t \frac{1}{k!} f^{(k)}(y_s) \left(\int_s^{r_1} \sum_n \frac{1}{n!} f^{(n)}(y_s)(y_{r_2} - y_s)^n dr_2 \right)^k dr_1 \end{aligned}$$

B-series

$$y_t - y_s = f(y_s)(t-s) + (f^{(1)}f)(y_s)\int_s^t \int_s^{r_1} dr_2 dr_1$$
$$+ \frac{1}{2}(f^{(2)}(f,f))(y_s)\int_s^t \left(\int_s^{r_1} dr_2 \int_s^{r_1} dr_2\right) dr_1 + \dots$$

Butcher Series

$$B(a,h,f,y_0) = a(\emptyset)y_0 + \sum_{\tau \in \mathrm{T}} \frac{h^{|\tau|}a(\tau)}{S(\tau)} F_f[\tau](y_0).$$

For a tree $\tau = B_+(\tau_1, \dots, \tau_n) =$ 

, the elementary differential is

$$F_f[\tau] = f^{(n)}(F_f[\tau_1], \dots, F_f[\tau_n]).$$

The symmetry factor is

$$S(\tau) = \prod_j r_j!(S(\tau_j))^{r_j}.$$

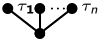
Motivation

The Ordinary Differential Equation (ODE)

$$dy = f(y_t)dt, \quad y(0) = y \in \mathbb{R}. \tag{1}$$

Butcher Series

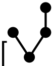
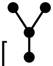
$$B(a, h, f, y_0) = a(\emptyset)y_0 + \sum_{\tau \in \mathrm{T}} \frac{h^{|\tau|} a(\tau)}{S(\tau)} F_f[\tau](y_0).$$

For a tree $\tau = B_+(\tau_1, ..., \tau_n) =$ 

, the elementary differential is

$$F_f[\tau] = f^{(n)}(F_f[\tau_1], ..., F_f[\tau_n]).$$

In 1-dimension (scalar equations), it is not injective from trees to elementary differentials.

$F_f[$  $] = F_f[$  $] = f^2 f^{(1)} f^{(2)}$

Multi-indices

The concept of multi-indices emerged initially within the context of studying singular stochastic partial differential equations by Otto, Sauer, Smith, and Weber [5].

Multi-indice

$$z^{\beta} := \prod_{k \in \mathbb{N}} z_k^{\beta(k)}$$

is a collection of abstract variables $(z_k)_{k \in \mathbb{N}}$.

- z_k: nodes within a rooted tree possessing k children.
- $\beta(k)$: number of nodes possessing k children within a rooted tree.
- We assume finite support for β, i.e., $|\{i \in \mathbb{N} \mid \beta(i) \neq 0\}| < \infty$.

Populated Multi-indices

Given our aim to exclusively examine multi-indices corresponding to non-planar rooted trees, we shall focus on those fulfilling the so-called "population" condition [3].

$$[\beta] := \sum_{k \in \mathbb{N}} (1-k)\beta(k) = |\beta| - \sum_{k \in \mathbb{N}} k\beta(k) = 1.$$

$$|\beta| = \sum_{k \in \mathbb{N}} \beta(k).$$

From a tree point of view, $|\beta|$ corresponds to the number of nodes and the sum $\sum_{j \in \mathbb{N}} j\beta(j)$ corresponds to the number of edges.

 : $[(1,1)] = |(1,1)| - 0 - 1 = 1$

 : $[(1,2)] = |(1,2)| - 0 - 2 = 1.$

Multi-indices B-series

$$B(a,h,f,y) = a(\emptyset)y + \sum_{z^\beta \in \mathrm{M}} \frac{h^{|z^\beta|} a(z^\beta)}{S(z^\beta)} F_f[z^\beta](y),$$

- $\mathrm{M} := \{z^\beta : [\beta] = 1\}$ is the set of populated multi-indices
- $\emptyset$: the empty forest of multi-indices.
- $S(z^\beta)$ is the symmetry factor given by

$$S(z^\beta) := \prod_{k \in \mathbb{N}} (k!)^{\beta(k)}.$$

- $F_f[z^\beta]$: elementary differentials

$$F_f[z^\beta](y) := \prod_{k \in \mathbb{N}} \left(f^{(k)}(y)\right)^{\beta(k)},$$

Connection with local affine equivariant methods

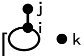
Theorem (Munthe-Kaas and Verdier, 2016 [4])

If a smooth mapping $\varphi : \mathfrak{X}(\mathbb{R}^d) \mapsto \mathfrak{X}(\mathbb{R}^d)$ is local and affine equivariant, then its Taylor development at the zero vector field is an aromatic B-series.

In 1-dimension case, the elementary differentials of aromatic trees collapse to

$$F_f[z^\beta](y) = \prod_{k \in \mathbb{N}} \left(f^{(k)}(y) \right)^{\beta(k)},$$

and all the aromatic trees can be described by the populated multi-indices as they also satisfy the population condition.

$F_f[$ 

$] = f^j f^k \partial_{ij} f^i.$

Connection with local affine equivariant methods

Theorem (Bruned, Ebrahimi-Fard, H. (2025) J. Lond. Math. Soc.)

If a map from $\mathcal{C}^\infty(\mathfrak{X}, \mathfrak{X})$ with $\mathfrak{X} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is local and affine equivariant, then its Taylor development is a multi-indice B-series. Moreover, the choice of the multi-indice B-series is unique.

- Elementary differentials encoded by aromatic trees are factorised by those encoded by multi-indices.
- Uniqueness: Multi-indices elementary differentials are linear independent.

Composition Law

The composition of two multi-indices B-series is defined as

$$B(a, h, f, \cdot) \circ B(b, h, g, y) = B(a, h, f, B(b, h, g, y))$$

Theorem (Bruned, Ebrahimi-Fard, H. (2025) J. Lond. Math. Soc.)

For linear maps a and b with $b(\emptyset) = 1$, the composition of two multi-indices B-series satisfies

$$B(a, h, f, \cdot) \circ B(b, h, f, y) = B(b \star_2 a, h, f, y)$$

where for $z^\mu \in \mathrm{M}$

$$(b \star_2 a)(z^\mu) := (b \otimes a)\Delta_2 z^\mu = b(z^{\mu_1})a(z^{\mu_2})$$

We have the duality

$$< z^\alpha, z^\beta >= \delta_{\alpha,\beta} S(z^\alpha)$$
$$< z^\beta \otimes z^\alpha, \Delta_2 z^\mu >=< z^\beta \star_2 z^\alpha, z^\mu >$$

Composition Law

$$B(a, h, f, \cdot) \circ B(b, h, g, y) = B(a, h, f, B(b, h, g, y))$$

Theorem (Bruned, Ebrahimi-Fard, H. (2025) J. Lond. Math. Soc.)

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$$\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z^{\alpha} := (D^n z^{\alpha}) \prod_{j=1}^{n} z^{\beta_j}, \quad D = z_{k+1}\partial_{z_k}$$

Analogue to the grating of trees.

Composition Law

$$\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z^{\alpha} := (D^n z^{\alpha}) \prod_{j=1}^{n} z^{\beta_j}, \quad D = z_{k+1}\partial_{z_k}$$

Taylor expansion of elementary differentials

$$F_f[z^{\alpha}]\,(B(b,h,f,y)) = \sum_{n\in\mathbb{N}} \frac{1}{n!}\partial^n F_f[z^{\alpha}](y)\,(B(b,h,f,y)-y)^n$$

$$= F_f[z^{\alpha}](y) + \sum_{k\in\mathbb{N}_+} \sum_{z^{\beta_1},\dots,z^{\beta_n}\in \mathrm{M}} \frac{1}{n!}\partial^n F_f[z^{\alpha}](y) \prod_{j=1}^{n} \left(\frac{b(z^{\beta_j})h^{|z^{\beta_j}|}}{S(z^{\beta_j})} F_f[z^{\beta_j}](y) \right).$$

Leibniz rule and Morphism property

$$\partial(f^{(k_1)}f^{(k_2)}\dots f^{(k_l)}) = \sum_{i=1}^{l} f^{(k_1)}f^{(k_2)}\dots f^{(k_l)} \frac{f^{(k_i+1)}}{f^{(k_i)}}$$

$$\partial \equiv D, \quad F_f\left[\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z^{\alpha}\right](y)$$

Composition Law

For $z^{\beta_j} \in \mathrm{M}$ and $z^\alpha \in \mathrm{M}$, define

$$\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z^\alpha := \left(\prod_{j=1}^{n} z^{\beta_j}\right) D^n z^\alpha, \quad D = \sum_{k \in \mathbb{N}} z_{k+1} \partial_{z_k}$$

$$\emptyset \star_2 \emptyset := \emptyset, \quad \emptyset \star_2 z^\alpha := z^\alpha, \quad \text{and} \quad z^\alpha \star_2 \emptyset := z^\alpha.$$

Proposition (morphism property of elementary differentials w.r.t. $\star_2$)

For every $z^{\beta_j} \in \mathrm{M}$ and $z^\alpha \in \mathrm{M}$, one has

$$F_f\left[\tilde{\prod}_{j=1}^{n} z^{\beta_j} \star_2 z^\alpha\right] = \left(\prod_{j=1}^{n} F_f[z^{\beta_j}]\right) \partial^n F_f[z^\alpha].$$

Theorem (Bruned, H. (2025) Trans. Am. Math. Soc.)

$$\Delta_2 z^{\beta} = \emptyset \otimes z^{\beta} + z^{\beta} \otimes \emptyset + \sum_{\substack{\beta=\beta_1+\dots+\beta_n+\hat{\beta} \\ n\in\mathbb{N}^*}} \frac{1}{S_{ext}(\tilde{\prod}_{i=1}^{n} z^{\beta_i})} \tilde{\prod}_{i=1}^{n} z^{\beta_i} \otimes \bar{D}^n z^{\hat{\beta}},$$

Substitution Law

Firstly, the substitution of two multi-indices B-series is defined as:

$$B(a,h,f,\cdot) \circ_s B(b,h,g,y) := B(a,h,B(b,h,g,y),y).$$

Since the substitution is replacing f by $B(b,h,g,y)$, by applying the definition of elementary differentials, one has

$$B(a,h,f,\cdot) \circ_s h^{-1}B(b,h,g,y) = a(\emptyset)y + \sum_{z^\alpha \in \mathrm{M}_0} \frac{a(z^\alpha)h^{|z^\alpha|}}{S(z^\alpha)} \hat{F}_g[z^\alpha](y),$$

where

$$\hat{F}_g[z^\alpha](y) = h^{-|z^\alpha|} \prod_{k\in\mathbb{N}} \left(\partial^k B(b,h,g,y)\right)^{\alpha(k)}.$$

Morphism property

Replace f by $B(b,h,g,y) \equiv$ Replace z_k by $\sum_{z^\beta} D^k z^\beta$.

Substitution Law

$$F_{B(g),f}\left[z^\beta \blacktriangleright z^\alpha\right] = \sum_{k\in\mathbb{N}} \alpha(k) F_f[z^\alpha] \frac{\partial^k F_g[z^\beta]}{f^{(k)}} = \sum_{k\in\mathbb{N}} \alpha(k) F_f[z^\alpha] \frac{F_g[D^k z^\beta]}{F_f[z_k]}$$

Definition

The insertion of $z^\beta \in \mathrm{M}$ into $z^\alpha \in \mathrm{M}$ is defined to be

$$z^\beta \blacktriangleright z^\alpha := \sum_{k\in\mathbb{N}} \left(D^k z^\beta\right)\left(\partial_{z_k} z^\alpha\right).$$

The simultaneous insertion of $z^{\beta_j} \in \mathrm{M}_0$ into $z^\alpha \in \mathrm{M}_0$ is

$$\tilde{\prod}_{j=1}^n z^{\beta_j} \star_1 z^\alpha := \sum_{k_1,\dots,k_n\in\mathbb{N}} \left(\prod_{j=1}^n D^{k_j} z^{\beta_j}\right)\left[\left(\prod_{j=1}^n \partial_{z_{k_j}}\right) z^\alpha\right].$$

and we also require $n = |z^\alpha|$ since all the $f^{(k)}$ should be replaced.

Substitution Law

Theorem (Bruned, Ebrahimi-Fard, H. (2025) J. Lond. Math. Soc.)

If $b(\emptyset) = 0$ and $(b \star_1 a)(\emptyset)$ is set to be $a(\emptyset)$, one has

$$B(a, h, f, y) \circ_s h^{-1} B(b, h, g, y) = B(b \star_1 a, h, g, y)$$

where

$$(b \star_1 a)(z^\gamma) := (b \otimes a)(\Delta_1 z^\gamma)$$

and the duality is

$$< b \star a, \Delta_1 z^\gamma > = < b \otimes a, \Delta_1 z^\gamma > .$$

- The proof is similar to the composition law and uses the morphism property
- The substitution law is closely related to renormalisation in singular SPDEs and QFT measures.

Theorem (Bruned, H. (2025) Trans. Am. Math. Soc.)

$$\Delta_1 z^\beta = \sum_{\tilde{\prod}_{i=1}^n z^{\beta_i} \in \mathcal{F}} \sum_{\substack{k_1,\dots,k_n \in \mathbb{N} \\ \prod_i z_{k_i} \in \mathcal{M}}} E\Big(\tilde{\prod}_{i=1}^n z^{\beta_i}, z^\alpha, z^\beta\Big) \tilde{\prod}_{i=1}^n z^{\beta_i} \otimes z^\alpha,$$

with

$$E\Big(\tilde{\prod}_{i=1}^n z^{\beta_i}, z^\alpha, z^\beta\Big) = \sum_{\beta = \hat{\beta}_1 + \cdots + \hat{\beta}_n} \frac{\alpha!}{S(z^\alpha) S_{ext}(\tilde{\prod}_{i=1}^n z^{\beta_i})} \prod_{i=1}^n \frac{\langle z^{\beta_i}, \bar{D}^{k_i} z^{\hat{\beta}_i} \rangle}{S(z^{\beta_i})}$$

where $z^\alpha := \prod_{i=1}^n z_{k_i}$ *and we have order among* $\hat{\beta}_1, \cdots, \hat{\beta}_n$. *Note that* $n \geq 1$ *since inserting the empty multi-indice is forbidden.*

Summary

- Multi-indices yield the unique Taylor development for scalar Butcher series.
- Morphism properties of elementary differentials with respect to $\star_2$ and $\star_1$ link the analytical objects with multi-indices.
- The duality between products and coproducts play an important role.
- This framework is also important in renormalisation of singular SPDEs and QFT measures.

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Thank you for your attention!