Asymptotic modeling of a random rough thin layer for electromagnetic wave scattering

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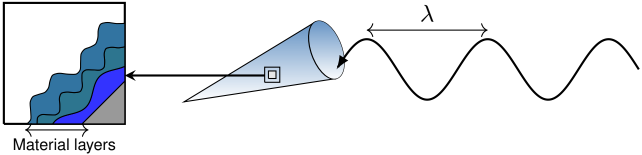
Pierre Boulogne [1 2] Sonia Fliss [2] Laure Giovangigli [2] Justine Labat [1]

[1] CEA-CESTA

[2] POEMS, CNRS, Inria, ENSTA, Institut Polytechnique de Paris, Palaiseau, France

Context and motivation

- Context: Stacks of thin layers are used to improve electromagnetic stealth of objects they cover. Small surface defects may affect this stealth, hence their impact must be studied.



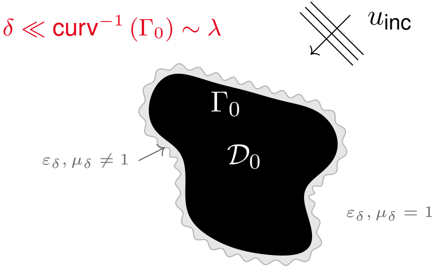
- Challenge: High computational costs due to the strongly multi-scale problem
- Proposed solution: Derivation of effective conditions replacing the complete stack which take roughness into account

Thin layers: Bendali and Lemrabet (1996), Duruflé, Haddar, and Joly (2006)

Periodic setting: Abboud and Ammari (1996), Ammari and He (1998), Madureira and Valentin (2007), Delourme and Claeys (2012), Chamaillard (2016)

Random setting: Gérard-Varet (2008), Boucart (2023)

Model problem



- The total field u_δ satisfies the 2D Helmholtz equation

$$\begin{cases} -\text{div}\left(\dfrac{1}{\mu_\delta}\nabla u_\delta\right) - \kappa_0^2 \varepsilon_\delta u_\delta = 0 & \text{in } \mathbb{R}^2 \setminus \mathcal{D}_0 \\ u_\delta = 0 & \text{on } \Gamma_0 = \partial\mathcal{D}_0 \\ u_\delta - u_{\text{inc}} & \text{is outgoing} \end{cases}$$

Local parametrization

$$\varepsilon_\delta(s,\nu) = \varepsilon\left(\frac{s}{\delta}, \frac{\nu}{\delta}\right)$$

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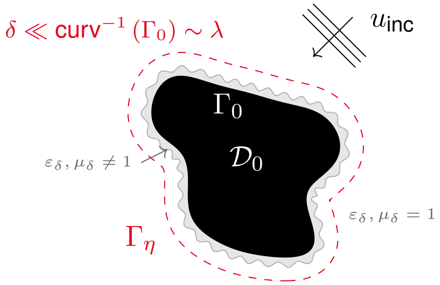

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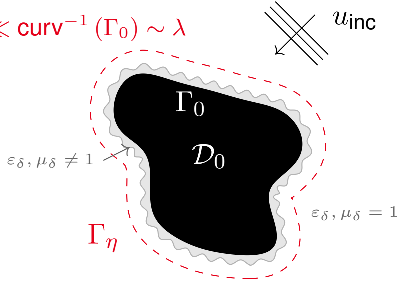
$\delta \ll \text{curv}^{-1}(\Gamma_0) \sim \lambda$



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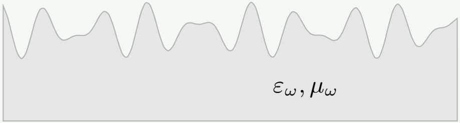
Random Case

$(X,\omega) \mapsto (\varepsilon,\mu)_\omega(X,\cdot)$ is **stationary** ergodic



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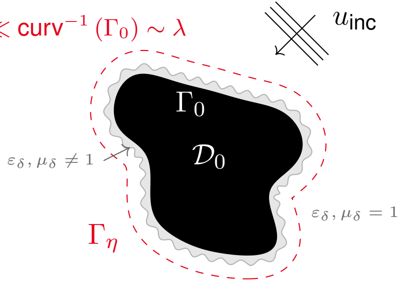

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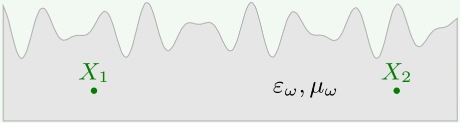

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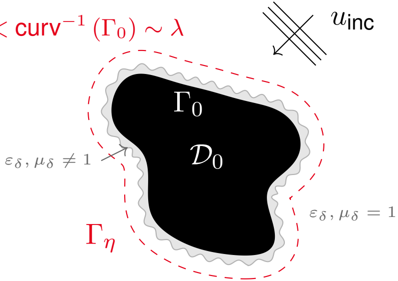

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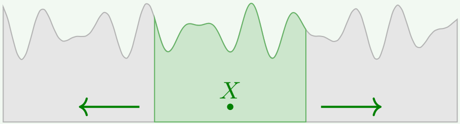

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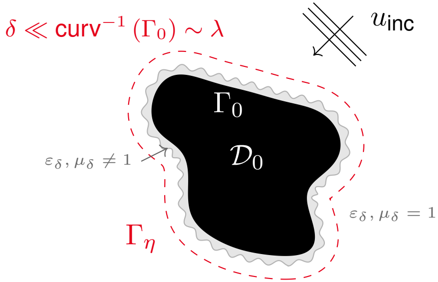

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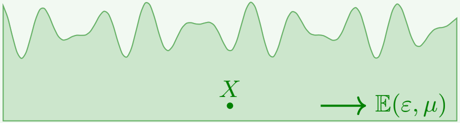

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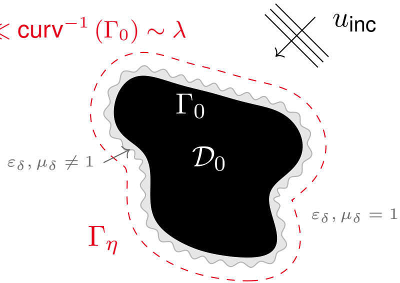

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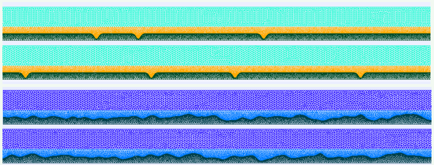


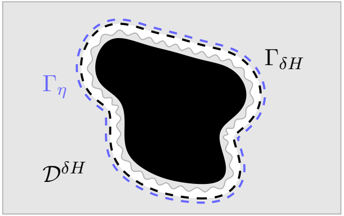
Randomly perturbated periodic

Generic random

Multi-scale expansions

- Near-field ansatz, $0 < \nu < \delta H$

$$u_{\omega,\delta}(s,\nu) = \sum_{n=0}^{\infty} \delta^n u_{n,\omega}^{\mathsf{NF}}\left(s; \frac{s}{\delta}, \frac{\nu}{\delta}\right)$$



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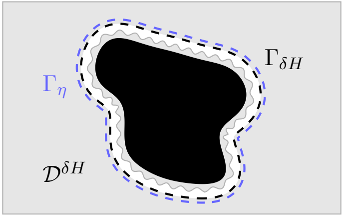
- Far-field ansatz, $\delta H < \nu < \eta$

$$u_{\omega,\delta}(s,\nu) = \sum_{n=0}^{\infty} \delta^n \left[u_{n,\omega}^{\mathsf{NF}} \left(s; \frac{s}{\delta}, \frac{\nu}{\delta}\right) + u_{n,\omega}^{\mathsf{FF}}(s,\nu)\right]$$



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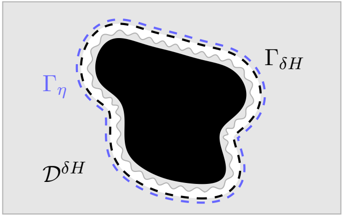

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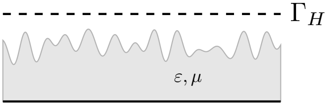
$\lim_{Y\to+\infty} u^{\mathsf{NF}}_{n,\omega}(\cdot;\cdot,Y) = 0$ and $(X,\omega) \mapsto u^{\mathsf{NF}}_{n,\omega}(\cdot;X,\cdot)$ is stationary ergodic



Far-field problems

- The far field terms satisfy the Helmholtz equation

$$\textbf{(FFn)} \begin{cases} -\Delta u^{\mathsf{FF}}_{n,\omega} - \kappa_0^2 u^{\mathsf{FF}}_{n,\omega} = 0 & \text{in } \mathcal{D}^{\delta H} \\ u^{\mathsf{FF}}_{n,\omega} - \delta_{n,0} u_{\mathsf{inc}} & \text{is outgoing} \end{cases}$$

A condition on $\Gamma_{\delta H}$ is missing

Far-field problems

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$$\textbf{(FFn)} \begin{cases} -\Delta u_{n,\omega}^{\text{FF}} - \kappa_0^2 u_{n,\omega}^{\text{FF}} = 0 & \text{in } \mathcal{D}^{\delta H} \\ u_{n,\omega}^{\text{FF}} - \delta_{n,0} u_{\text{inc}} & \text{is outgoing} \end{cases}$$

A condition on $\Gamma_{\delta H}$ is missing

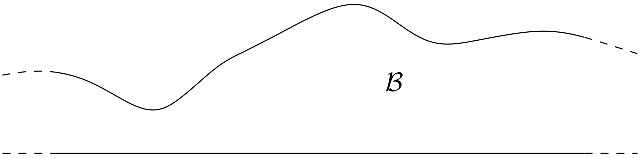
Condition on $\Gamma_{\delta H}$

This condition will be obtained by ensuring the existence of near-field terms satisfying our assumptions

Near-field problems

- The near fields satisfy the problems parametrized by s:

$$
\textbf{(NFn)} \begin{cases} -\nabla_{X,Y}\left(\frac{1}{\mu_\omega}\nabla_{X,Y} u^{\text{NF}}_{n,\omega}(s;\cdot,\cdot)\right) = \mathcal{F}_s\left[\left(u^{\text{NF}}_{i,\omega}(s;\cdot,\cdot)\right)_{i\leq n-1}\right] & \text{in } \mathcal{B} := \mathbb{R}\times\mathbb{R}_+ \\ \left[u^{\text{NF}}_{n,\omega}\right]_{\Gamma_H}(s) = -u^{\text{FF}}_{n,\omega}|_{\Gamma_{\delta H}}(s) \qquad \left[\partial_Y u^{\text{NF}}_{n,\omega}\right]_{\Gamma_H}(s) = -\partial_\nu u^{\text{FF}}_{n-1,\omega}|_{\Gamma_{\delta H}}(s) & \\ u^{\text{NF}}_{n,\omega}(s;\cdot,\cdot) = 0 \qquad \text{on } \Sigma & \\ (\omega, X) \mapsto u^{\text{NF}}_{n,\omega}(s; X,\cdot) \text{ is stationary} & \end{cases}
$$

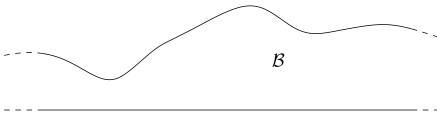


Near-field problems - Random case

- The general corrector problem is:

(Boucart (2023))

$$
\textbf{(NF)}\begin{cases}
-\,\mathrm{div}\left(\dfrac{1}{\mu_\omega}\nabla U_\omega\right) = F_\omega & \text{in } \mathcal{B} \\
[U_\omega]_{\Gamma_H} = \alpha_\omega^D \qquad [\partial_Y U_\omega]_{\Gamma_H} = \alpha_\omega^N & \\
U_\omega = 0 & \text{on } \Sigma \\
(\omega, X) \mapsto U_\omega(X, \cdot) & \text{is stationary}
\end{cases}
$$



The Dirichlet condition allows the use of a Hardy inequality in

$$
V_0 := \Big\{\text{a.s.} \quad v_\omega \in H^1_{\text{loc}}\left(\overline{\mathcal{B}}\right), v_\omega|_\Sigma = 0, (\omega, X) \mapsto v_\omega(X,\cdot) \text{ is stationary}
$$

$$
\text{and } \mathbb{E}\left[\int_0^{+\infty} |\nabla v|^2 + \frac{|v|^2}{1+Y^2}\right] < +\infty \Big\}
$$

in the form

$$
\mathbb{E}\left[\int_0^{\infty} \frac{|v|^2}{1+Y^2}\right] \lesssim \mathbb{E}\left[\int_0^{\infty} |\nabla v|^2\right]
$$

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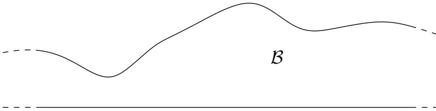

Provided that $\sqrt{1+Y^2}F_\omega \in L^2\left(\Omega, L^2(\mathcal{B})\right)$, there exists a unique solution in $V_0 + \alpha_\omega^D \chi_{Y>H}$.

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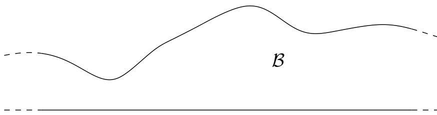

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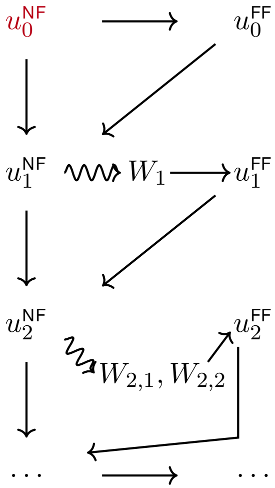
When $F_\omega = 0$, $\lim\limits_{Y\to\infty} U_\omega = 0$ iff a relation between α_ω^D and α_ω^N is satisfied.
In this case, we can simply show that:

$$\mathbb{E}\left[|U_\omega|^2 + Y^2|\nabla U_\omega|^2\right] \underset{Y\to\infty}{\to} 0$$

Formal recursive construction



- Problem satisfied by $u_{0,\omega}^{\mathsf{NF}}$:

$$\begin{cases} -\,\mathrm{div}_{X,Y}\left(\frac{1}{\mu_\omega}\nabla_{X,Y} u_{0,\omega}^{\mathsf{NF}}\right) = 0 \qquad \text{in } \mathcal{B} \\ \left[u_{0,\omega}^{\mathsf{NF}}\right]_{\Gamma_H} = -u_{0,\omega}^{\mathsf{FF}}|_{\Gamma_{\delta H}} \qquad \left[\partial_Y u_{0,\omega}^{\mathsf{NF}}\right]_{\Gamma_H} = 0 \\ u_{0,\omega}^{\mathsf{NF}} = 0 \qquad \text{on } \Sigma \end{cases}$$

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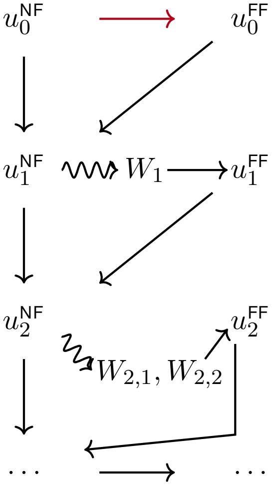

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Solution of (NF0)

$$u_{0,\omega}^{\mathsf{NF}} + u_{0,\omega}^{\mathsf{FF}}|_{\Gamma_{\delta H}} \chi_{Y>H} = 0$$

$$Y \to \infty \implies u_{0,\omega}^{\mathsf{FF}} = 0 \qquad \text{on } \Gamma_{\delta H}$$

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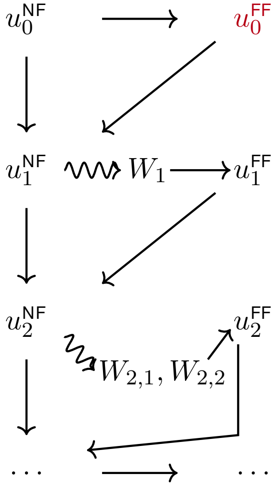

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Solution of (NF0)

$$u_{0,\omega}^{\mathsf{NF}} + u_{0,\omega}^{\mathsf{FF}}|_{\Gamma_{\delta H}}\chi_{Y>H} = 0$$

$$Y \to \infty \implies u_{0,\omega}^{\mathsf{FF}} = 0 \qquad \text{on } \Gamma_{\delta H}$$

$\implies u_{0,\omega}^{\mathsf{NF}} = 0$ and $u_{0,\omega}^{\mathsf{FF}}$ satisfies:

$$\text{Deterministic} \leftarrow \begin{cases} -\Delta u_{0,\omega}^{\mathsf{FF}} - \kappa_0^2 u_{0,\omega}^{\mathsf{FF}} = 0 & \text{in } \mathcal{D}^{\delta H} \\ u_{0,\omega}^{\mathsf{FF}} = 0 & \text{on } \Gamma_{\delta H} \\ u_{0,\omega}^{\mathsf{FF}} - u_{\mathsf{inc}} & \text{is outgoing} \end{cases}$$

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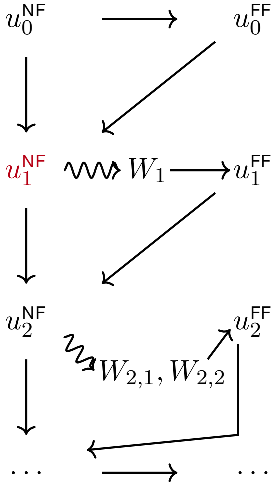

- Problem satisfied by $u_{1,\omega}^{\text{NF}}$:

$$\begin{cases} -\operatorname{div}_{X,Y}\left(\dfrac{1}{\mu_\omega}\nabla_{X,Y} u_{1,\omega}^{\text{NF}}\right) = {\color{red}0} & \text{in } \mathcal{B} \\ \left[u_{1,\omega}^{\text{NF}}\right]_{\Gamma_H} = -u_{1,\omega}^{\text{FF}}|_{\Gamma_{\delta H}} & \left[\partial_Y u_{1,\omega}^{\text{NF}}\right]_{\Gamma_H} = -\partial_\nu u_{0,\omega}^{\text{FF}}|_{\Gamma_{\delta H}} \\ u_{1,\omega}^{\text{NF}} = 0 & \text{on } \Sigma \end{cases}$$

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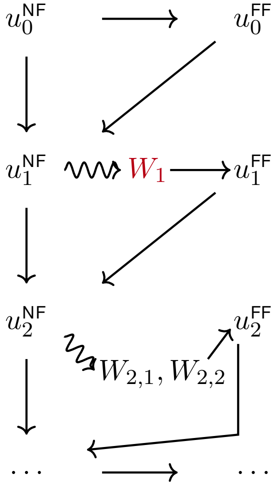

- We can write $\left(u_{1,\omega}^{NF} + u_{1,\omega}^{FF}|_{\Gamma_{\delta H}} \chi_{Y>H}\right) = \partial_\nu u_{0,\omega}^{FF}|_{\Gamma_{\delta H}} W_{1,\omega}$ where $W_{1,\omega}$ is solution of:

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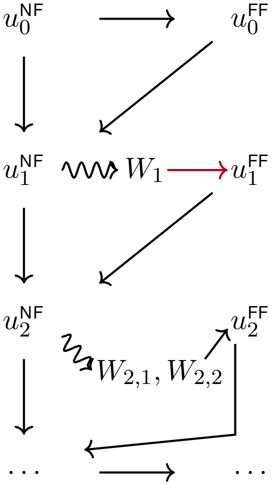


- We can write $\left(u_{1,\omega}^{NF} + u_{1,\omega}^{FF}|_{\Gamma_{\delta H}} \chi_{Y>H}\right) = \partial_\nu u_{0,\omega}^{FF}|_{\Gamma_{\delta H}} W_{1,\omega}$ where $W_{1,\omega}$ is solution of:

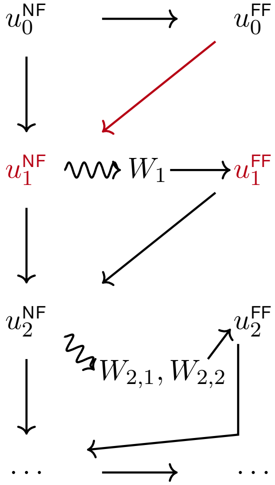
$$\begin{cases} -\operatorname{div}_{X,Y}\left(\frac{1}{\mu_\omega}\nabla_{X,Y} W_{1,\omega}\right) = 0 & \text{in } \mathcal{B} \\ [W_{1,\omega}]_{\Gamma_H} = 0 \qquad [\partial_Y W_{1,\omega}]_{\Gamma_H} = -1 & \\ W_{1,\omega} = 0 & \text{on } \Sigma \end{cases}$$

Boundary Condition

$$u_{1,\omega}^{NF}(\cdot, Y) \underset{Y\to\infty}{\longrightarrow} 0 \implies u_{1,\omega}^{FF}|_{\Gamma_{\delta H}} = \partial_\nu u_{0,\omega}^{FF}|_{\Gamma_{\delta H}} \lim_{Y\to\infty} W_{1,\omega}(\cdot, Y)$$

$$c_1 := \lim_{Y\to\infty} W_{1,\omega}(\cdot, Y) = \mathbb{E}\left[W_{1,\omega}(\cdot, H_\infty)\right]$$

Formal recursive construction

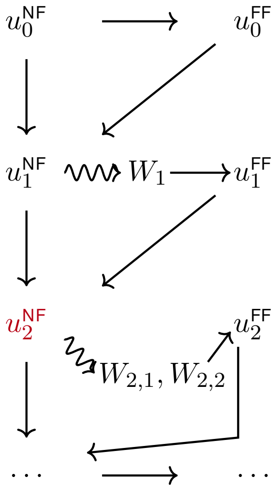


- u_1^{FF} satisfies:

$$\text{Deterministic} \leftarrow \begin{cases} -\Delta u_1^{\mathsf{FF}} - \kappa_0^2 u_1^{\mathsf{FF}} = 0 & \text{in } \mathcal{D}^{\delta H} \\ u_1^{\mathsf{FF}} = c_1 \partial_\nu u_0^{\mathsf{FF}} & \text{on } \Gamma_{\delta H} \\ u_1^{\mathsf{FF}} & \text{is outgoing} \end{cases}$$

- $u_{1,\omega}^{\mathsf{NF}}$ is written as $u_{1,\omega}^{\mathsf{NF}} = \left(W_{1,\omega} - c_1 \chi_{Y>H}\right) \partial_\nu u_0^{\mathsf{FF}}|_{\Gamma_{\delta H}}$

Formal recursive construction



- Problem satisfied by $u_{2,\omega}^{\mathsf{NF}}$:

$$\begin{cases} -\operatorname{div}_{X,Y}\left(\frac{1}{\mu_\omega}\nabla_{X,Y} u_{2,\omega}^{\mathsf{NF}}\right) = \mathcal{F}_s\left(u_{1,\omega}^{\mathsf{NF}}\right) & \text{in } \mathcal{B} \\ \left[u_{2,\omega}^{\mathsf{NF}}\right]_{\Gamma_H} = -u_{2,\omega}^{\mathsf{FF}}|_{\Gamma_{\delta H}} \qquad \left[\partial_Y u_{2,\omega}^{\mathsf{NF}}\right]_{\Gamma_H} = -\partial_\nu u_{1,\omega}^{\mathsf{FF}}|_{\Gamma_{\delta H}} & \\ u_{2,\omega}^{\mathsf{NF}} = 0 & \text{on } \Sigma \end{cases}$$

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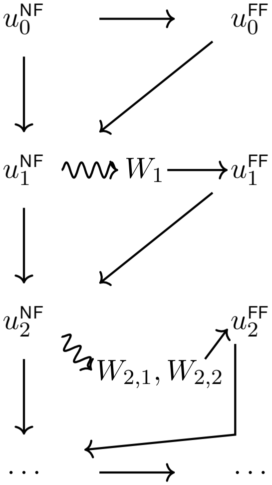

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Du to the lack of higher decrease of the solution, we generally cannot define $u_{2,\omega}^{\mathsf{NF}}$ and $u_{2,\omega}^{\mathsf{FF}}$.

Effective models and error estimates

- In practice, we introduce truncated asymptotic expansions, which we approximate by functions satisfying effective impedance conditions at different orders

$$u^{\mathsf{FF}}_{\delta,N} = \sum_{k=0}^{N} \delta^k u^{\mathsf{FF}}_k$$

Order 1 $u^{\mathsf{FF}}_{\delta,0} = 0$ on $\Gamma_{\delta H} \longrightarrow u_{\delta,0} = u^{\mathsf{FF}}_0$

Order 2 $\partial_\nu u^{\mathsf{FF}}_{\delta,1} + \frac{1}{\delta c_1} u^{\mathsf{FF}}_{\delta,1} = O(\delta)$ on $\Gamma_{\delta H} \longrightarrow \partial_\nu u_{\delta,1} + \frac{1}{\delta c_1} u_{\delta,1} = 0$ on $\Gamma_{\delta H}$

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Effective models and error estimates

Error Estimates

For H big enough and every compact set K in $\mathbb{R}^2 \setminus \mathcal{D}_0$ such that dist $(K, \mathcal{D}_0) > 0$,

$$\sqrt{\mathbb{E}\left[\left\|u_{\omega,\delta} - u^{\text{FF}}_{\delta,0}\right\|^2_{H^1(K)}\right]} = O\left(\delta\right)$$

Under certain hypotheses on the randomness (Boucart, Fliss, and Giovangigli (2026)):

$$\sqrt{\mathbb{E}\left[\left\|u_{\omega,\delta} - u^{\text{FF}}_{\delta,1}\right\|^2_{H^1(K)}\right]} = O\left(\delta^{\frac{3}{2}}\sqrt{|\log(\delta)|}\right)$$

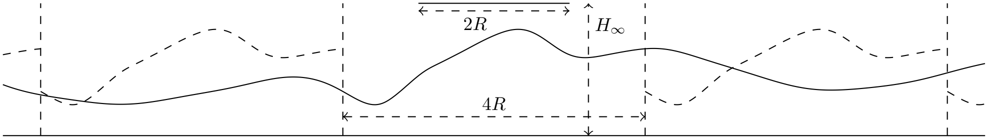
Computation of c_1: random case

- $W_{1,\omega}$ is not computable in practice but using ergodicity we have

$$c_1 = \mathbb{E}\left[W_{1,\omega}\left(\cdot, H_\infty\right)\right] = \lim_{R \to \infty} \fint_{-R}^{R} W_{1,\omega}\left(\cdot, H_\infty\right)$$

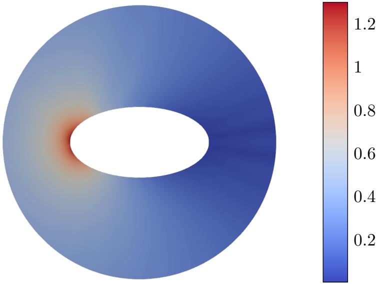
- Truncation of the computational domain of width $4R$ in the X direction by periodic conditions $\rightarrow W^{\#}_{1,\omega,R}$. Let $c^{\#}_{1,R} := \fint_{-R}^{R} W^{\#}_{1,\omega,R}\left(\cdot, H_\infty\right)$
- We can estimate the error made by considering $W^{\#}_{1,\omega,R}$ instead of $W_{1,\omega}$ (relying on the work of A. Gloria)

$$\sqrt{\mathbb{E}\left[\left|c_1 - c^{\#}_{1,R}\right|^2\right]} \leq \underbrace{\sqrt{\mathbb{E}\left[\left|c_1 - \fint_{-R}^{R} W_{1,\omega}\left(\cdot, H_\infty\right)\right|^2\right]}}_{\lesssim \frac{1}{\sqrt{R}} \text{ under quantitative assumptions on the ergodicity}} + \underbrace{\sqrt{\mathbb{E}\left[\left|\fint_{-R}^{R} W_{1,\omega}\left(\cdot, H_\infty\right) - c^{\#}_{1,R}\right|^2\right]}}_{\lesssim \frac{\log(R)^2}{\sqrt{R}} \text{ under regularity hypothesis on } \mu_\omega}$$

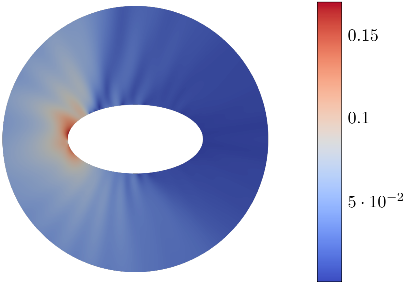


Error plots

$$\kappa_0 = 10,\ \mu^{-1} = 1/3,\ \varepsilon = 1.4,\ \delta = 0.026$$



$|u_\delta - u_{\delta,0}|$



$|u_{\omega,\delta} - u_{\delta,1}|$

$\rightarrow$ Implementation with Xlife++

Error plots

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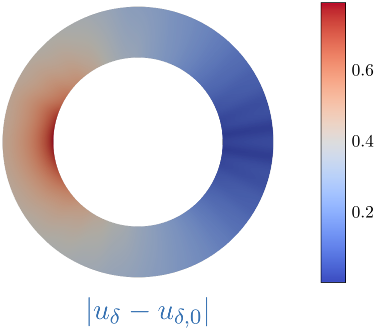
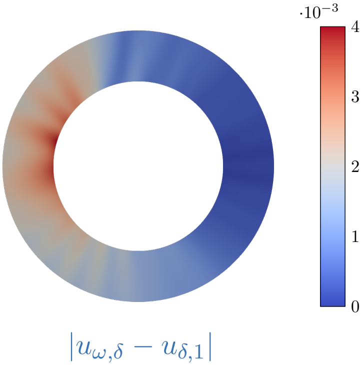




$\rightarrow$ Implementation with Xlife++

Convergence plot

- Monte-Carlo method for calculating the expectation
- Reference solutions must be computed for a large number of realizations

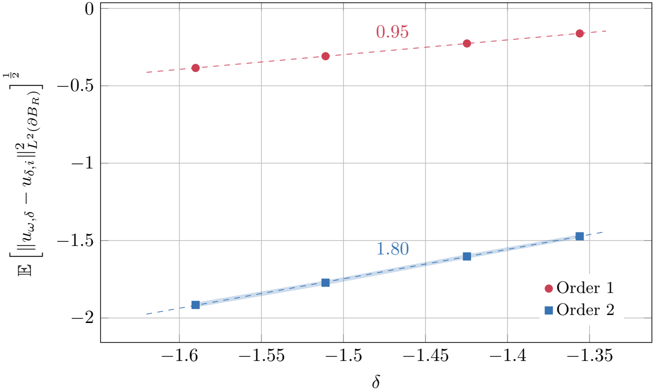
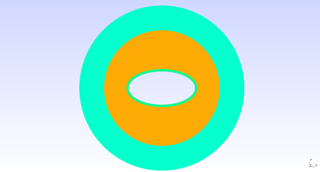


Figure: $L^2(\Omega, L^2(\partial B_R))$-norm of the error between u_δ and $u_{\delta,i}$ for $R \gg 1$ ($N_{simu} = 220$)



Conesphere

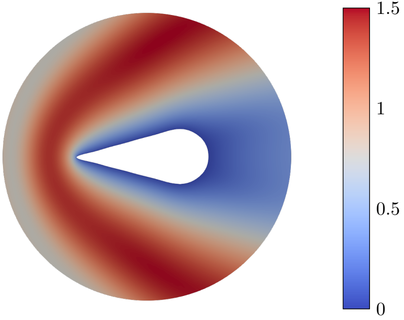


Figure: $|u_\delta|$, $\kappa_0 = 4$, $\mu^{-1} = 1/3$, $\varepsilon = 1.4$, $\delta = 0.026$

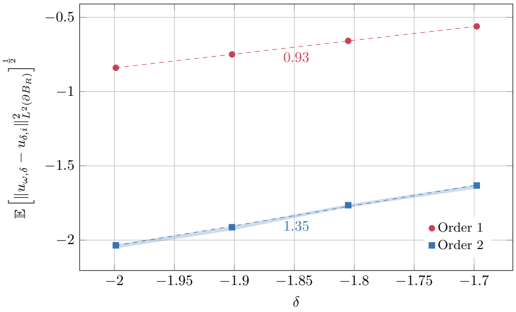
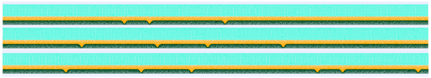


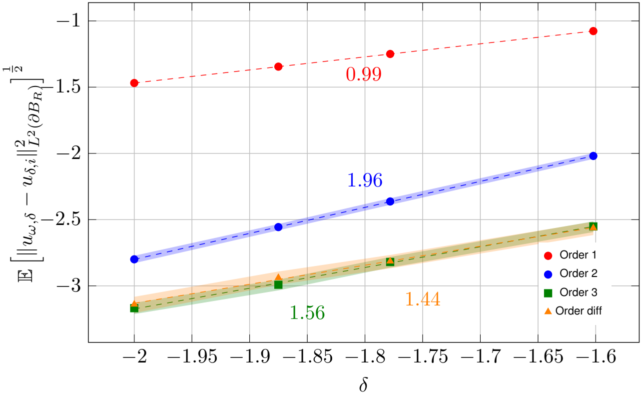
Figure: $L^2(\Omega, L^2(\partial B_R))$-norm of the error between u_δ and $u_{\delta,i}$ for $R \gg 1$ $(N_{simu} = 130)$

Convergence rates have deteriorated due to the singular point (Vial (2003), Baudet (2024))

Higher orders of approximation

Simulations seem to show no higher order is defined, even with "kind" stochastic laws

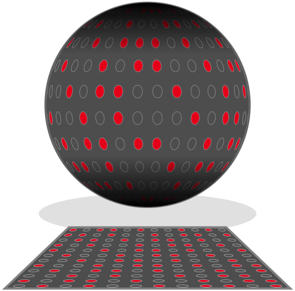




$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|u_{\omega_i,\delta} - u_{\omega_{N+i},\delta}\right\|^2} \leq \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|u^{\mathsf{FF}}_{\delta,2} - u_{\omega_i,\delta}\right\|^2} + \sqrt{\frac{1}{N}\sum_{i=N+1}^{2N}\left\|u^{\mathsf{FF}}_{\delta,2} - u_{\omega_i,\delta}\right\|^2}$$

Transition to 3D

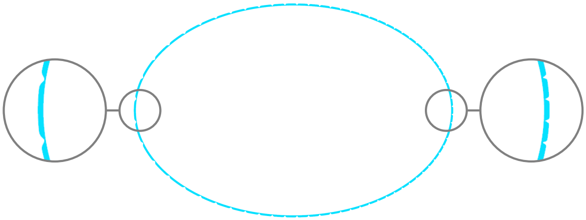
- Difficulty in defining stationarity on a 3D object (Chamaillard (2016))

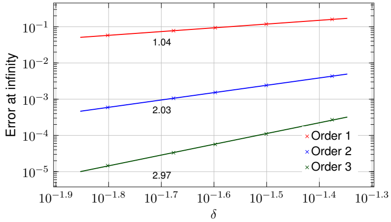


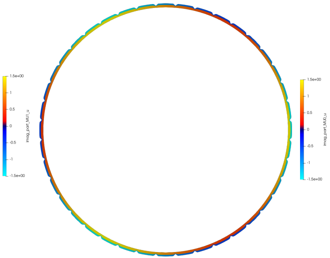
- Introduction of deformations and patches
- Correctors' dependence on tangential variables $\rightarrow$ Calculation of correctors at each point on the surface
- High numerical cost

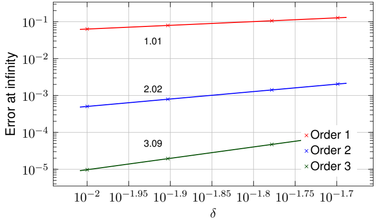
For a given realization, cell problems correspond to parametric PDEs. We use **basis reduction** techniques to reduce computation time (S. Hesthaven, Rozza, and Stamm (2016))

3D difficulties in 2D with basis reduction, $\mu\left(s; \frac{f(s)}{\delta}, \frac{\nu}{\delta}\right)$









$\mu^{-1}_{\delta,\omega}(s; \frac{s}{\delta}, \frac{\nu}{\delta})$

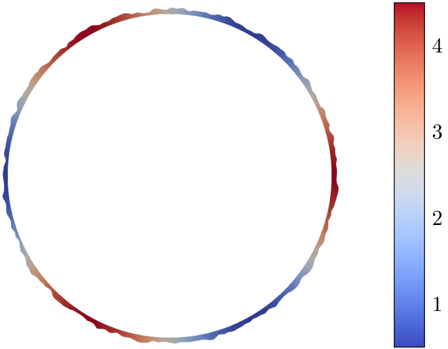


Figure: $\mu^{-1}_{\delta,\omega}$, $\kappa_0 = 1$, $\varepsilon = 1.4$, $\delta = 0.033$

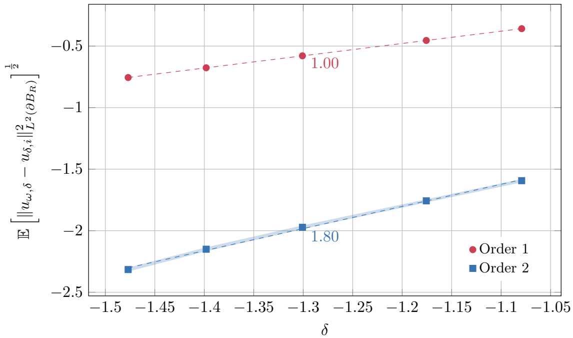


Figure: $L^2(\Omega, L^2(\partial B_R))$-norm of the error between u_δ and $u_{\delta,i}$ for $R \gg 1$ ($N_{simu} = 393$)

Conclusion and Perspectives

Problems under study

- ✔ 2D Helmholtz problem for any smooth geometry
- ✖ 3D Helmholtz problem for any smooth geometry
- ✖ 3D Maxwell problem for any smooth geometry
- ✖ Basis reduction for cell problems in high dimensions (Oseledets (2011))

Thanks for your attention!