DOMAIN DECOMPOSITION BASED PRECONDITIONING FOR A UNIQUE CONTINUATION PROBLEM SUBJECT TO THE WAVE EQUATION

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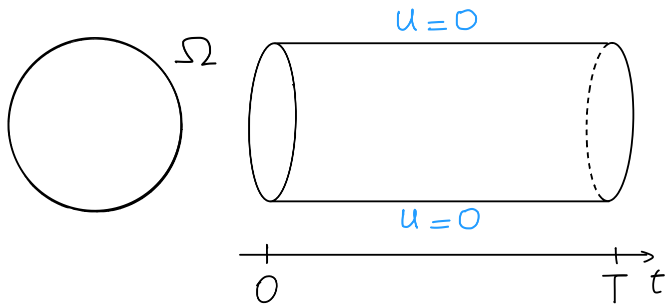
THE MODEL PROBLEM

$\Omega \subseteq \mathbb{R}^d$, $\partial\Omega$ smooth,
$T > 0$,

$$Q := \Omega \times [0,T],$$



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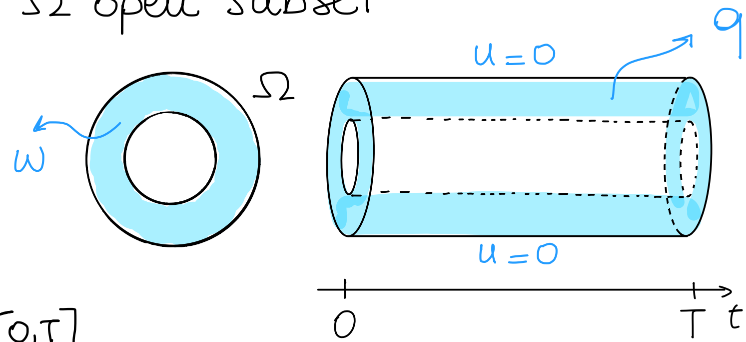

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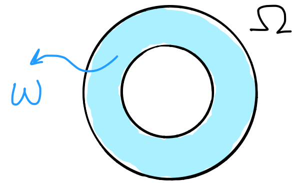

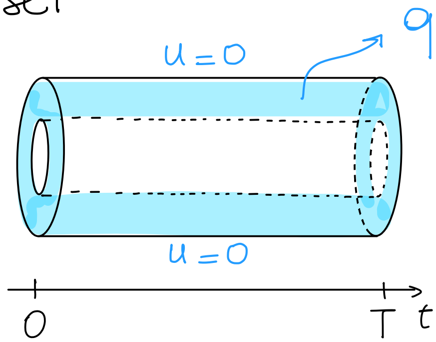

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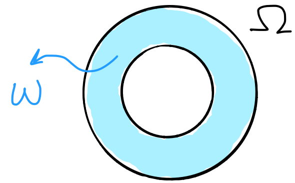

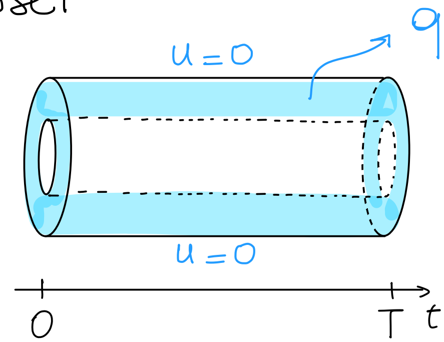

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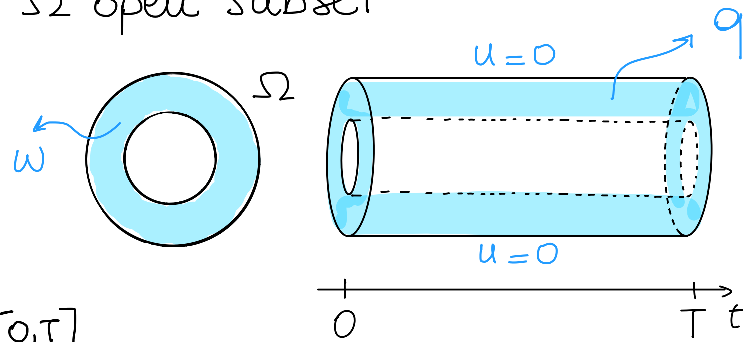

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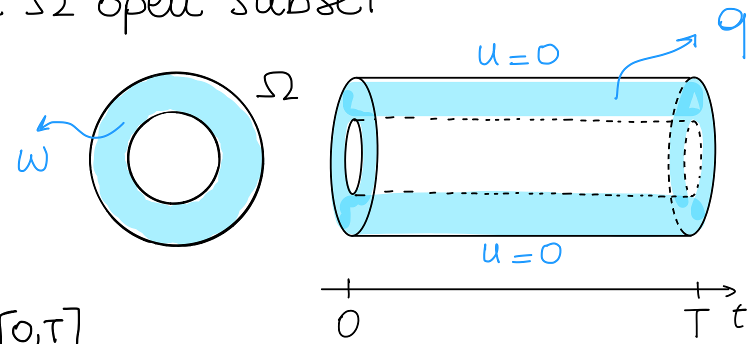

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$$(1) \quad \boxed{\min_{u \in W} J(u) := \frac{1}{2} \| u - u_d \|^2_{L^2(q)}.}$$

We look for the $u \in W$ that minimises the L^2-misfit with $u_d \in L^2(q)$

FIRST ORDER OPTIMALITY CONDITIONS

The PDE constraint is weakly imposed using a Lagrange multiplier $\lambda \in Y := L^2(0,T,H_0^1(\Omega))$

$$\mathcal{L}(u,\lambda) := \frac{1}{2}\|u-u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle_{H^{-1},H_0^1} .$$

$$\Big\downarrow \; \nabla \mathcal{L} = 0$$

Find $u \in Z, \lambda \in Y$ s.t.

$$(2) \quad \begin{aligned} (u,v)_q + \int_0^T \langle \Box v, \lambda \rangle_{H^{-1},H_0^1} &= (u_d, v)_q && \forall v \in Z, \\ \int_0^T \langle \Box u, \psi \rangle_{H^{-1},H_0^1} &= 0 && \forall \psi \in Y. \end{aligned}$$

Lemma [Cîndea, Münch 2015]

Under GCC, Problem (2) is well-posed and the unique solution is the unique saddle point of $\mathcal{L}$ and the unique minimiser in (1)

MOTIVATION BEHIND THE WORK

Find $u \in Z, \lambda \in Y$ s.t.

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- Time lost its connotation of unidirectional variable.
  - ⇨ The problem to solve is globally coupled in space AND time.
- We switched to a mixed formulation.
  - ⇨ Additional degrees of freedom for the Lagrange multiplier and indefinite saddle point system.
- It is an inverse problem
  - ⇨ We expect strong ill-conditioning at the discrete level

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⇨ We expect strong ill-conditioning at the discrete level

Linear system of large size + bad spectral properties ⇨ Both direct and iterative methods struggle

DECOMPOSE-THEN-OPTIMIZE APPROACH(ES)

[Coquet, Gander, Vieira, 2025, arXiv]

DECOMPOSE then OPTIMISE	OPTIMISE then DECOMPOSE
1) Decompose the domain Ω	1) Write the first order optimality conditions
2) Write the PDE in a multidomain fashion + Introduce continuity conditions at the subdomains' interfaces	2) Decompose the domain Ω
3) Treat each constraint using some optimization technique	3) Apply some DD paradigm on the resulting system

DECOMPOSITION OF THE TIME AXIS

Let us subdivide $[0,T]$ into N subintervals

$$I_n := [t_n, t_{n+1}], \quad n \in [\![0, N-1]\!]$$

$$Q_n := \Omega \times I_n$$

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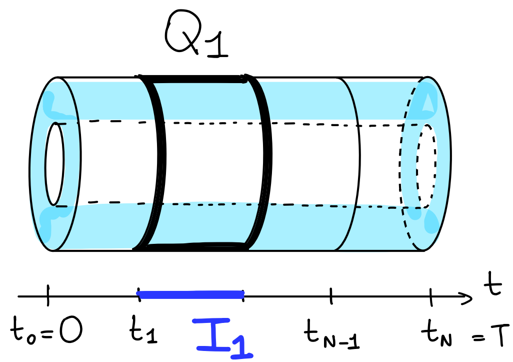
$$\underline{U} := (u_n)_n, \quad u_n : Q_n \to \mathbb{R}$$



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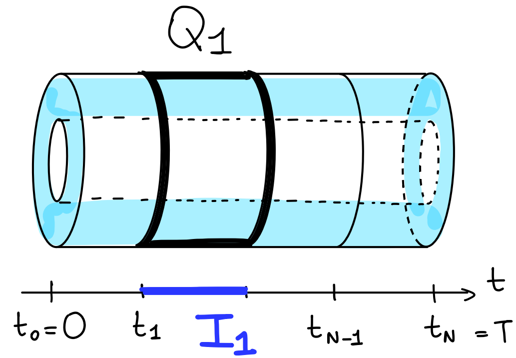

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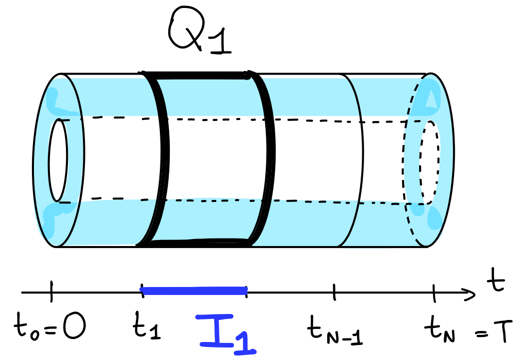

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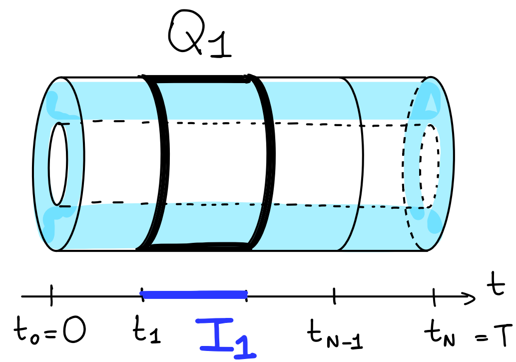

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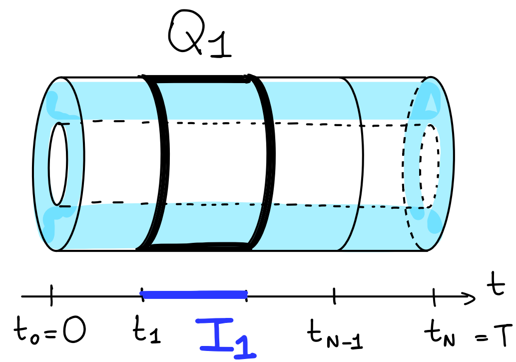

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regularity + continuity conditions

locally enforced PDE

MULTIDOMAIN FORMULATION

(Decompose-then-optimise approach)

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multidomain PDE constrained minimisation

Lemma

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- $\Box u_n = 0,\ \forall n = 0, \dots, N-1$ ⇨ Lagrange multipliers $\underline{\Lambda} = (\lambda_n)_n \in Y$
- $[\![\underline{U}]\!]_n := u_n(t_n) - u_{n-1}(t_n) = 0$
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$\Rightarrow$ "Least-squares" penalisation

"LEAST-SQUARES" PENALISATION

For $\sigma > 0$, let us consider the following Lagrangian function

$$\mathcal{L}_\sigma^{P}(\underline{U}, \underline{\Delta}) := \frac{1}{2} \sum_{n=0}^{N-1} \| u_n - u_{d,n} \|_{q_n}^2$$

$$+ \frac{1}{2\sigma} \sum_{n=1}^{N-1} \left\{ \| [\![\underline{U}]\!]_n \|_{L^2(\Omega)}^2 + \| \partial_t [\![\underline{U}]\!]_n \|_{H^{-1}(\Omega)}^2 \right\}$$

$$+ \sum_{n=0}^{N-1} \int_{I_n} \langle \Box u_n , \lambda_n \rangle_{H^{-1}, H_0^1}$$

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NO continuity in time

penalisation parameter

N local multipliers

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$$\Big\downarrow \ \nabla \mathcal{L}_\sigma^p = 0$$

Find $(\underline{U}_\sigma, \underline{\Lambda}_\sigma) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$(4) \qquad \begin{aligned} \sum_{n=0}^{N-1} (u_n, v_n)_{q_n} + \frac{1}{\sigma} \mathbf{s}(\underline{U}_\sigma, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Lambda}_\sigma) &= \mathbf{L}(\underline{V}) \quad && \forall \underline{V} \in \mathbf{Z}, \\ \mathbf{b}(\underline{U}_\sigma, \underline{\Psi}) &= 0 && \forall \underline{\Psi} \in \mathbf{Y}. \end{aligned}$$

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"LEAST-SQUARES" PENALISATION

For $\sigma > 0$, let us consider the following Lagrangian function

$$\mathcal{L}_\sigma^p(\underline{U}, \underline{\Delta}) := \frac{1}{2}\sum_{n=0}^{N-1} \| u_n - u_{d,n} \|_{q_n}^2 + \frac{1}{2\sigma}\sum_{n=1}^{N-1} \left\{ \| [\![\underline{U}]\!]_n \|_{L^2(\Omega)}^2 + \| [\![\partial_t \underline{U}]\!]_n \|_{H^{-1}(\Omega)}^2 \right\} + \sum_{n=0}^{N-1} \int_{I_n} \langle \Box u_n, \lambda_n \rangle_{H^{-1}, H_0^1}$$

for $\underline{U} \in \mathbf{Z}$, $\underline{\Delta} \in \mathbf{Y}$

$$\Big\downarrow \ \nabla \mathcal{L}_\sigma^p = 0$$

Find $(\underline{U}_\sigma, \underline{\Delta}_\sigma) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$(4) \quad \begin{aligned} \sum_{n=0}^{N-1} (u_n, v_n)_{q_n} + \frac{1}{\sigma}\mathbf{s}(\underline{U}_\sigma, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Delta}_\sigma) &= \mathbf{L}(\underline{V}) \quad && \forall \underline{V} \in \mathbf{Z}, \\ \mathbf{b}(\underline{U}_\sigma, \underline{\Psi}) &= 0 && \forall \underline{\Psi} \in \mathbf{Y}. \end{aligned}$$

$$\mathbf{s}(\underline{U}_\sigma, \underline{V}) := \sum_{n=1}^{N-1} \left\{ \left([\![\underline{U}_\sigma]\!]_n, [\![\underline{V}]\!]_n \right)_\Omega + \langle [\![\partial_t \underline{U}_\sigma]\!]_n, [\![\partial_t \underline{V}]\!]_n \rangle_{H^{-1}(\Omega)} \right\}$$

Coupling between neighbouring slabs $\Rightarrow$ "block-tridiagonal" structure

"LEAST-SQUARES" PENALISATION

For $\sigma > 0$, let us consider the following Lagrangian function

$$\mathcal{L}_\sigma^P(\underline{U}, \underline{\Lambda}) := \frac{1}{2}\sum_{n=0}^{N-1} \| u_n - u_{d,n} \|_{q_n}^2 + \frac{1}{2\sigma}\sum_{n=1}^{N-1} \left\{ \| [\![\underline{U}]\!]_n \|_{L^2(\Omega)}^2 + \| [\![\partial_t \underline{U}]\!]_n \|_{H^{-1}(\Omega)}^2 \right\} + \sum_{n=0}^{N-1} \int_{I_n} \langle \Box u_n, \lambda_n \rangle_{H^{-1}, H_0^1}$$

for $\underline{U} \in \mathbf{Z}$, $\underline{\Lambda} \in \mathbf{Y}$

$$\downarrow \; \nabla \mathcal{L}_\sigma^P = 0$$

Find $(\underline{U}_\sigma, \underline{\Lambda}_\sigma) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$(4) \qquad \begin{aligned} \sum_{n=0}^{N-1} (u_n, v_n)_{q_n} + \frac{1}{\sigma}\mathbf{s}(\underline{U}_\sigma, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Lambda}_\sigma) &= \mathbf{L}(\underline{V}) \quad \forall \underline{V} \in \mathbf{Z}, \\ \mathbf{b}(\underline{U}_\sigma, \underline{\Psi}) &= 0 \quad \forall \underline{\Psi} \in \mathbf{Y}. \end{aligned}$$

Lemma

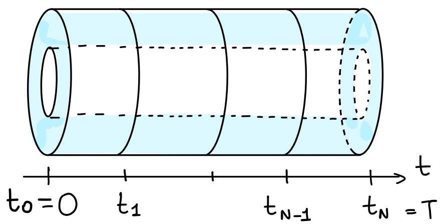
For $\sigma > 0$, problem (4) is well-posed and the unique solution is the unique minimiser of the penalised functional.

Lemma

Given $\underline{U}$ solution of (3), $\underline{U}_\sigma \to \underline{U}$ in $\mathbf{Z}$ for $\sigma \to 0$.

SPLITTING THE COUPLING TERM

Notation : $y, z \in L^2(\Omega)$, $\partial_t y, \partial_t z \in H^{-1}(\Omega)$ $\Rightarrow$ $\langle y, z \rangle := (y,z)_\Omega + \langle \partial_t y, \partial_t z \rangle_{H^{-1}}$



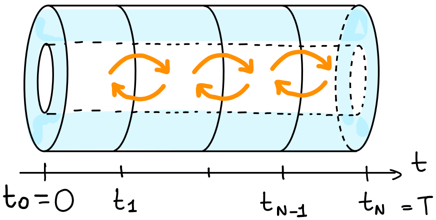
$$\mathbf{s}(\underline{U}, \underline{V}) = \sum_{n=1}^{N-1} \langle [\![\underline{U}]\!]_n, [\![\underline{V}]\!]_n \rangle$$

inter-subdomain coupling

$$= \sum_{n=1}^{N-1} \Big\{ \langle u_{n-1}(t_n), v_{n-1}(t_n) \rangle + \langle u_n(t_n), v_{n-1}(t_n) \rangle + \langle u_{n-1}(t_n), v_n(t_n) \rangle + \langle u_n(t_n), v_n(t_n) \rangle \Big\}$$

SPLITTING THE COUPLING TERM

Notation : $y, z \in L^2(\Omega)$, $\partial_t y, \partial_t z \in H^{-1}(\Omega)$ $\Rightarrow$ $\langle y, z \rangle := (y,z)_\Omega + \langle \partial_t y, \partial_t z \rangle_{H^{-1}}$



$$\mathbf{S}(\underline{U}, \underline{V}) = \sum_{n=1}^{N-1} \langle [\![\underline{U}]\!]_n, [\![\underline{V}]\!]_n \rangle$$

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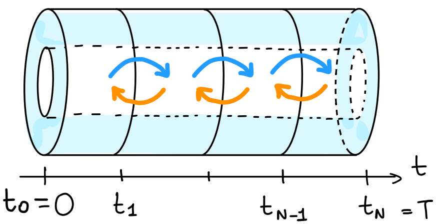
$$\Rightarrow \mathbf{S}(\underline{U}, \underline{V}) = \mathbf{S}_d(\underline{U}, \underline{V}) - \mathbf{S}_c(\underline{U}, \underline{V}).$$

Jacobi-like splitting

SPLITTING THE COUPLING TERM

Notation : $y, z \in L^2(\Omega)$, $\partial_t y, \partial_t z \in H^{-1}(\Omega)$ $\Rightarrow$ $\langle y, z \rangle := (y,z)_\Omega + \langle \partial_t y, \partial_t z \rangle_{H^{-1}}$



$$\mathbf{S}(\underline{U}, \underline{V}) = \sum_{n=1}^{N-1} \langle [\![\underline{U}]\!]_n, [\![\underline{V}]\!]_n \rangle$$

inter-subdomain coupling

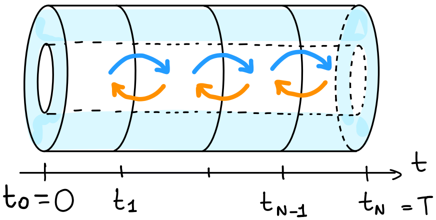
$$= \sum_{n=1}^{N-1} \Big\{ \langle u_{n-1}(t_n), v_{n-1}(t_n) \rangle + \langle u_n(t_n), v_{n-1}(t_n) \rangle + \langle u_{n-1}(t_n), v_n(t_n) \rangle + \langle u_n(t_n), v_n(t_n) \rangle \Big\}$$

$\Rightarrow$ $\mathbf{S}(\underline{U}, \underline{V}) = \mathbf{S}_d(\underline{U}, \underline{V}) - \mathbf{S}_c(\underline{U}, \underline{V}).$ Jacobi-like splitting

$\Rightarrow$ $\mathbf{S}(\underline{U}, \underline{V}) = \mathbf{S}_f(\underline{U}, \underline{V}) - \mathbf{S}_b(\underline{U}, \underline{V}).$ Gauss-Seidel-like splitting

SPLITTING THE COUPLING TERM

Notation : $y, z \in L^2(\Omega)$, $\partial_t y, \partial_t z \in H^{-1}(\Omega)$ $\Rightarrow$ $\langle y, z\rangle := (y,z)_\Omega + \langle \partial_t y, \partial_t z\rangle_{H^{-1}}$



$$\mathbf{S}(\underline{U}, \underline{V}) = \sum_{n=1}^{N-1} \langle [\![\underline{U}]\!]_n, [\![\underline{V}]\!]_n \rangle$$

inter-subdomain coupling

$$= \sum_{n=1}^{N-1} \Big\{ \langle u_{n-1}(t_n), v_{n-1}(t_n)\rangle + \langle u_n(t_n), v_{n-1}(t_n)\rangle + \langle u_{n-1}(t_n), v_n(t_n)\rangle + \langle u_n(t_n), v_n(t_n)\rangle \Big\}$$

$\Rightarrow$ $\mathbf{S}(\underline{U}, \underline{V}) = \mathbf{S}_d(\underline{U}, \underline{V}) - \mathbf{S}_c(\underline{U}, \underline{V}).$ Jacobi-like splitting

$\Rightarrow$ $\mathbf{S}(\underline{U}, \underline{V}) = \mathbf{S}_f(\underline{U}, \underline{V}) - \mathbf{S}_b(\underline{U}, \underline{V}).$ Gauss-Seidel-like splitting

For $\gamma \geq 0$

$\Rightarrow$ $\mathbf{S}(\underline{U}, \underline{V}) = (1+\gamma)\, \mathbf{S}_d(\underline{U}, \underline{V}) - [\gamma\, \mathbf{S}_d(\underline{U}, \underline{V}) + \mathbf{S}_c(\underline{U}, \underline{V})]$ JOR-like splitting

$\Rightarrow$ $\mathbf{S}(\underline{U}, \underline{V}) = \mathbf{S}_f(\underline{U}, \underline{V}) + \gamma\, \mathbf{S}_d(\underline{U}, \underline{V}) - [\gamma\, \mathbf{S}_d(\underline{U}, \underline{V}) + \mathbf{S}_b(\underline{U}, \underline{V})].$

SOR-like splitting

SPLITTING ALGORITHMS

Algorithm 1 [JOR-like]

▷ Find $(\underline{U}_\sigma^{\ell+1}, \underline{\Delta}_\sigma^{\ell+1}) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$
\begin{aligned}
\mathbf{a}(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{1+\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Delta}_\sigma^{\ell+1}) &= \mathbf{L}(\underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell}, \underline{V}) + \frac{1}{\sigma}\mathbf{s}_c(\underline{U}_\sigma^{\ell}, \underline{V}), \\
\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) &= 0,
\end{aligned}
$$

$$\forall \underline{V} \in \mathbf{Z},\ \forall \underline{\Psi} \in \mathbf{Y}$$

SPLITTING ALGORITHMS

Algorithm 1 [JOR-like]

▷ Find $(\underline{U}_\sigma^{\ell+1}, \underline{\Delta}_\sigma^{\ell+1}) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

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$$\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) = 0,$$

$$\forall \underline{V} \in \mathbf{Z},\ \forall \underline{\Psi} \in \mathbf{Y}$$

Algorithm 2 [SOR-like]

▷ Find $(\underline{U}_\sigma^{\ell+1}, \underline{\Delta}_\sigma^{\ell+1}) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$\mathbf{a}(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{1}{\sigma}\mathbf{s}_f(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Delta}_\sigma^{\ell+1}) = \mathbf{L}(\underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell}, \underline{V}) + \frac{1}{\sigma}\mathbf{s}_b(\underline{U}_\sigma^{\ell}, \underline{V}),$$

$$\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) = 0,$$

SPLITTING ALGORITHMS

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$$\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) = 0,$$

$$\forall \underline{V} \in \mathbf{Z},\ \forall \underline{\Psi} \in \mathbf{Y}$$

Algorithm 2 [SOR-like]

▷ Find $(\underline{U}_\sigma^{\ell+1}, \underline{\Delta}_\sigma^{\ell+1}) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$\mathbf{a}(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{1}{\sigma}\mathbf{s}_f(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Delta}_\sigma^{\ell+1}) = \mathbf{L}(\underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell}, \underline{V}) + 1/\sigma\, \mathbf{s}_b(\underline{U}_\sigma^{\ell}, \underline{V}),$$

$$\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) = 0,$$

Theorem [Convergence]

For $\sigma > 0$, Algorithms **1** and **2** converge $\forall \gamma > 0$, namely $\|\underline{U}_\sigma^\ell - \underline{U}_\sigma\|_{\mathbf{Z}} \to 0$ for $\ell \to \infty$, where $\underline{U}_\sigma$ is the solution of (4).
In particular, Algorithm **2** converges also for $\gamma = 0$.

SPLITTING ALGORITHMS

Algorithm 1 [JOR-like]

▷ Find $(\underline{U}_\sigma^{\ell+1}, \underline{\Delta}_\sigma^{\ell+1}) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$\mathbf{a}(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{1+\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Delta}_\sigma^{\ell+1}) = \mathbf{L}(\underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell}, \underline{V}) + \frac{1}{\sigma}\mathbf{s}_c(\underline{U}_\sigma^{\ell}, \underline{V}),$$

$$\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) = 0,$$

⇨ PARALLEL

$$\forall \underline{V} \in \mathbf{Z},\ \forall \underline{\Psi} \in \mathbf{Y}$$

Algorithm 2 [SOR-like]

▷ Find $(\underline{U}_\sigma^{\ell+1}, \underline{\Delta}_\sigma^{\ell+1}) \in \mathbf{Z} \times \mathbf{Y}$ s.t.

$$\mathbf{a}(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{1}{\sigma}\mathbf{s}_f(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell+1}, \underline{V}) + \mathbf{b}(\underline{V}, \underline{\Delta}_\sigma^{\ell+1}) = \mathbf{L}(\underline{V}) + \frac{\gamma}{\sigma}\mathbf{s}_d(\underline{U}_\sigma^{\ell}, \underline{V}) + {}^{1}\!/_{\sigma}\,\mathbf{s}_b(\underline{U}_\sigma^{\ell}, \underline{V}),$$

⇨ SEQUENTIAL

$$\mathbf{b}(\underline{U}_\sigma^{\ell+1}, \underline{\Psi}) = 0,$$

Theorem [Convergence]

For $\sigma > 0$, Algorithms **1** and **2** converge $\forall \gamma > 0$, namely $\| \underline{U}_\sigma^\ell - \underline{U}_\sigma \|_{\mathbf{Z}} \to 0$ for $\ell \to \infty$, where $\underline{U}_\sigma$ is the solution of (4). In particular, Algorithm **2** converges also for $\gamma = 0$.

PRECONDITIONED FIXED POINT ITERATION

As for linear system of equation $Au = b$

splitting algorithm $\Longleftrightarrow$ left-preconditioned fixed point iteration

$$Mu^{\ell+1} = Nu^{\ell} + b \qquad u^{\ell+1} = u^{\ell} + M^{-1}(Au^{\ell} - b)$$

with $A = M - N$

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$$\boxed{\mathbf{b}}(\underline{U}_{\sigma}^{\ell+1}, \underline{\Psi}) = 0,$$

$\forall \underline{V} \in \mathbf{Z}, \forall \underline{\Psi} \in \mathbf{Y}$.

N

M

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Algorithm 2 [SOR-like]

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$$\boxed{\mathbf{b}}(\underline{U}_{\sigma}^{\ell+1}, \underline{\Psi}) = 0,$$

$\forall \underline{V} \in \mathbf{Z}, \forall \underline{\Psi} \in \mathbf{Y}.$

N

M

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z, \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

$\updownarrow$ DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^\circ, \underline{\Lambda} \in \mathbf{Y}$

$$\mathcal{L}_{MD}(\underline{U},\underline{\Lambda}) = \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + \int_{I_n} \langle \Box u_n, \lambda_n \rangle \right\}$$

$\downarrow \uparrow \ \sigma \to 0$

$\underline{U} \in \mathbf{Z}, \underline{\Lambda} \in \mathbf{Y}$

$$\mathcal{L}^P(\underline{U},\underline{\Lambda}) = \mathcal{L}_{MD}(\underline{U},\underline{\Lambda}) + \frac{1}{2\sigma}\sum_{n=1}^{N-1}\left\{ \|[\![\underline{U}]\!]_n\|_\Omega^2 + \|[\![\partial_t \underline{U}]\!]_n\|_{H^{-1}}^2 \right\}$$

$\ell \to \infty \ \uparrow \downarrow$

JOR / SOR - like algorithms $\Longleftrightarrow$ preconditioner

Discrete level

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z,\ \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

$\updownarrow$ DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^0,\ \underline{\Lambda} \in \mathbf{Y}$

$$\mathcal{L}_{MD}(\underline{U},\underline{\Lambda}) = \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + \int_{I_n} \langle \Box u_n, \lambda_n \rangle \right\}$$

$\updownarrow$ $\sigma \to 0$

M_σ^{-1}

$\underline{U} \in \mathbf{Z},\ \underline{\Lambda} \in \mathbf{Y}$

$$\mathcal{L}^P(\underline{U},\underline{\Lambda}) = \mathcal{L}_{MD}(\underline{U},\underline{\Lambda}) + \frac{1}{2\sigma}\sum_{n=1}^{N-1} \left\{ \|[\![\underline{U}]\!]_n\|_\Omega^2 + \|[\![\partial_t \underline{U}]\!]_n\|_{H^{-1}}^2 \right\}$$

$\updownarrow$ $\ell \to \infty$

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Discrete level

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$\updownarrow$ $\sigma \to 0$

$\underline{U} \in \mathbf{Z}, \underline{\Lambda} \in \mathbf{Y}$

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$\updownarrow$ $\ell \to \infty$

JOR / SOR - like algorithms

$\Longleftrightarrow$ preconditioner

$h \to 0$

$\rightleftarrows$

Discrete level

$u_h \in Z_h, \lambda_h \in Y_h$

$$\mathcal{L}_h(u_h, \lambda_h) := \frac{1}{2}\|u_h - u_d\|_q^2 + S(u_h, u_h) + b(u_h, \lambda_h) - S^*(\lambda_h, \lambda_h)$$

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z,\ \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^\circ,\ \underline{\Delta} \in \mathbf{Y}$

$$\mathcal{L}_{MD}(\underline{U},\underline{\Delta}) = \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + \int_{I_n} \langle \Box u_n, \lambda_n \rangle \right\}$$

$\sigma \to 0$

$\underline{U} \in \mathbf{Z},\ \underline{\Delta} \in \mathbf{Y}$

$$\mathcal{L}^P(\underline{U},\underline{\Delta}) = \mathcal{L}_{MD}(\underline{U},\underline{\Delta}) + \frac{1}{2\sigma}\sum_{n=1}^{N-1}\left\{ \|[\![\underline{U}]\!]_n\|_\Omega^2 + \|[\![\partial_t \underline{U}]\!]_n\|_{H^{-1}}^2 \right\}$$

$\ell \to \infty$

JOR / SOR - like algorithms

$\Longleftrightarrow$ preconditioner

$h \to 0$

$\longleftarrow$

$\longrightarrow$

Discrete level

$u_h \in Z_h,\ \lambda_h \in Y_h$

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stabilisation term for u_h

constraint

stabilisation term for λ_h

A RECIPE FOR THE DISCRETE LEVEL

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$u \in Z, \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

DECOMPOSE then OPTIMIZE

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$$\mathcal{L}_{MD}(\underline{U}, \underline{\Delta}) = \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + \int_{I_n} \langle \Box u_n, \lambda_n \rangle \right\}$$

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$\ell \to \infty$

JOR / SOR - like algorithms

$\Longleftrightarrow$ preconditioner

$h \to 0$ $\quad \longleftarrow \quad \longrightarrow$

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$u_h \in Z_h, \lambda_h \in Y_h$

$$\mathcal{L}_h(u_h, \lambda_h) := \frac{1}{2}\|u_h - u_d\|_q^2 + S(u_h, u_h) + b(u_h, \lambda_h) - S^*(\lambda_h, \lambda_h)$$

DECOMPOSE then OPTIMIZE

$\underline{U}_h \in \mathbf{Z}_h^0, \underline{\Delta}_h \in \mathbf{Y}_h$

$$\mathcal{L}_h^{MD}(\underline{U}_h, \underline{\Delta}_h) := \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + S_n(u_h, u_h) + b_n(u_h, \lambda_h) - S_n^*(\lambda_h, \lambda_h) \right\}$$

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z, \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^0, \underline{\Delta} \in \mathbf{Y}$

$$\mathcal{L}_{MD}(\underline{U}, \underline{\Delta}) = \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + \int_{I_n} \langle \Box u_n, \lambda_n \rangle \right\}$$

$\sigma \to 0$

$\underline{U} \in \mathbf{Z}, \underline{\Delta} \in \mathbf{Y}$

$$\mathcal{L}^P(\underline{U}, \underline{\Delta}) = \mathcal{L}_{MD}(\underline{U}, \underline{\Delta}) + \frac{1}{2\sigma} \sum_{n=1}^{N-1} \left\{ \|[\![U]\!]_n\|_\Omega^2 + \|[\![\partial_t U]\!]_n\|_{H^{-1}}^2 \right\}$$

$\ell \to \infty$

JOR / SOR - like algorithms

$\Longleftrightarrow$ preconditioner

$h \to 0$

$\longleftarrow$ / $\longrightarrow$

Discrete level

$u_h \in Z_h, \lambda_h \in Y_h$

$$\mathcal{L}_h(u_h, \lambda_h) := \frac{1}{2}\|u_h - u_d\|_q^2 + S(u_h, u_h) + b(u_h, \lambda_h) - S^*(\lambda_h, \lambda_h)$$

DECOMPOSE then OPTIMIZE

$\underline{U}_h \in \mathbf{Z}_h^0, \underline{\Delta}_h \in \mathbf{Y}_h$

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regularity + continuity conditions

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z, \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_Q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^0, \underline{\Lambda} \in \mathbf{Y}$

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$\ell \to \infty$

JOR / SOR - like algorithms $\Leftrightarrow$ preconditioner

$h \to 0$

$$\xleftarrow{\hspace{1cm}} \qquad \xrightarrow{\hspace{1cm}}$$

Discrete level

$u_h \in Z_h, \lambda_h \in Y_h$

$$\mathcal{L}_h(u_h,\lambda_h) := \frac{1}{2}\|u_h - u_d\|_Q^2 + S(u_h,u_h) + b(u_h,\lambda_h) - S^*(\lambda_h,\lambda_h)$$

DECOMPOSE then OPTIMIZE

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$\sigma \to 0$

$\underline{U}_h \in \mathbf{Z}_h, \underline{\Lambda}_h \in \mathbf{Y}_h$

$$\mathcal{L}_h(\underline{U}_h,\underline{\Lambda}_h) := \mathcal{L}_h^{MD} + \boxed{S_\sigma^{\downarrow\uparrow}(\underline{U}_h,\underline{U}_h)}$$

penalisation of the continuity conditions

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z, \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^0, \underline{\Delta} \in \mathbf{Y}$

$$\mathcal{L}_{MD}(\underline{U},\underline{\Delta}) = \sum_{n=0}^{N-1} \left\{ \frac{1}{2}\|u_n - u_{d,n}\|_{q_n}^2 + \int_{I_n} \langle \Box u_n, \lambda_n \rangle \right\}$$

$\sigma \to 0$

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$\ell \to \infty$

JOR / SOR - like algorithms $\Longleftrightarrow$ preconditioner

$h \to 0$

Discrete level

$u_h \in Z_h, \lambda_h \in Y_h$

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$\ell \to \infty$

preconditioner $\Longleftrightarrow$ JOR/SOR - like algorithms

A RECIPE FOR THE DISCRETE LEVEL

Continuous level

$u \in Z, \lambda \in Y$

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\|u - u_d\|_q^2 + \int_0^T \langle \Box u, \lambda \rangle$$

DECOMPOSE then OPTIMIZE

$\underline{U} \in \mathbf{Z}^0, \underline{\Lambda} \in \mathbf{Y}$

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$\sigma \to 0$

$\underline{U} \in \mathbf{Z}, \underline{\Lambda} \in \mathbf{Y}$

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$\ell \to \infty$

JOR / SOR - like algorithms $\Longleftrightarrow$ preconditioner

$h \to 0$

Discrete level

$u_h \in Z_h, \lambda_h \in Y_h$

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M_σ^{-1}

$\ell \to \infty$

preconditioner $\Longleftrightarrow$ JOR/SOR - like algorithms

EXAMPLES OF DISCRETIZATIONS

- [Cîndea, Münch. 2015] Conforming space-time discretisation $Z_h \subset Z$, $Y_h \subset Y$, using C^1-conforming elements
- [Burman et al. 2020] Non-conforming discretisation using FEM in space + finite differences in time
- [Burman et al. 2021] Non-conforming space-time FEM discretisation
- [Burman, Preuss 2025] Non-conforming discretisation using FEM in space + DG in time on a tensorized space-time mesh

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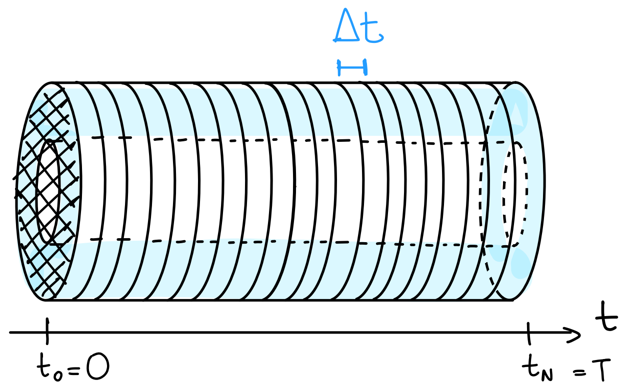
Jumps at the time interfaces are already penalised

$$
\begin{aligned}
S_{\Delta t}^{\downarrow\uparrow}(\underline{U}_h, \underline{V}_h) = \sum_{n=1}^{N-1} \Big\{ & \frac{1}{2\Delta t} \left([\![\underline{U}]\!]_n, [\![\underline{U}]\!]_n \right)_\Omega \\
& + \frac{\Delta t}{2} \left(\nabla [\![\underline{U}]\!]_n, \nabla [\![\underline{U}]\!]_n \right)_\Omega \\
& + \frac{1}{2\Delta t} \left([\![\partial_t \underline{U}]\!]_n, [\![\partial_t \underline{U}]\!]_n \right)_\Omega \Big\}
\end{aligned}
$$



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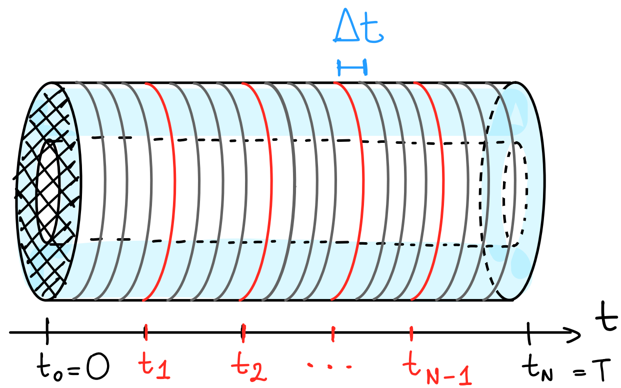
Adding a domain decomposition framework

$$S_\sigma^{\downarrow\uparrow}(\underline{U}_h, \underline{V}_h) = \sum_{n=1}^{N-1} \left\{ \frac{1}{2(\Delta t+\sigma)} \left([\![\underline{U}]\!]_n, [\![\underline{U}]\!]_n \right)_\Omega + \frac{(\Delta t+\sigma)}{2} \left(\nabla [\![\underline{U}]\!]_n, \nabla [\![\underline{U}]\!]_n \right)_\Omega + \frac{1}{2(\Delta t+\sigma)} \left([\![\partial_t \underline{U}]\!]_n, [\![\partial_t \underline{U}]\!]_n \right)_\Omega \right\}$$



NUMERICAL EXPERIMENTS

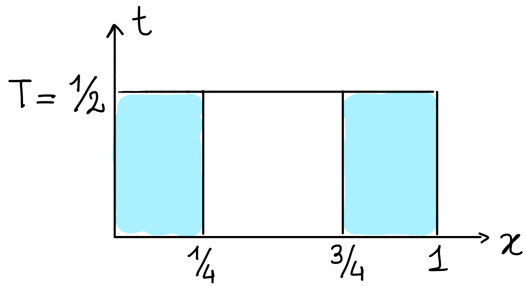
1D + time test case

q

$\Omega = [0,1]$

$\omega = [0, 1/4] \cup [3/4, 1]$

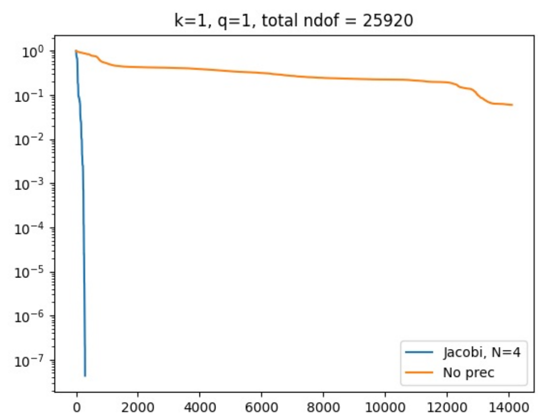
$C = 1$



We consider the discretisation from [Burman, Preuss, 2025]. And we use the preconditioner from the fixed point in GMRES

⇨ Unpreconditioned GMRES cannot be applied

- space : FEM order 1
- time : DG order 1
- total ndofs : 25920



NUMERICAL EXPERIMENTS

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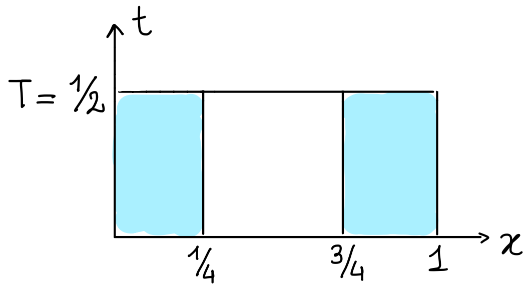
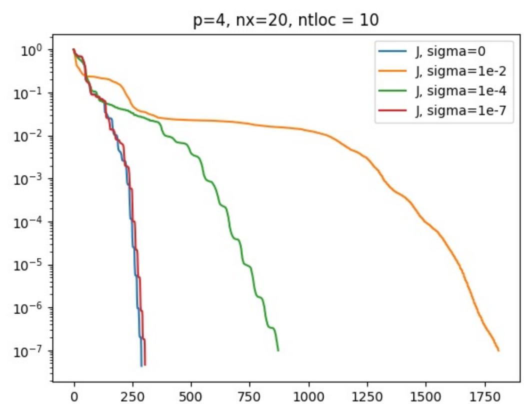


We consider the discretisation from [Burman, Preuss, 2025]. with $S_{\sigma}^{\downarrow\uparrow}$ at the subdomain interfaces

⇨ We observe numerically that the best choice is $\sigma = 0$

- space : FEM order 1
- time : DG order 1
- total ndofs : 25920



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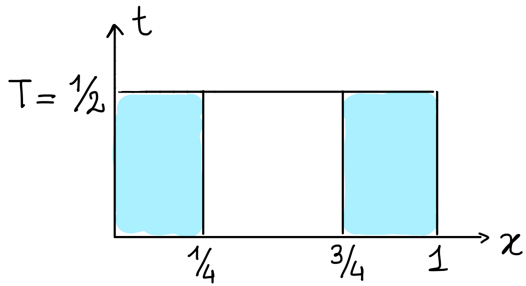
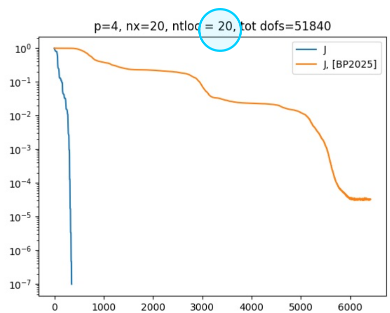
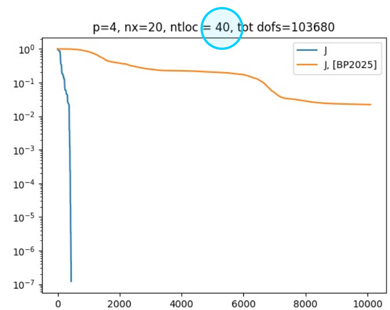


We consider the discretisation from [Burman, Preuss, 2025]. with $S_\sigma^{\downarrow\uparrow}$ at the subdomain interfaces with $\sigma = 0$

⇨ We perform better than the monolithic preconditioners proposed in [Burman, Preuss, 2025]





NUMERICAL EXPERIMENTS

1D + time test case

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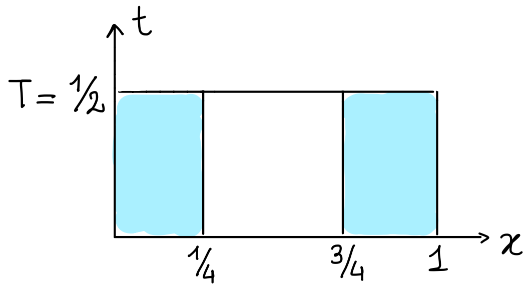

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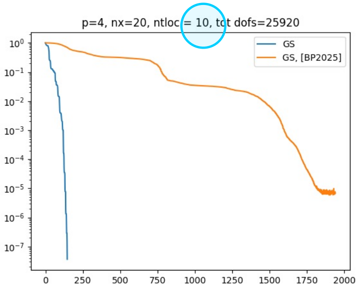

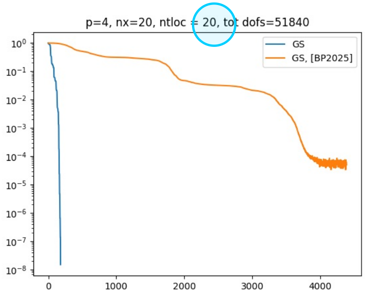

CONCLUSIONS & PROSPECTS

- We proposed a decompose-then-optimise paradigm to define preconditioners, given some discretisation of a unique continuation problem subject to the wave equation.

- The performance of the preconditioner depends on the penalisation parameter σ

CONCLUSIONS & PROSPECTS

- We proposed a decompose-then-optimise paradigm to define preconditioners, given some discretisation of a unique continuation problem subject to the wave equation.
- The performance of the preconditioner depends on the penalisation parameter σ

Future work

- More complete numerical investigation on different discretisations
- Introduce Lagrange multipliers for the continuity conditions instead of least-squares penalisation
- Adding a decomposition with respect to space
- Introducing a second level to speed up convergence and (hopefully) gain scalability

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GEOMETRIC CONTROL CONDITION

[Bardos, Lebeau, Rauch 1992]

We consider the case of stationary cylindrical observation domains

$$\omega(t) \equiv \omega \,, \quad q = \omega \times [0,T]$$

"Roughly speaking, it says that every geodesic propagating in Ω at unit speed, and reflecting at the boundary according to the classical laws of geometrical optics, so-called generalized geodesics, should meet the open set ω within time T."

[Le Rousseau et al. 2017]

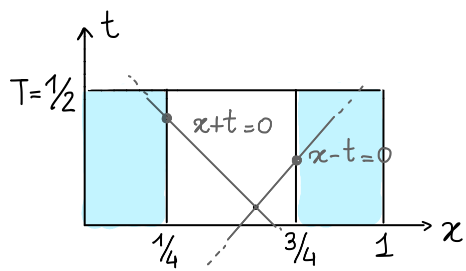
1D + time example

- q

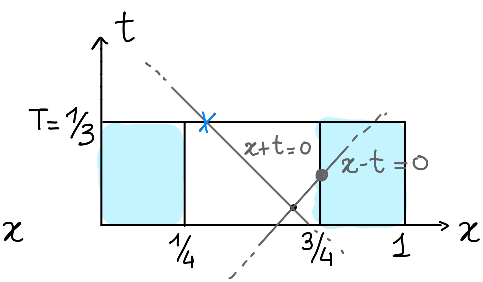
$\Omega = [0,1]$

$\omega = [0,1/4] \cup [3/4,1]$

$c = 1\ m/s$



GCC verified



GCC not verified

Assumption

The GCC holds for the triplet (Ω, ω, T).

OBSERVABILITY ESTIMATE

Lemma (Generalised observability inequality - GOI)

For Ω with smooth boundary and (Ω, ω, T) satisfying the geometric control condition there exists a constant $C = C(\Omega, \omega, T, c)$ such that $\forall u \in Z$

$$\|u(0), \partial_t u(0)\|_{H} \leq C \left(\|u\|_q + \|\Box u\|_{L^2(0,T;H^{-1}(\Omega))} \right)$$

⇨ The proof uses a lifting argument from the classical result of [Bardos, Lebeau, Rauch 1992].

Lemma [Castro, Cîndea, Münch, 2014]

The space Z, equipped with the scalar product

$$(u,v)_Z := (u,v)_q + \eta \int_0^T < \Box u, \Box v >_{H^{-1}}$$

is an Hilbert space.