Shape optimization using φ-FEM

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Context and setting

Shape optimization problems arise in many engeneering applications where the objective is to design optimal structures. In this context, the level-set approach have become a widely used tool to describe evolving complex geometries during the optimization process (see [1]).

In this work, we couple the level-set framework with φ-FEM, an unfitted Finite Element Method based on an implicite description of the geometry of the domain by a level-set (see [3]). Such FEM scheme avoids remeshing as the domain evolves, making it particularly well suited for shape optimization problems.

We focus on the minimization of the compliance functional for a elastic cantilever (see Figure 1). We use a shape optimization algorithm based on the level-set method and the gradient descent, working with FormOpt (see [4]). Our long-term objective is to develop efficient numerical tools for studies in three-dimensions and design of patient-specific angioplastic balloons.

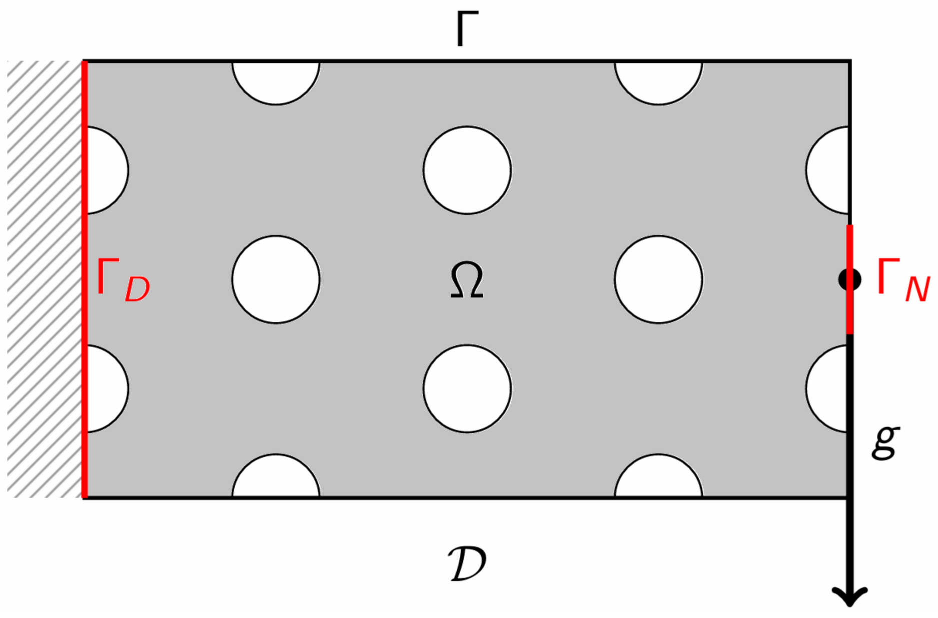

Fig. 1: The cantilever $\Omega \subset \mathcal{D} \subset \mathbb{R}^2$. $\Gamma := \partial\Omega \setminus (\Gamma_D \cup \Gamma_N)$.

I. Shape optimization problem

The displacement u satisfies:

$$-\mathrm{div}(\sigma(u)) = 0 \quad \text{in } \Omega, \tag{1}$$
$$u = 0 \quad \text{on } \Gamma_D, \tag{2}$$
$$\sigma(u)n = g \quad \text{on } \Gamma_N, \tag{3}$$
$$\sigma(u)n = 0 \quad \text{on } \Gamma, \tag{4}$$

with:

- $g \in L^2(\Gamma_N)^2$ a given traction force on Γ_N;
- n the exterior unit normal to $\partial\Omega$;
- $\sigma(u) := 2\mu\varepsilon(u) + \lambda\mathrm{div}(u)I$, where λ, μ are the Lamé parameters, $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ and I the identity matrix of $\mathcal{M}_2(\mathbb{R})$.

Variational formulation:

Find $u_\Omega \in H(\Omega) := \{u \in H^1(\Omega)^2 \mid u_{|\Gamma_D} = 0\}$:

$$\forall v \in H(\Omega), \quad \int_\Omega \sigma(u_\Omega) : \varepsilon(v)\mathrm{d}x = \int_{\Gamma_N} g \cdot v\mathrm{d}s.$$

Shape optimization problem:

$$\min_{\Omega \in \mathcal{O}} J(\Omega),$$

where $J(\Omega) = \int_\Omega \sigma(u_\Omega) : \varepsilon(u_\Omega)\mathrm{d}x$ and $\mathcal{O}$ the shape admissible set.

The shape derivative of J is (see [4]):

$$\mathrm{d}J_\Omega(\theta) = \int_\Omega [2\nabla u_\Omega^T \sigma(u_\Omega) - \sigma(u_\Omega) : \varepsilon(u_\Omega) I] : \nabla\theta.$$

Remark. We penalize the volume by adding the term $\Lambda|\Omega|$, where Λ acts as a material cost and $|\Omega|$ is the volume of Ω.

II. Shape optimization algorithm

Via the gradient descent and the level-set method: assume $\Omega = \{\varphi < 0\}$, $\partial\Omega = \{\varphi = 0\}$.

1. Elastic problem approximation:

   φ-FEM scheme given in Section **III**.

2. Perturbation computation: find θ s.t.

   $$\forall \xi, \quad \mathcal{B}(\theta, \xi) = -\mathrm{d}J_{\Omega_0}(\xi),$$

   where $\mathcal{B}$ is a continuous coercive bilinear form. Thus, $\mathrm{d}J_{\Omega_0}(\theta) = -\mathcal{B}(\theta,\theta) < 0$.

   See Section **IV**.

3. Level-set transportation: find ϕ s.t.

   $$\partial_t\phi + \theta \cdot \nabla\phi = 0, \ \phi(0,\cdot) = \varphi.$$

4. Level-set reinitialization: find ψ s.t.

   $$\partial_t\psi + \mathcal{S}(\phi)(|\nabla\psi| - 1) = 0, \ \psi(0,\cdot) = \phi,$$

   where $\mathcal{S}$ is the sign function.

III. Resolution of the elastic problem

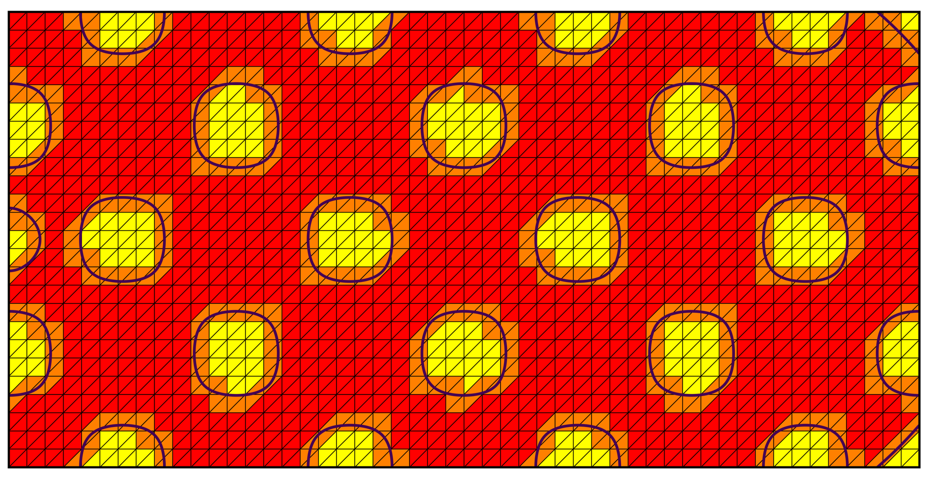

Fig. 2: Ω_h: red + orange | Ω_h^Γ: orange
Initialization mesh.

To solve (1)-(4), we use φ-FEM (Neumann case [3]).

Assume φ is a signed distance function: $n = \frac{\nabla\varphi}{|\nabla\varphi|}$ on Γ.

Formally, we introduce auxiliary variables:

- $y := \sigma(u)$ on Ω_h^Γ.
- Relaxation: $y\nabla\varphi + p\varphi = 0$ on Ω_h^Γ (so that $yn = 0$ on Γ).

Let $\mathcal{F}_{h,1}^\Gamma$ be the facets between orange and red cells.

Goal: find (u_h, y_h, p_h) such that for all (v_h, z_h, q_h),

$$\int_{\Omega_h} \sigma(u_h) : \varepsilon(v_h)\mathrm{d}s + \int_{\partial\Omega_h} (y_h n) \cdot v_h \mathrm{d}s$$
$$+\gamma_u \int_{\Omega_h^\Gamma} (y_h + \sigma(u_h)) : (z_h + \sigma(v_h))\mathrm{d}x + \frac{\gamma_p}{h^2}\int_{\Omega_h^\Gamma}\left(y_h \cdot \nabla\varphi_h + \frac{1}{h}p_h\varphi_h\right)\cdot\left(z_h \cdot \nabla\varphi_h + \frac{1}{h}q_h\varphi_h\right)\mathrm{d}x$$
$$+\gamma_{div}\int_{\Omega_h^\Gamma} \mathrm{div}(y_h)\,\mathrm{div}(z_h)\mathrm{d}x + \sigma h \sum_{F \in \mathcal{F}_{h,1}^\Gamma} \int_F [\sigma(u_h)n_F][\sigma(v_h)n_F]\mathrm{d}s = \int_{\Gamma_N} g \cdot v_h \mathrm{d}s.$$

Classic terms | Penalization | Stabilization | Stabilization (Ghost penalty)

IV. Resolution of the perturbation problem

When u_h is obtained, we compute $\mathrm{d}J_\Omega$. This part is based on gradient descent.

Goal: find $\theta \in \Theta := \left\{\theta \in W^{1,\infty}(\mathcal{D})^2, \ \theta_{|\Gamma_D \cup \Gamma_N} = 0\right\}$ such that

$$\forall \xi \in \Theta, \quad \mathcal{B}(\theta, \xi) = -\mathrm{d}J_{\Omega_0}(\xi),$$

where $\mathcal{B}$ is a continuous coercive bilinear form. We take $\mathcal{B}(\theta,\xi) = \int_\Omega \left(\nabla\theta : \nabla\xi + \frac{1}{10}\theta\cdot\xi\right)\mathrm{d}x$.

Thus, taking $\xi = \theta$, $\mathrm{d}J_{\Omega_0}(\theta) = -\mathcal{B}(\theta,\theta) < 0$ (coercivity of $\mathcal{B}$).

To solve this perturbation problem, we use classical FEM (unfitted).

V. Results and future work

We take $\mathcal{D} = [0,2] \times [0,1]$, $\lambda \approx 0.5769$, $\mu \approx 0.3846$, $g = (0,-2)$, $\Lambda = 1$.

We compute the optimal shape Ω_{opt} in Figure 3.

Using φ-FEM for the elastic problem, the optimal value of the functional is $J(\Omega_{\text{opt}}) \approx 3.0486$. Using FEM for both problems (ersatz material for $\bar{\mathcal{D}} \setminus \Omega$, see [4]), we obtain $J(\Omega_{\text{opt}}) \approx 3.1744$.

Future work:

- Implementation of φ-FEM for the perturbation problem (interface problem).
- Validation on other problems (e.g. L-shape, ...).
- Generalizing in three dimensions.
- Final goal: design of patient-specific angioplastic balloons by optimizing their thickness.

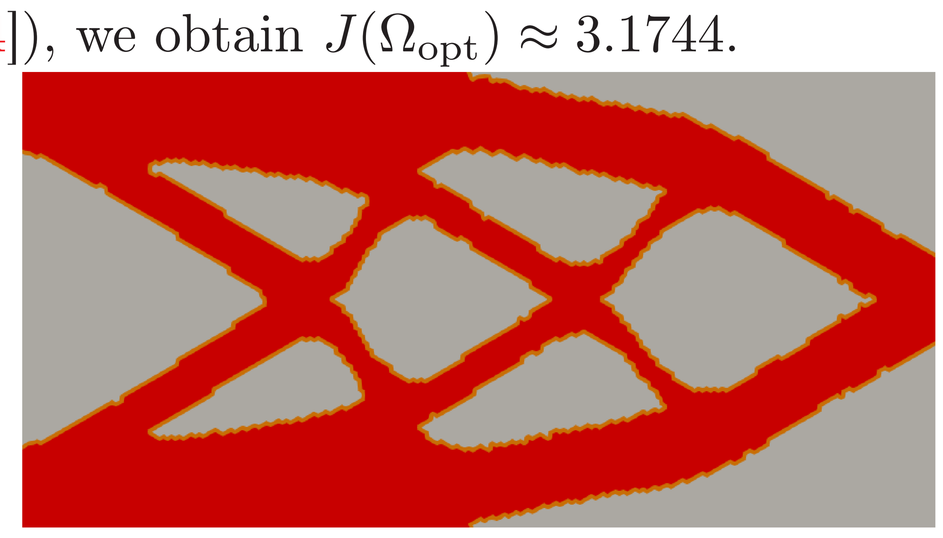

Fig. 3: Result of the shape optimization.

Bibliography

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