A shock-capturing numerical scheme for a non-conservative self-organized hydrodynamics model

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Introduction

The Self-Organized Hydrodynamics (SOH) model[1,2] in 1D:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho(\partial_t u + c u \partial_x u) + \lambda(1-u^2)\partial_x \rho = 0, \\ \rho(\partial_t v + c u \partial_x v) - \lambda u v \partial_x \rho = 0, \\ \sqrt{u^2+v^2} = 1. \end{cases}$$

- $\rho > 0$ the density of individuals,
- $\Omega = \begin{pmatrix} u \\ v \end{pmatrix}$ their mean velocity,
- $\lambda > 0$ and $0 < c < 1$ two constants.

Non-conservative system $\rightarrow$ How to define shock wave solutions?

[1] Degond and Motsch: "Continuum limit of self-driven particles with orientation interaction" (2008)
[2] Degond, Frouvelle, et al.: "Macroscopic models of collective motion and self-organization" (2014)

Introduction

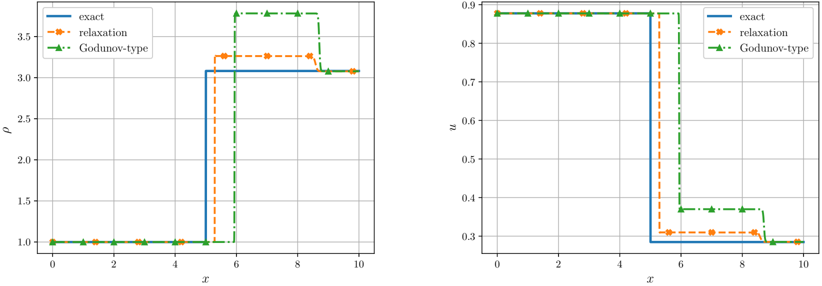


Figure: Numerical results for the case of a stationnary shock at time $T = 5$ with 5000 cells. In ━━: Exact solution. In - ✖ - : Relaxation method[2]. In - ▲ · : Godunov-type scheme.

Non-convergence of schemes[1] $\rightarrow$ How to properly recover shock waves numerically?

[1] Hou and LeFloch: "Why Nonconservative Schemes Converge to Wrong Solutions: Error Analysis" (1994)

[2] Motsch and Navoret: "Numerical Simulations of a Nonconservative Hyperbolic System with Geometric Constraints Describing Swarming Behavior" (2011)

Contents

Theoretical analysis of the SOH model

Reformulation of the system

Let $u = \cos\theta$ and $v = \sin\theta$, with $\theta \in (-\pi, \pi]$:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\cos\theta) = 0, \\ \partial_t \theta + \partial_x(c\sin\theta) - \lambda \frac{\sin\theta}{\rho}\partial_x \rho = 0. \end{cases}$$

It recasts as[1]

$$\partial_t \begin{pmatrix} \rho \\ \theta \end{pmatrix} + \underbrace{\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ -\lambda\frac{\sin\theta}{\rho} & c\cos\theta \end{pmatrix}}_{A} \partial_x \begin{pmatrix} \rho \\ \theta \end{pmatrix} = 0.$$

$\rightarrow$ The SOH model is **hyperbolic** and **non-conservative**.

[1] Motsch and Navoret: "Numerical Simulations of a Nonconservative Hyperbolic System with Geometric Constraints Describing Swarming Behavior" (2011)

Rarefaction waves

$\to$ **Composite fields.**

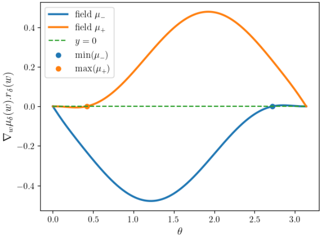


Figure: Change of nature of the fields for $d = 0.2$.

Rarefaction waves : Integral curve[1] starting from (ρ_L, θ_L):

$$\rho_\pm(\theta) = \rho_L \exp\left(\int_{\theta_L}^{\theta} \frac{\sin s}{\cos s - \mu_\pm(s)} ds\right).$$

[1] Motsch and Navoret: "Numerical Simulations of a Nonconservative Hyperbolic System with Geometric Constraints Describing Swarming Behavior" (2011)

Study of shock waves

Viscous model

Two ways of defining shocks:

- Paths theory[1],
- **Traveling wave solutions and viscous model**[2].

Given a non-conservative hyperbolic PDE system:

$$\partial_t w + A(w)\partial_x w = 0.$$

Its associated viscous model with a rescaling:

$$\partial_t w^\varepsilon + A(w^\varepsilon)\partial_x w^\varepsilon = \varepsilon D_0(w^\varepsilon)\partial_x(D_1(w^\varepsilon)\partial_x w^\varepsilon),$$

where $\varepsilon > 0$ and $\varepsilon \to 0$.

[1] Dal Maso et al.: “Definition and weak stability of nonconservative products” (1995)

[2] Sainsaulieu: “Travelling waves solutions of convective diffusive systems and non conservative hyperbolic systems” (1991)

Traveling waves

Definition (Traveling wave[1])

A **traveling wave** solution of the viscous model is a solution such that

$$w^\varepsilon(x,t) = \hat{w}\left(\frac{x-\sigma t}{\varepsilon}\right),$$

where σ is the wave speed and

$$\lim_{\xi\to-\infty} \hat{w}(\xi) = w_L \text{ and } \lim_{\xi\to+\infty} \hat{w}(\xi) = w_R.$$

[1] Sainsaulieu: "Travelling waves solutions of convective diffusive systems and non conservative hyperbolic systems" (1991)

Viscous shock waves

Proposition[1]

A shock wave w solution of $\partial_t w + A(w)\partial_x w = 0$ compatible with the viscosity (D_0, D_1) is the limit when $\varepsilon \to 0$ of a traveling wave $\hat{w}$ solution of the viscous model.

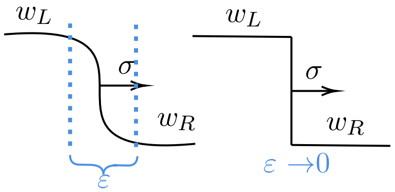


Figure: Viscous profile of a shock.

[1] Sainsaulieu: "Travelling waves solutions of convective diffusive systems and non conservative hyperbolic systems" (1991)

Viscous SOH model

The viscous SOH model[1] reads

$$\partial_t w^\varepsilon + A(w^\varepsilon)\partial_x w^\varepsilon = \varepsilon D_0(w^\varepsilon)\partial_x(D_1(w^\varepsilon)\partial_x w^\varepsilon),$$

where

$$w = \begin{pmatrix} \rho \\ \theta \end{pmatrix}, \; A(w) = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ -\lambda\frac{\sin\theta}{\rho} & c\cos\theta \end{pmatrix}, \; D_0(w) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}, \text{ and } D_1(w) = \begin{pmatrix} 0 & 0 \\ 0 & \rho^2 \end{pmatrix}.$$

[1] Degond, Frouvelle, et al.: "Macroscopic models of collective motion and self-organization" (2014)

Generalized Rankine-Hugoniot conditions

Proposition (Generalized Rankine-Hugoniot conditions)

For $\sigma \neq \cos\theta$, let us introduce G_σ such that

$$G'_\sigma(\theta) = \frac{-\sigma + c\cos\theta - \lambda\frac{\sin^2\theta}{\cos\theta-\sigma}}{(\cos\theta-\sigma)^2}.$$

Given two initial states $w_L, w_R \in \mathcal{D} = \mathbb{R}_+^* \times (-\pi, \pi]$ and a wave speed σ, suppose that:

1. w_L, w_R and σ verify
$$\begin{cases} -\sigma(\rho_R - \rho_L) + (\rho_R\cos\theta_R - \rho_L\cos\theta_L) = 0, \\ G_\sigma(\theta_R) - G_\sigma(\theta_L) = 0. \end{cases}$$
2.
   - if $\theta_L < \theta_R$, for all $\theta \in (\theta_L, \theta_R)$, $G_\sigma(\theta) - G_\sigma(\theta_L) > 0$,
   - if $\theta_L > \theta_R$, for all $\theta \in (\theta_R, \theta_L)$, $G_\sigma(\theta) - G_\sigma(\theta_L) < 0$.
3. for all θ between θ_L and θ_R, $\cos\theta \neq \sigma$.

Then w_L and w_R define a shock wave of speed σ compatible with the viscous model.

Generalized Rankine-Hugoniot conditions

Sketch of proof. We consider traveling wave solutions $\hat{w}$ of the viscous SOH system:

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon \cos\theta^\varepsilon) = 0, \\ \partial_t \theta^\varepsilon + \partial_x(c \sin\theta^\varepsilon) - \lambda \frac{\sin\theta^\varepsilon}{\rho^\varepsilon} \partial_x \rho^\varepsilon = \varepsilon c_v \left(\frac{2}{\rho^\varepsilon} \partial_x \theta^\varepsilon \partial_x \rho^\varepsilon + \partial_{xx} \theta^\varepsilon \right). \end{cases}$$

With $\xi = \frac{x - \sigma t}{\varepsilon}$: $\partial_t . = -\frac{\sigma}{\varepsilon} d_\xi .$ and $\partial_x . = \frac{1}{\varepsilon} d_\xi .$
Second equation on θ:

$$-\sigma d_\xi \hat{\theta} + c \cos\hat{\theta} d_\xi \hat{\theta} - \lambda \sin\hat{\theta} \frac{d_\xi \hat{\rho}}{\hat{\rho}} = c_v \left(2 \frac{d_\xi \hat{\rho}}{\hat{\rho}} d_\xi \hat{\theta} + d_{\xi\xi} \hat{\theta} \right).$$

First equation on ρ, since $\cos\theta \neq \sigma$:

$$\frac{d_\xi \hat{\rho}}{\hat{\rho}} = \frac{\sin\hat{\theta} d_\xi \hat{\theta}}{\cos\hat{\theta} - \sigma}.$$

Generalized Rankine-Hugoniot conditions

This leads to:

$$\underbrace{\frac{-\sigma + c\cos\hat{\theta} - \lambda \frac{\sin^2\hat{\theta}}{\cos\hat{\theta}-\sigma}}{(\cos\hat{\theta}-\sigma)^2}}_{G'_\sigma(\hat{\theta})} d_\xi \theta = c_v d_\xi \left(\frac{d_\xi \hat{\theta}}{(\cos\hat{\theta}-\sigma)^2} \right).$$

Integrating this equation between θ_L and ξ:

$$G_\sigma(\theta) - G_\sigma(\theta_L) = c_v \frac{d_\xi \theta}{(\cos\theta - \sigma)^2}.$$

$\rightarrow$ Autonomous ODE with critical points θ_L and θ_R.

$\rightarrow$ Assumption 2 (Lax): existence of an orbit connecting θ_L to θ_R.

A viscous shock-preserving numerical scheme

Finite volume scheme

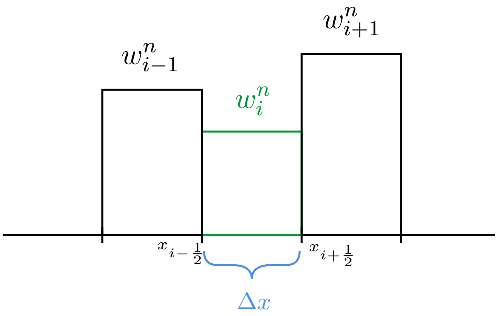


Figure: Illustration of the FV method.

- ⋆ Uniform stencil: $\left\{x_{i+\frac{1}{2}}\right\}_{i\in\mathbb{Z}}$,
- ⋆ $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$,
- ⋆ Time step: $\Delta t = t^{n+1} - t^n$.

→ On cell $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$ at time t^n,

$w = \begin{pmatrix}\rho\\ \theta\end{pmatrix}$ approximated such that

$$w_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n)dx.$$

Discretization of the SOH model

Godunov-type scheme[1]

$\rightarrow$ Approximate Riemann solver $\tilde{w}_{\mathcal{R}}$:

$$\tilde{w}_{\mathcal{R}}\left(\frac{x}{t}; w_L, w_R\right) = \begin{cases} w_L & \text{if } x < -\mu t, \\ w^* & \text{if } -\mu t < x < \mu t, \\ w_R & \text{if } x > \mu t, \end{cases}$$

w^* to be determined, $\mu = \max(|\mu_\pm(\theta_L)|, |\mu_\pm(\theta_R)|)$.

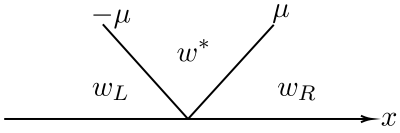


Figure: Illustration of an approximate Riemann solver with one intermediate state.

[1] Harten et al.: "On Upstream Differencing and Godunov-Type Schemes for Hyperbolic Conservation Laws" (1983)

Discretization of the SOH model

Integral consistency condition[1]

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}_{\mathcal{R}}\left(\frac{x}{\Delta t}; w_L, w_R\right) dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_{\mathcal{R}}^{ex}\left(\frac{x}{\Delta t}; w_L, w_R\right) dx,$$

where $w_{\mathcal{R}}^{ex}$ is the exact Riemann solver.

Mean value of $\tilde{w}_{\mathcal{R}}$ over $\left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right]$:

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}_{\mathcal{R}}\left(\frac{x}{\Delta t}; w_L, w_R\right) dx = \left(\frac{1}{2} - \mu\frac{\Delta t}{\Delta x}\right) w_L + 2\mu\frac{\Delta t}{\Delta x} w^* + \left(\frac{1}{2} - \mu\frac{\Delta t}{\Delta x}\right) w_R.$$

[1] Harten et al.: "On Upstream Differencing and Godunov-Type Schemes for Hyperbolic Conservation Laws" (1983)

Discretization of the SOH model

Given the SOH model:

$$\begin{cases} \partial_t \rho + \partial_x(\rho \cos\theta) = 0, \\ \partial_t \theta + \partial_x(c \sin\theta) - \lambda \frac{\sin\theta}{\rho} \partial_x \rho = 0. \end{cases}$$

Integrating first equation on $\left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right] \times [0, \Delta t]$:

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \rho_{\mathcal{R}}^{ex} dx = \frac{1}{2}(\rho_L + \rho_R) - \frac{\Delta t}{\Delta x}(\rho_R \cos\theta_R - \rho_L \cos\theta_L).$$

Integrating second equation on $\left[-\frac{\Delta x}{2}, \frac{\Delta x}{2}\right] \times [0, \Delta t]$ and using the mean-value theorem:

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \theta_{\mathcal{R}}^{ex} dx = \frac{1}{2}(\theta_L + \theta_R) - \frac{\Delta t}{\Delta x} c(\sin\theta_R - \sin\theta_L) + \lambda \frac{\Delta t}{\Delta x} \overline{\left(\frac{\sin\theta}{\rho}\right)}_{LR} (\rho_R - \rho_L).$$

Discretization of the SOH model

Choice of $\overline{\left(\frac{\sin\theta}{\rho}\right)}_{LR}$:

$$\overline{\left(\frac{\sin\theta}{\rho}\right)}_{LR} \approx \frac{\overline{\sin\theta}_{LR}}{\overline{\rho}_{LR}},$$

with $\overline{\sin\theta}_{LR} = \frac{\sin\theta_L + \sin\theta_R}{2}$ and $\overline{\rho}_{LR} = \frac{\rho_L+\rho_R}{2}$.

Using integral consistency condition:

$$\rho^* = \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2\mu}(\rho_R\cos\theta_R - \rho_L\cos\theta_L),$$

$$\begin{aligned}\theta^* &= \frac{1}{2}(\theta_L + \theta_R) - \frac{c}{2\mu}(\sin\theta_R - \sin\theta_L) + \frac{\lambda}{2\mu}\overline{\left(\frac{\sin\theta}{\rho}\right)}_{LR}(\rho_R - \rho_L) \\ &\approx \frac{1}{2}(\theta_L + \theta_R) - \frac{c}{2\mu}(\sin\theta_R - \sin\theta_L) + \frac{\lambda}{2\mu}\frac{\overline{\sin\theta}_{LR}}{\overline{\rho}_{LR}}(\rho_R - \rho_L).\end{aligned}$$

Discretization of the SOH model

Juxtaposition of Riemann solvers: $w_\Delta(., t^n + t)$, for $t \in (0, \Delta t]$ that satisfies:

$$\forall x \in [x_i, x_{i+1}), \forall t \in (0, \Delta t], w_\Delta(x, t^n + t) = \tilde{w}_{\mathcal{R}}\left(\frac{x - x_{i+\frac{1}{2}}}{t}, w_i^n, w_{i+1}^n\right).$$

$\rightarrow$ CFL condition: $\Delta t \leq \frac{\Delta x}{2 \max_{i \in \mathbb{Z}}(|\mu_\pm(w_i^n)|, |\mu_\pm(w_i^n)|)}$.

Then w_i^{n+1} defined as $w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w_\Delta(x, t^n + \Delta t) \, \mathrm{d}x$.

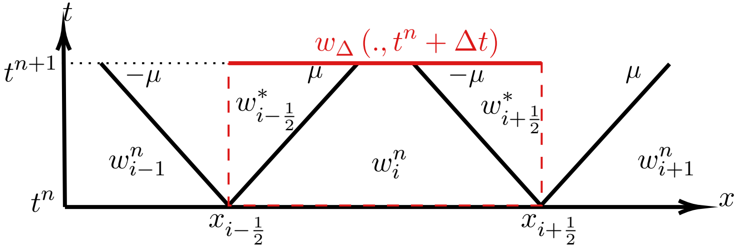


Figure: Illustration of $w_\Delta(., t^n + \Delta t)$ over the cell $\left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$.

Discretization of the SOH model

Godunov-type scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f(w_i^n, w_{i+1}^n) - f(w_{i-1}^n, w_i^n) + \frac{1}{2}\left(a_{i-\frac{1}{2}} + a_{i+\frac{1}{2}}\right)\right),$$

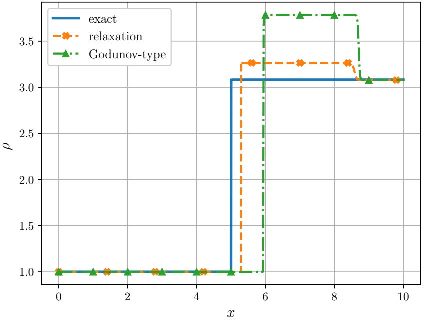
where f is the numerical flux defined as

$$f(w_L, w_R) = \begin{pmatrix} \frac{1}{2}(\rho_L \cos\theta_L + \rho_R \cos\theta_R) - \frac{\mu}{2}(\rho_R - \rho_L) \\ \frac{c}{2}(\sin\theta_L + \sin\theta_R) - \frac{\mu}{2}(\theta_R - \theta_L) \end{pmatrix},$$

and a_{LR} is the non-conservative contribution given by

$$a_{LR} = \begin{pmatrix} 0 \\ -\lambda \frac{\overline{\sin\theta}_{LR}}{\overline{\rho}_{LR}}(\rho_R - \rho_L) \end{pmatrix}.$$

Numerical results for the Godunov-type scheme



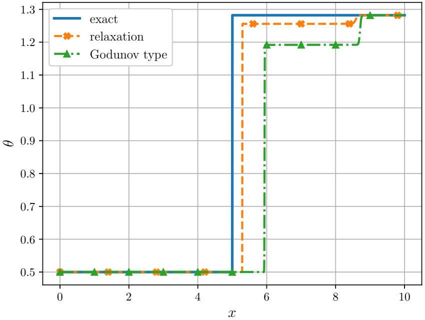


Figure: Numerical results for the case of a stationnary shock at time $T = 5$ with 5000 cells. In : Exact solution. In : Relaxation method. In : Godunov-type scheme.

Viscous correction

Discretization of the physical viscosity:

$$V^n_{phys,i} = \frac{\Delta t}{\Delta x}\frac{1}{(\rho^n_i)^2}\left(\frac{(\rho^n_i)^2+(\rho^n_{i+1})^2}{2}(\theta^n_{i+1}-\theta^n_i) - \frac{(\rho^n_i)^2+(\rho^n_{i-1})^2}{2}(\theta^n_i-\theta^n_{i-1})\right).$$

Natural viscosity of the Godunov-type scheme:

$$V^n_{HLL,i} = \frac{\Delta t}{2\Delta x}\mu(\theta^n_{i+1}-2\theta^n_i+\theta^n_{i-1}).$$

$\Longrightarrow$ Scheme for θ:

$$\theta^{n+1}_i = \theta^n_i - \frac{\Delta t}{\Delta x}\left(f^\theta(w^n_i,w^n_{i+1}) - f^\theta(w^n_{i-1},w^n_i) + \frac{1}{2}\left(a^\theta_{i-\frac{1}{2}} + a^\theta_{i+\frac{1}{2}}\right)\right) - \alpha_{HLL}V^n_{HLL,i} + \alpha_{phys}V^n_{phys,i}.$$

Viscous correction

Viscous shock-capturing scheme

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f(w_i^n, w_{i+1}^n) - f(w_{i-1}^n, w_i^n) + \frac{1}{2}\left(a_{i-\frac{1}{2}} + a_{i+\frac{1}{2}}\right)\right) + \frac{\Delta t}{2\Delta x} V_i^n,$$

where f and a defined previously, V_i^n is the artificial numerical viscosity that reads

$$V_i^n = \begin{pmatrix} 0 \\ \nu_L(w_i^n, w_{i+1}^n)(\theta_{i+1}^n - \theta_i^n) - \nu_R(w_{i-1}^n, w_i^n)(\theta_i^n - \theta_{i-1}^n) \end{pmatrix},$$

with ν_L and ν_R defined as follows:

$$\nu_L(w_L, w_R) = -\alpha_{HLL}\mu + \alpha_{phys}\frac{(\rho_L)^2 + (\rho_R)^2}{(\rho_L)^2},$$

$$\nu_R(w_L, w_R) = -\alpha_{HLL}\mu + \alpha_{phys}\frac{(\rho_L)^2 + (\rho_R)^2}{(\rho_R)^2}.$$

Viscous correction

CFL-like conditions

The time step Δt is restricted according to a CFL-like condition given by

$$\Delta t \leq \frac{\Delta x}{2 \max_{i\in\mathbb{Z}} (\mu + \nu_L(w_i^n, w_{i+1}^n), \mu + \nu_R(w_{i-1}^n, w_i^n))}.$$

Moreover, the physical viscosity coefficient must satisfy

$$\alpha_{phys} \geq (\alpha_{HLL}\mu + \varepsilon) \max_{i\in\mathbb{Z}} \left(\frac{(\rho_i^n)^2}{(\rho_i^n)^2 + (\rho_{i+1}^n)^2} \right),$$

where $\varepsilon > 0$.

Viscous correction

Sketch of proof.

$$\tilde{\theta}^{visc}_{\mathcal{R}}\left(\frac{x}{t};\theta_L,\theta_R\right)=\begin{cases}\theta_L & \text{if } x<-\mu t-\nu_L(w_L,w_R),\\ \overline{\theta}_{LR} & \text{if } -\mu t-\nu_L(w_L,w_R)<x<\mu t,\\ \theta^* & \text{if } -\mu t<x<\mu t,\\ \overline{\theta}_{LR} & \text{if } \mu t<x<\mu t+\nu_R(w_L,w_R),\\ \theta_R & \text{if } x>\mu t+\nu_R(w_L,w_R),\end{cases}$$

where $\overline{\theta}_{LR}=\frac{1}{2}(\theta_L+\theta_R)$, $\nu_L(w_L,w_R)=-\alpha_{HLL}\mu+\alpha_{phys}\frac{(\rho_L)^2+(\rho_R)^2}{(\rho_L)^2}$ and $\nu_R(w_L,w_R)=-\alpha_{HLL}\mu+\alpha_{phys}\frac{(\rho_L)^2+(\rho_R)^2}{(\rho_R)^2}$.

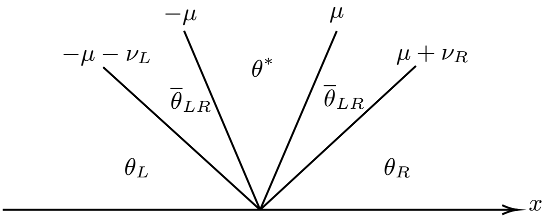


Figure: Illustration of the new Riemann solver for θ.

Numerical results

Numerical results: parametric test for α_{HLL}

Numerical parameters: $d = 0.2$ ($c = 0.77797037$ and $\lambda = 0.22090198$), Neumann boundary conditions, $\varepsilon = 10^{-5}$.

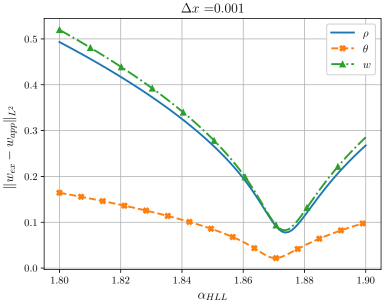
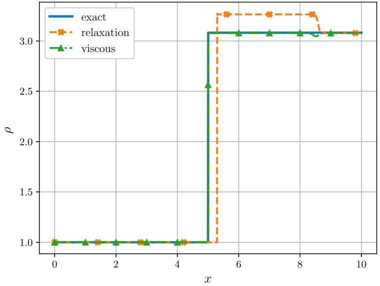


Figure: Stationnary shock at time $T = 5$ for $N_{cells} = 10^4$. In ━━: $\|\rho_{ex} - \rho_{app}\|_{L^2}$. In -✖-: $\|\theta_{ex} - \theta_{app}\|_{L^2}$. In -▲·: $\|w_{ex} - w_{app}\|_{L^2}$.

Optimal value: $\alpha_{HLL} \approx 1.871$.

Numerical results: stationnary shock test case

$$(\rho, \theta)_0^T(x) = \begin{cases} (1.0, 0.5)^T \text{ if } x < 5, \\ (3.0780090, 1.2816713)^T \text{ else.} \end{cases}$$



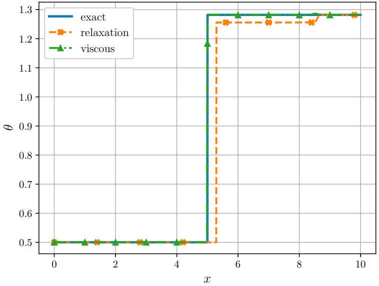
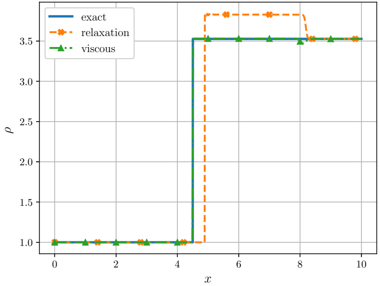


Figure: Stationnary shock at time $T = 5$ for $N_{cells} = 5000$. In : Exact solution. In : Relaxation method. In : Viscous scheme.

Numerical results: shock of speed $\sigma = -0.1$ test case

$$(\rho, \theta)_0^T(x) = \begin{cases} (1.0, 0.5)^T & \text{if } x < 5, \\ (3.5269793, 1.3926833)^T & \text{else.} \end{cases}$$



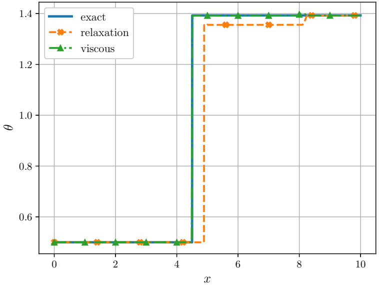


Figure: Shock of speed $\sigma = -0.1$ at time $T = 5$ for $N_{cells} = 5000$. In : Exact solution. In : Relaxation method. In : Viscous scheme.

Numerical results: 1-shock and 2-rarefaction test case

$$(\rho,\theta)_0^T(x) = \begin{cases} (1.0, 0.5)^T \text{ if } x < 5, \\ (4.44534486, 1.371887117)^T \text{ else.} \end{cases}$$

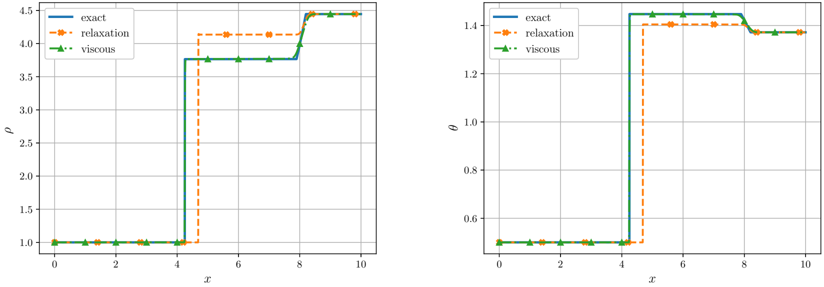


Figure: Shock of speed $\sigma = -0.15$ and rarefaction at time $T = 5$ for $N_{cells} = 5000$. In —: Exact solution. In -✖-: Relaxation method. In -▲-: Viscous scheme.

Conclusions and perspectives

Conclusions and perspectives

→ Characterization of shock waves for the SOH model;

→ Construction of reference solutions thanks to the generalized Rankine-Hugoniot conditions;

→ Derivation of a numerical scheme that captures properly the shock solutions.

Perspectives of future work:

⋆ Derivation of an in-cell discontinuous reconstruction scheme[1];

⋆ Derivation of a micro-macro scheme[2];

⋆ Extension of the numerical schemes to 2D;

⋆ Complexification of the model (addition of forces)[3].

[1] Pimentel-García: "Numerical analysis of some nonlinear hyperbolic systems of Partial Differential Equations arising from Fluid Mechanics." (2021)

[2] Crestetto et al.: "Kinetic/fluid micro-macro numerical schemes for Vlasov-Poisson-BGK equation using particles" (2012)

[3] Albi and Pareschi: "Modeling of self-organized systems interacting with a few individuals" (2013)

Thank you!

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