An optimal control based numerical method for transmission problems with sign-changing coefficients

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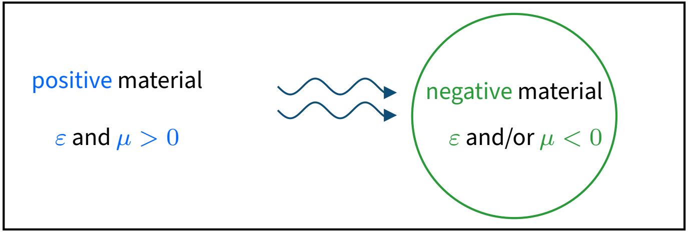
47th National Congress on Numerical Analysis, June 1-6, 2026

Introduction

Negative materials

- The scattering of electromagnetic waves by a negative material at frequency ω.



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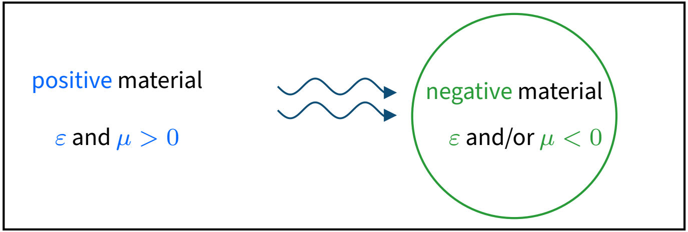

- Examples of negative materials:
  - Metals, such as silver or gold, at optical frequencies ($\varepsilon < 0$ and $\mu > 0$).
  - Negative metamaterials for well-chosen frequency ranges ($\varepsilon < 0$ and $\mu < 0$).

Introduction

Applications

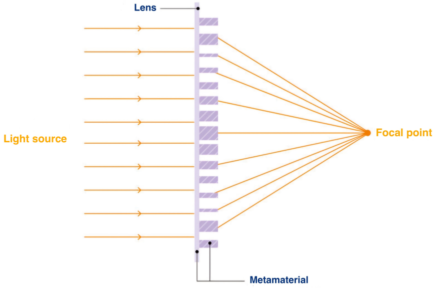

Figure 1: Superlens

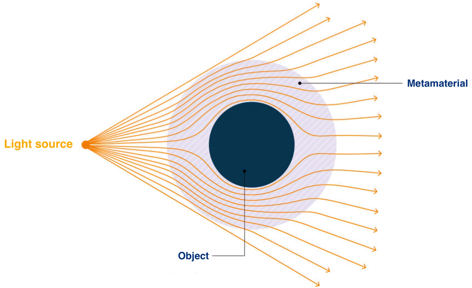

Figure 2: Invisibility cloack

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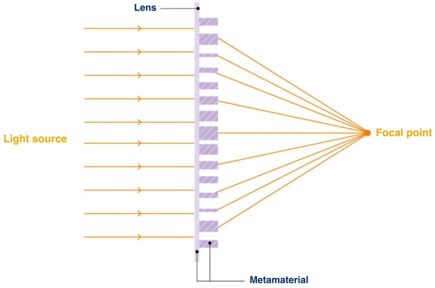

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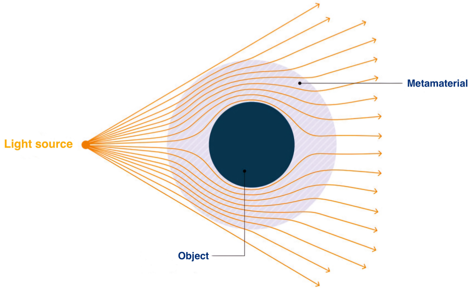

Figure 2: Invisibility cloack

- Combining a positive material with a negative material is essential for many applications.

Helmholtz problem

Theoretical difficulty

- Given $\omega \in \mathbb{R}$ and $g \in \mathrm{L}^2(\Omega)$, we introduce the Helmholtz problem:

$$\begin{cases} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\varepsilon \nabla u) - \omega^2 \mu\, u = g \qquad \text{in } \Omega. \end{cases}$$

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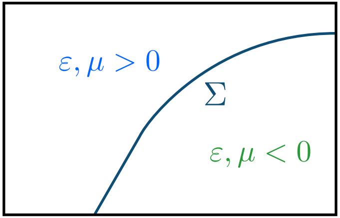

$\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$

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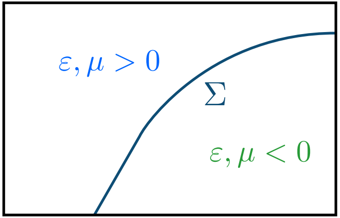

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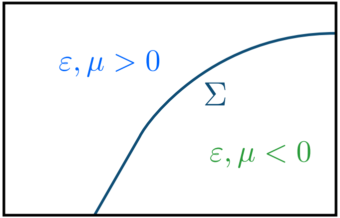

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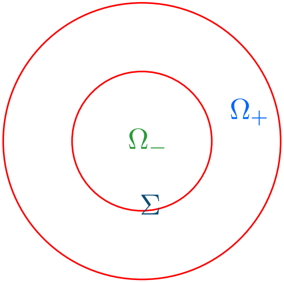
- Assume that $\kappa_\varepsilon := \frac{\varepsilon_-}{\varepsilon_+} \neq -1$. The problem for u is well-posed in the Hadamard sense for almost all frequencies.



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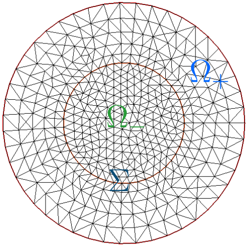
Discretization with conforming Lagrange finite elements:



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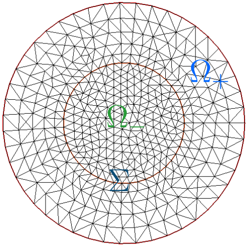

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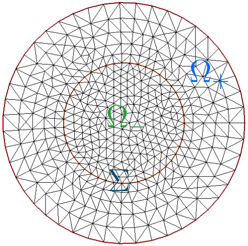

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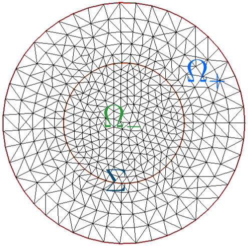

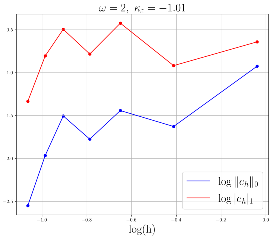

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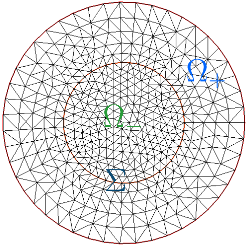

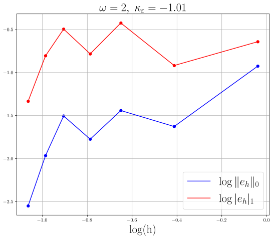

- Instability of the method was established by M. Halla, R. Oberender (2025).

Helmholtz problem

Existing approaches for the numerical approximation

- Many numerical methods have been proposed to solve:

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- **Discrete T-coercivity method**
  - L. Chesnel and P. Ciarlet Jr (2013), A.-S. Bonnet-Ben Dhia, C. Carvalho, P. Ciarlet Jr (2018).
- **Optimal control reformulation method with a surface control** $(\omega = 0)$
  - A. Abdulle, Huber, S. Lemaire (2017), A. Abdulle, S. Lemaire (2024).
- **Optimal control reformulation method with a volume control** $(\omega = 0)$
  - P. Ciarlet, D. Lassounon, M. Rihani (2023).
- **A hybridized Nitsche method** $(\omega = 0)$
  - E. Burman, J. Preuss, A. Ern (2025).
- **Explicit T-coercivity method**
  - M. Halla, T. Hohage, F. Oberender (2024).

Helmholtz problem

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Stable numerical method for Helmholtz problem with two sign-changing coefficients.

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Stable numerical method for Helmholtz problem with two sign-changing coefficients.

Solve the problem using the Volume Optimal Control (VOC) method.

Outline of the Talk

1. An optimal control reformulation

2. Discretization strategy

3. Numerical experiments

4. Conclusion and perspectives

An optimal control reformulation

VOC for the Helmholtz problem

First decomposition of the problem

$$\begin{cases} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\varepsilon \nabla u) - \omega^2 \mu\, u = g \quad \text{in } \mathrm{L}^2(\Omega), \qquad (P_u). \end{cases}$$

- **Assumption:** The problem (P_u) has a unique solution for the given source term.

VOC for the Helmholtz problem

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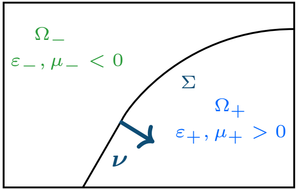
- **Assumption:** The problem (P_u) has a unique solution for the given source term.
- Split the unknown u into (u_+, u_-) with

$$u_i = u|_{\Omega_i}, \quad u_i \in \mathrm{H}^1(\Omega_i), \quad u_i = 0 \text{ on } \partial\Omega_i \setminus \Sigma, \quad i = \pm.$$



$$\Omega := \Omega_+ \cup \Sigma \cup \Omega_-$$

VOC for the Helmholtz problem

Inria

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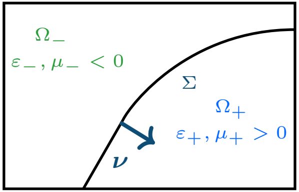
- The coupling conditions at the interface are:

$$\left| \begin{array}{ll} u_+ = u_- & \text{in } \mathrm{H}_{00}^{1/2}(\Sigma), \\ \varepsilon_+ \nabla u_+ \cdot \nu = \varepsilon_- \nabla u_- \cdot \nu & \text{in } (\mathrm{H}_{00}^{1/2}(\Sigma))^*. \end{array} \right.$$



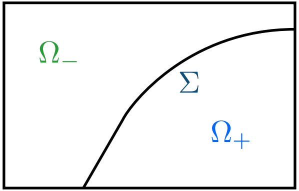
$\Omega := \Omega_+ \cup \Sigma \cup \Omega_-$

VOC for the Helmholtz problem

Toward an optimal control problem

- We apply the smooth extension method to (P_u):

  Look for $(E(u_+), u_-)$ instead of (u_+, u_-),
  where $E(u_+)$ is a continuous extension in Ω_-.



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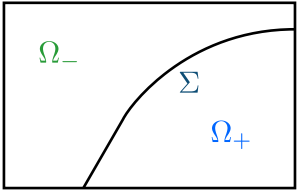

How to find a suitable extension of u_+?

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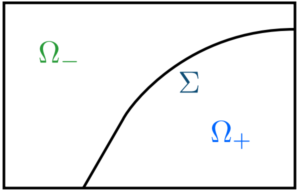

How to find a suitable extension of u_+?

Defined using an optimal control reformulation of (P_u).

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- Let $\tilde{\varepsilon}$ and $\tilde{\mu}$ be some extensions of ε_+ and μ_+ to Ω such that:

$$0 < c < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\mu} < -c \quad \text{in } \Omega_-, \text{ for some } c > 0.$$

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- With this choice: $\tilde{\varepsilon} > 0$ in Ω ; and one can define the inner product $\langle \cdot, \cdot \rangle_{\Omega_-}$ such that:

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$$\Bigg| \int_{\Omega} \tilde{\varepsilon} \nabla E(u_+) \cdot \nabla v - \omega^2 \int_{\Omega} \tilde{\mu}\, E(u_+)\, v \quad = \int_{\Omega_+} g_+\, v + \langle E(u_+), v \rangle_{\Omega_-} - \langle \varepsilon_+ \nabla u_+ \cdot \nu, v \rangle_{\Sigma}$$

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coercive + compact

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$$\left| \begin{aligned} \int_{\Omega} \tilde{\varepsilon} \nabla E(u_+) \cdot \nabla v - \omega^2 \int_{\tilde{\Omega}} \tilde{\mu}\, E(u_+)\, v &= \int_{\Omega_+} g_+\, v + \langle E(u_+), v \rangle_{\Omega_-} - \underline{\langle \varepsilon_+ \nabla u_+ \cdot \nu, v \rangle_{\Sigma}}, \\ \int_{\Omega_-} \varepsilon_- \nabla u_- \cdot \nabla v_- - \omega^2 \int_{\Omega_-} \mu_-\, u_-\, v_- &= \int_{\Omega_-} g_-\, v_- + \underline{\langle \varepsilon_+ \nabla u_+ \cdot \nu, v_- \rangle_{\Sigma}}. \end{aligned} \right.$$

VOC for the Helmholtz problem

Toward an optimal control problem

- Let $\tilde{\varepsilon}$ and $\tilde{\mu}$ be some extensions of ε_+ and μ_+ to Ω such that:

$$0 < c < \tilde{\varepsilon} \quad \text{and} \quad \tilde{\mu} < -c \quad \text{in } \Omega_-, \text{ for some } c > 0.$$

- With this choice: $\tilde{\varepsilon} > 0$ in Ω ; and one can define the inner product $\langle \cdot, \cdot \rangle_{\Omega_-}$ such that:

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- For each $E(u_+) \in \mathrm{H}^1_0(\Omega)$, $\exists ! z_{E(u_+)} \in \mathrm{H}^1_{0,\partial\Omega_- \setminus \Sigma}(\Omega_-)$ such that:

$$\langle z_{E(u_+)}, v_- \rangle_{\Omega_-} = \overbrace{\langle E(u_+), v_- \rangle_{\Omega_-} - \langle \varepsilon_+ \, \nabla u_+ \cdot \nu, v_- \rangle_{\Sigma}}.$$

VOC for the Helmholtz problem

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VOC for the Helmholtz problem

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- Assuming uniqueness for both variational formulations from now on.

VOC for the Helmholtz problem

Toward an optimal control problem

- Given $z_- \in \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)$, we define $(u^{z_-}, u^{z_-}_-) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)$ such that:

$$\left|\begin{array}{ll} \displaystyle\int_\Omega \tilde{\varepsilon}\,\nabla u^{z_-}\cdot\nabla v - \omega^2\int_\Omega \tilde{\mu}\,u^{z_-}\,v & = \displaystyle\int_{\Omega_+} g_+\,v + \langle z_-, v\rangle_{\Omega_-}, \\ \displaystyle\int_{\Omega_-} \varepsilon_-\,\nabla u^{z_-}_-\cdot\nabla v_- - \omega^2\int_{\Omega_-} \mu_-\,u^{z_-}_-\,v_- & = \displaystyle\int_{\Omega_-} g_-\,v_- + \langle u^{z_-} - z_-, v_-\rangle_{\Omega_-}. \end{array}\right. \qquad (P_J)$$

VOC for the Helmholtz problem

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- $\forall z_- \in \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)$, we define $(u^{z_-}, u^{z_-}_-) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)$ such that:

$$\left|\begin{array}{ll} -\mathrm{div}(\varepsilon_+\nabla u^{z_-}{}_{|\Omega_+}) - \omega^2\,\mu_+\, u^{z_-}{}_{|\Omega_+} = g|_{\Omega_+} & \text{in } \Omega_+, \\ -\mathrm{div}(\varepsilon_-\nabla u^{z_-}_-) - \omega^2\,\mu_-\, u^{z_-}_- = g|_{\Omega_-} & \text{in } \Omega_-, \\ \varepsilon_+\,\nabla u^{z_-}{}_{|\Omega_+}\cdot\nu = \varepsilon_-\,\nabla u^{z_-}_-\cdot\nu & \text{on } \Sigma. \end{array}\right.$$

VOC for the Helmholtz problem

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- Given $z_- \in \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)$, we define $(u^{z_-}, u^{z_-}_-) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)$ such that:

$$\left|\begin{aligned} \int_\Omega \tilde{\varepsilon}\,\nabla u^{z_-}\cdot\nabla v - \omega^2\int_\Omega \tilde{\mu}\,u^{z_-}\,v &= \int_{\Omega_+} g_+\,v + \langle z_-, v\rangle_{\Omega_-}, \\ \int_{\Omega_-} \varepsilon_-\,\nabla u^{z_-}_-\cdot\nabla v_- - \omega^2\int_{\Omega_-} \mu_-\,u^{z_-}_-\,v_- &= \int_{\Omega_-} g_-\,v_- + \langle u^{z_-} - z_-, v_-\rangle_{\Omega_-}. \end{aligned}\right. \qquad (P_J)$$

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⚠ $u^{z_-}|_{\Omega_+} = u^{z_-}_-$ on Σ **is missing!**

VOC for the Helmholtz problem

Optimal control reformulation

How to choose z_- such that $u^{z_-}|_{\Omega_+} = u_-^{z_-}$ on Σ?

VOC for the Helmholtz problem

Optimal control reformulation

How to choose z_- such that $u^{z_-}|_{\Omega_+} = u_-^{z_-}$ on Σ?

- We consider the optimal control problem:

$$\min_{z_- \in \mathrm{H}^1_{0,\partial\Omega_-\setminus\Sigma}(\Omega_-)} J(z_-) := \frac{1}{2}\|u^{z_-} - u_-^{z_-}\|^2_{\mathrm{L}^2(\Sigma)}, \quad (P_{oc})$$

VOC for the Helmholtz problem

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- The problem (P_{oc}) has an infinite number of solutions.

VOC for the Helmholtz problem

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How to choose z_- such that $u^{z_-}|_{\Omega_+} = u_-^{z_-}$ on Σ?

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- For uniqueness, one considers the regularized optimal control problem $(\lambda > 0)$:

$$\min_{z_- \in \mathrm{H}^1_{0,\partial\Omega_- \setminus \Sigma}(\Omega_-)} J^\lambda(z_-) := \frac{1}{2} \| u^{z_-} - u_2^{z_-} \|^2_{\mathrm{L}^2(\Sigma)} + \lambda \| z_- \|^2_{\Omega_-}, \quad (P^\lambda_{oc})$$

Discretization strategy

VOC for the Helmholtz problem

Discretization by conforming Lagrange finite elements

1. $\mathrm{V}_h(\Omega) := \{v_h \in \mathrm{H}^1_0(\Omega), \,|\, v_{h|T} \in \mathrm{P}^k(T), \forall T \in \mathcal{T}_h\}.$
2. $\mathrm{V}_h(\Omega_i)_{i=\pm} := \{v_{i,h} \in \mathrm{H}^1_{0,\partial\Omega_i\setminus\Sigma}(\Omega_i) \,|\, v_{i,h|T} \in \mathrm{P}^k(T), \forall T \in \mathcal{T}^i_h\}.$

VOC for the Helmholtz problem

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- One considers the discrete regularized optimal control problem ($\lambda_h > 0$):

$$\min_{z_{-,h} \in \mathrm{V}_h(\Omega_-)} J^{\lambda_h}(z_{-,h}) := \frac{1}{2}\|u_h^{z_{-,h}} - u_{-,h}^{z_{-,h}}\|^2_{\mathrm{L}^2(\Sigma)} + \lambda_h \|z_{-,h}\|^2_{\Omega_-}, \quad (P_{oc}^{\lambda_h})$$

VOC for the Helmholtz problem

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- We denote by $z^\sharp_{-,h}$ the unique solution of $(P_{oc}^{\lambda_h})$.

VOC for the Helmholtz problem

Convergence result

- **First shift value:** Assume that $u|_{\Omega_i} \in \mathrm{H}^{1+\tau}(\Omega_i),\ i = \pm$, for some $\tau > 0$.

Theorem

Taking the regularization parameters as

$$\lambda_h = C\,h^q, \qquad C > 0, \qquad q \in (0, 2\tau + \sigma),$$

where $\sigma \in (0,1]$ denotes the ***second shift value****, ensures that:*

$$(u_h^{z^\sharp_{-,h}}|_{\Omega_+}, u_{-,h}^{z^\sharp_{-,h}}) \text{ converges to } (u_+, u_-) \text{ as } h \to 0.$$

VOC for the Helmholtz problem

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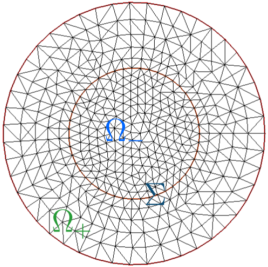
- For the circular interface case presented earlier: $\tau = \sigma = 1$.

Numerical experiments

VOC for the Helmholtz problem

Numerical results: Circular interface

- We set $\omega = 2$, $\varepsilon_+ = 1$, $\mu_+ = 2$, $\mu_- = -1$ and we let $\varepsilon_- < 0$ vary.
  We choose $\tilde{\varepsilon}|_{\Omega_-} = 10$, $\tilde{\mu}|_{\Omega_-} = -10$.
- We study the behavior of the error $e_h := u - u_h$ with respect to h.



VOC for the Helmholtz problem

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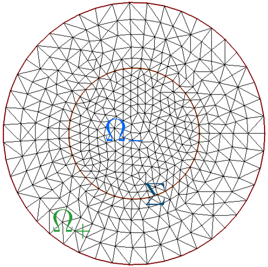
- Assume that $\kappa_\varepsilon = \dfrac{\varepsilon_-}{\varepsilon_+} \neq -1$. (P_u) is well-posed in the Hadamard sense for almost all frequencies.



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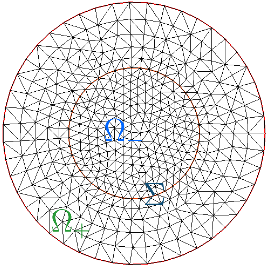

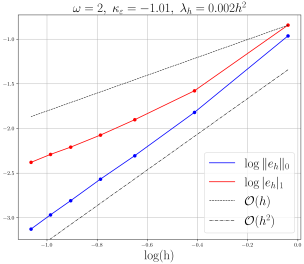

VOC for the Helmholtz problem

Numerical results: Interface with corner

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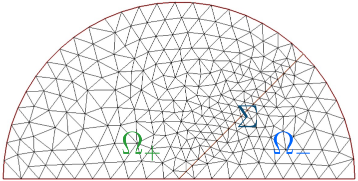

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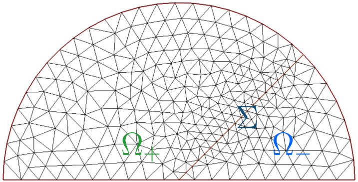
- Assume that $\kappa_\varepsilon = \dfrac{\varepsilon_-}{\varepsilon_+} \notin [-3; -\dfrac{1}{3}]$. (P_u) is well-posed in the Hadamard sense for almost all frequencies.



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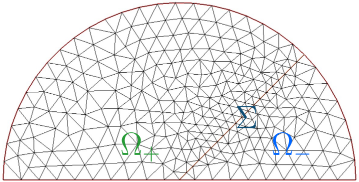

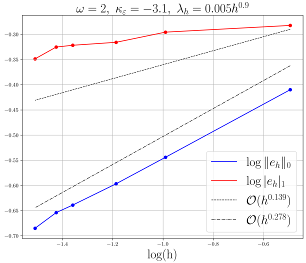

Conclusion and perspectives

Conclusion and perspectives

Conclusion:

- Efficient method: no conditions on solution regularity or mesh.
- General framework applicable to: Helmholtz problem and Maxwell problem.
- Works with elliptic tensor coefficients ε and μ.

F. Chaaban, Unconditionally stable numerical methods for solving transmission problems with sign-changing coefficients. PhD thesis, Institut Polytechnique de Paris (2025).

Perspectives:

- Optimize λ_h in order to accelerate the method's convergence.
- Estimate the convergence rate of the method.

Thank you!