Localization length for elastic waves propagating in layered random media

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We present mathematical tools to study attenuation and mode exchange of elastic waves propagating in 3D random media. We consider a solid medium Ω approximated by random layers where an elastic plane wave arrives obliquely and propagates through the layers. From the equilibrium equation.

$$-\nabla \cdot \sigma(\mathbf{x}, z, t) = \rho \frac{\partial}{\partial t}\mathbf{v}(\mathbf{x}, z, t), \qquad (\mathbf{x}, z) \in \Omega^- \cup \Omega \cup \Omega^+, t \in \mathbb{R}_+$$

relating stress σ and velocity $\mathbf{v}$, along with the constitutive relation of the slab

$$\frac{\partial}{\partial t}\sigma(\mathbf{x}, z, t) = \lambda(z)\left(\nabla \cdot \mathbf{v}(\mathbf{x}, z, t)\right)\mathbb{I} + \mu^\epsilon(z)\left(\nabla \mathbf{v}(\mathbf{x}, z, t) + \nabla^T \mathbf{v}(\mathbf{x}, z, t)\right), \qquad (\mathbf{x}, z) \in \Omega, t \in \mathbb{R}_+$$

$(t, \mathbf{x})$-Fourier transform yields the linear system of differential equations with random coefficients

$$\frac{d}{dz}\begin{bmatrix}\hat{v}_x \\ \hat{\sigma}_{xz} \\ \hat{v}_z \\ \hat{\sigma}_{zz}\end{bmatrix}(\kappa, z, \omega) = \frac{i\omega}{\epsilon}A^\epsilon(\kappa, z)\begin{bmatrix}\hat{v}_x \\ \hat{\sigma}_{xz} \\ \hat{v}_z \\ \hat{\sigma}_{zz}\end{bmatrix}(\kappa, z, \omega), \qquad (\kappa, z, \omega) \in (\mathbb{R}_+)^3$$

where A^ϵ is a random matrix and ϵ is an homogenization parameter that we let tend towards 0. Because of the randomness, under the weakly or strongly heterogeneous regime, wave localization in space can happen. We are interested in characterizing the localization length with regards to the parameters : wave frequency ω, horizontal wave number κ, mass density ρ but also mean, standard deviation and autocorrelation of Lamé coefficients λ and μ. Indeed, when standard deviation is sufficiently small compared to wavelength and sufficiently small compared to slab size, the power transmission coefficient decays exponentially with distance and is characterized by the localization length. Our work generalizes the acoustic case (SH waves) presented in [2] to the two-mode (P and SV) transfer matrix. We consider transverse isotropy which is more general than the hypotheses made in previous P-SV works such as [3] and unlike [1] we do not restrict to small angles of incidence. Thanks to physical properties, the system matrix A^ϵ lies in $sp(4, \mathbb{C})$ and $su(2, 2)$ Lie algebra. However, the difficulty of the P-SV case lies in the mode couplings. Instead of directly breaking the complex 4×4 transfer matrix into scalar quantities, we choose to work at the 2×2 block matrices level to preserve Lie groups structure exhibited by matrix Riccati equations. This allows us to change variables and work on differential equations with complex matrix-valued functions of complex matrix variables. From stochastic homogenization methods, we obtain limit stochastic differential equations. We use Ito lemma generalized to complex tensors and pass to the limit when the slab size goes to infinity in order to compute Lyapunov exponents which lead to a formula for the localization length.

[1] M. Colvez, R. Cottereau. *High frequency attenuation of elastic waves transmitted at an angle through a randomly-fluctuating horizontally-layered slab.* Wave Motion, 2023.

[2] J. P. Fouque, J. Garnier, G. Papanicolaou, K. Solna. *Wave Propagation and Time Reversal in Randomly Layered Media*, vol. 56. Springer Science and Business Media LLC, 2007.

[3] W. Kohler, G. Papanicolaou, B. White. *Localization and mode conversion for elastic waves in randomly layered media i and ii.* Wave Motion, 1996.

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