Numerical stability of Hermite moments system coupled to the Poisson equation

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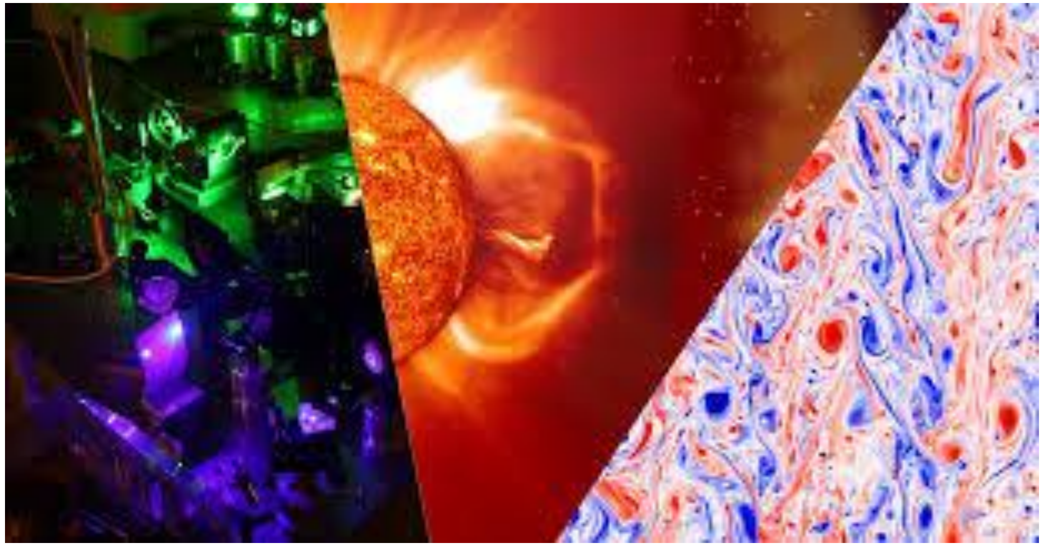
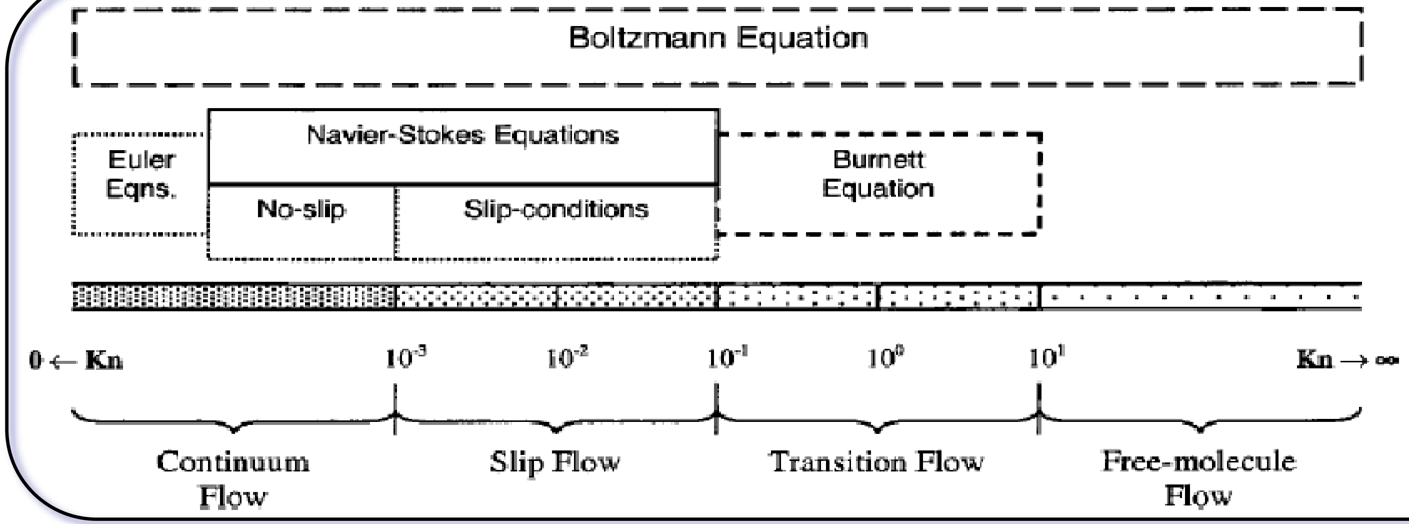
Motivation and context



- Industrial plasmas
- Spatial plasmas
- Fusion plasmas

Turbulence
Instabilities
Collisions
Non equilibrium
Surface interactions
Electromagnetism
Wave propagation



- Kinetic methods
- Hybrid methods
- Fluid methods

Scientific context

Approximation of kinetic equation through spectral methods:

- *H. Grad, 1949: "On the kinetic theory of rarefied gases"*
- *J. Parker and P. Dellar, 2014: "Fourier-Hermite spectral representation for the Vlasov-Poisson system in the weakly collisional limit"*
- *Z. Cai and Y. Wang, 2017: "Suppression of recurrence in the Hermite spectral method for transport equations"*
- *A. Zocco, 2020: "Linear collisionless Landau damping in Hilbert space"*

Numerical properties of such methods:

- *S. Fabre, 1992: "Stability analysis of the Euler-Poisson equations"*
- *F. Filbet and T. Xiong, 2020: "Conservative discontinuous Galerkin/Hermite spectral method for the Vlasov-Poisson system"*

About the Hermite moments

- Target system: Vlasov-Poisson for electrons with collisionless static background.

$$\begin{cases} \dfrac{\partial f_e}{\partial t} + v\dfrac{\partial f_e}{\partial x} - \dfrac{eE}{m_e}\dfrac{\partial f_e}{\partial v} = 0 & \forall t \geq 0, \quad \forall x \in [0, L_0], \quad \forall v \in \mathbb{R}, \\ E = -\dfrac{\partial \phi}{\partial x} & \forall x \in [0, L_0], \\ \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{-e}{\varepsilon_0}\left(n_0 - \displaystyle\int_{\mathbb{R}} f_e(x,v,t)\mathrm{d}v\right) & \forall x \in [0, L_0]. \end{cases}$$

- Speed space discretisation: The set of Hermite polynomials is chosen as a basis and constrains the distribution:

$$f(x,v,t) = \frac{1}{\sqrt{2\pi}}\sum_{i=0}^{\infty} f_i(x,t)H_i(v)e^{-v^2/2}.$$

- With properties: completeness, orthonormality, recurrence formula…

$$\sqrt{i+1}H_{i+1}(v) = vH_i(v) - \sqrt{i}H_{i-1}(v), \quad \forall i \in \mathbb{N}^*.$$

Linearized Hermite hierarchy

- <u>Moment hierarchy:</u>

$$\begin{cases} \dfrac{\partial f_i}{\partial t} + \sqrt{i}\dfrac{\partial f_{i-1}}{\partial x} + \sqrt{i+1}\dfrac{\partial f_{i+1}}{\partial x} = \sqrt{i}Ef_{i-1} & \forall t \geq 0, \quad \forall x \in [0, L_0], \quad \forall i \in \mathbb{N}, \\ E = -\dfrac{\partial \phi}{\partial x} & \forall x \in [0, L_0], \\ \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{-1}{\lambda_D^2}(1 - f_0) & \forall x \in [0, L_0]. \end{cases}$$

- <u>Linearize and truncate:</u> $(f_0, f_1, \dots, f_M) = \underbrace{(1, 0, \dots, 0)}_{U_0} + U, \quad \text{with} \quad \|U\| \ll \|U_0\|.$

$$\begin{cases} \dfrac{\partial U}{\partial t} + A\dfrac{\partial U}{\partial x} = EBU_0 & \forall t \geq 0, \quad \forall x \in [0, L_0], \\ E = -\dfrac{\partial \phi}{\partial x} & \forall x \in [0, L_0], \\ \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{-1}{\lambda_D^2}[U]_0 & \forall x \in [0, L_0]. \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \sqrt{2} & & \vdots \\ 0 & \sqrt{2} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \sqrt{M} \\ 0 & \dots & 0 & \sqrt{M} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & & \vdots \\ 0 & \sqrt{2} & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sqrt{M} & 0 \end{pmatrix}.$$

Semi discretized scheme & Von Neumann analysis

- Flux splitting upwind scheme for transport: $$\frac{\mathrm{d}U_j}{\mathrm{d}t} = \frac{-1}{\Delta x}\left(A^+(U_j - U_{j-1}) + A^-(U_{j+1} - U_j)\right) + S_j$$
- Finite difference scheme for Poisson: $$E_j = -\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} \quad \text{and} \quad \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2} = -\frac{1}{\lambda_D^2}[U_j]_0.$$

- Principle of Von Neumann stability analysis:

How are the errors amplified through the scheme ?

- Tools: Fourier modes decomposition and amplification scheme.

$$U_j^n = \widehat{U^n} e^{\mathrm{i}kj\Delta x}, \quad k \in \left[\frac{2\pi}{L_0}, \frac{4\pi}{L_0}, \dots, \frac{2\pi}{2\Delta x}\right].$$

$$\widehat{U^{n+1}} = \mathcal{G}(k, \Delta x, \Delta t)\widehat{U^n} \quad \Longrightarrow \quad \widehat{U^{n+1}} = \mathcal{G}^n(k, \Delta x, \Delta t)\widehat{U^0}.$$

$$\Longrightarrow \quad \|\widehat{U^{n+1}}\| \leq C\|\mathcal{G}^n(k, \Delta x, \Delta t)\|\|\widehat{U^0}\|.$$

- IMEX Theta Schemes: $$\mathcal{G}(k) = (I - \theta\Delta t\mathcal{S}(k))^{-1}(I + \Delta t(\mathcal{T}(k) + (1-\theta)\mathcal{S}(k)))$$

Von Neumann condition, limits and warnings

- The Von Neumann necessary condition of stability: $\rho(\mathcal{G}(k)) \leq 1, \quad \forall k \in \mathcal{K}.$

- Non normality: eigenmodes are no longer orthogonal -> high coupling between modes that Von Neumann condition may not control.

$$U_j^n = \sum_{k \in \mathcal{K}} \mathcal{G}^n(k) \widehat{U^0}(k) e^{ikj\Delta x}$$

- Diagonalisability: the presence of Jordan blocks leads to a temporary growth.

$$\|\mathcal{G}^n(k)\| \sim n^p \rho(\mathcal{G}(k))^n$$

- Classical approach: one consider the continuous limit for wave numbers -> infinite domain and infinite resolution

$$\theta = k\Delta x \in [0, \pi] \quad \Longleftrightarrow \quad k \in \mathbb{R}^+$$

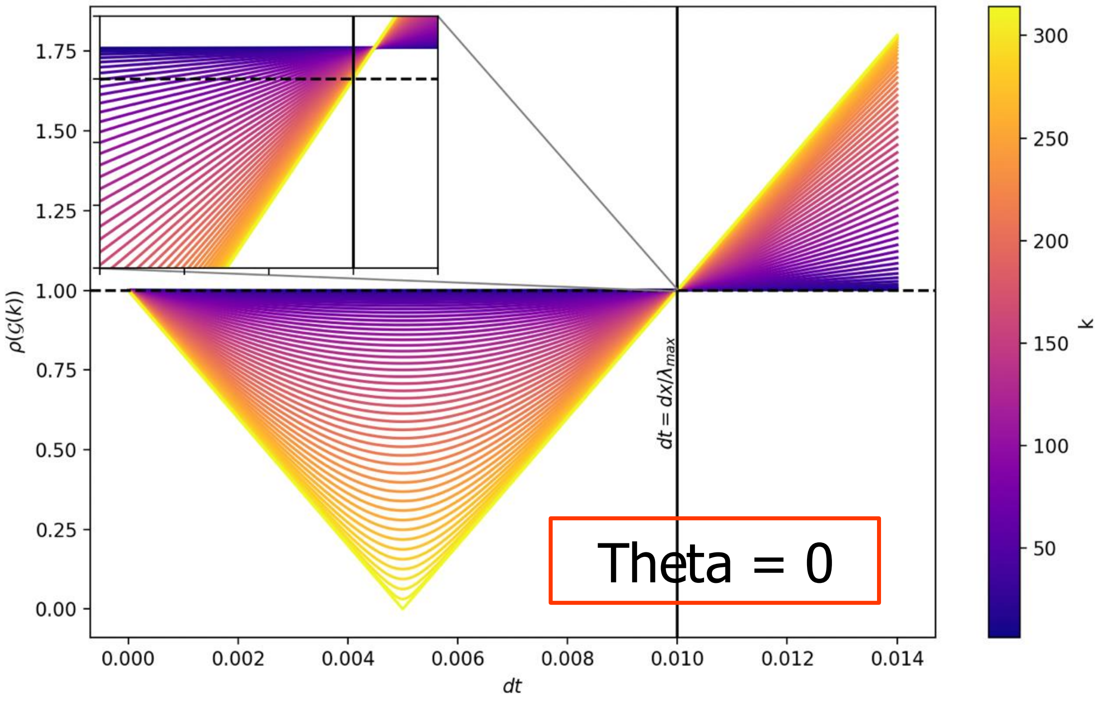
- Admissible wave numbers: the periodicity of our domain constraints the waves + Nyquist limit relax the condition.

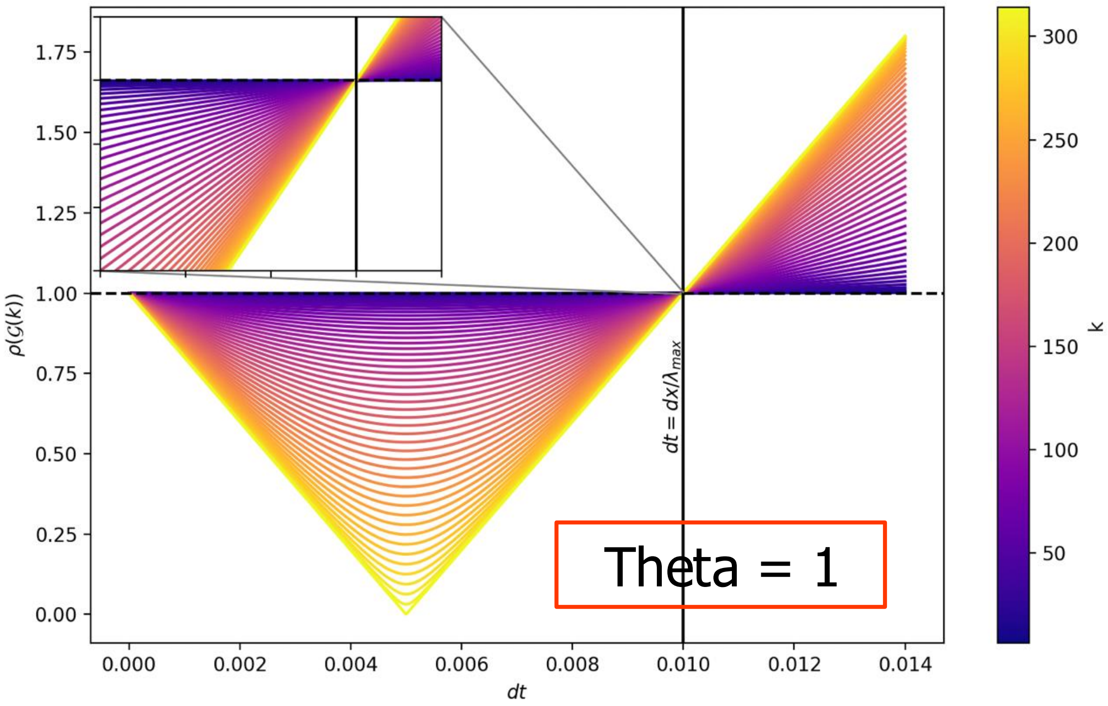
$$k_n \in \left\{ \frac{2n\pi}{L_0}; 1 \leq \ n \leq \lfloor L_0 / 2\Delta x \rfloor \right\}.$$

Euler-Poisson analysis

➢ <u>Proposition (Fabre 1992):</u>

A necessary condition of stability for IMEX theta-scheme of Euler-Poisson system is theta=1 (+standard CFL condition for transport).





➢ Figures: Spectral radius of the IMEX amplification matrix for all admissible waves and time steps.

Euler-Poisson analysis

- <u>Proposition (Fabre 1992):</u>
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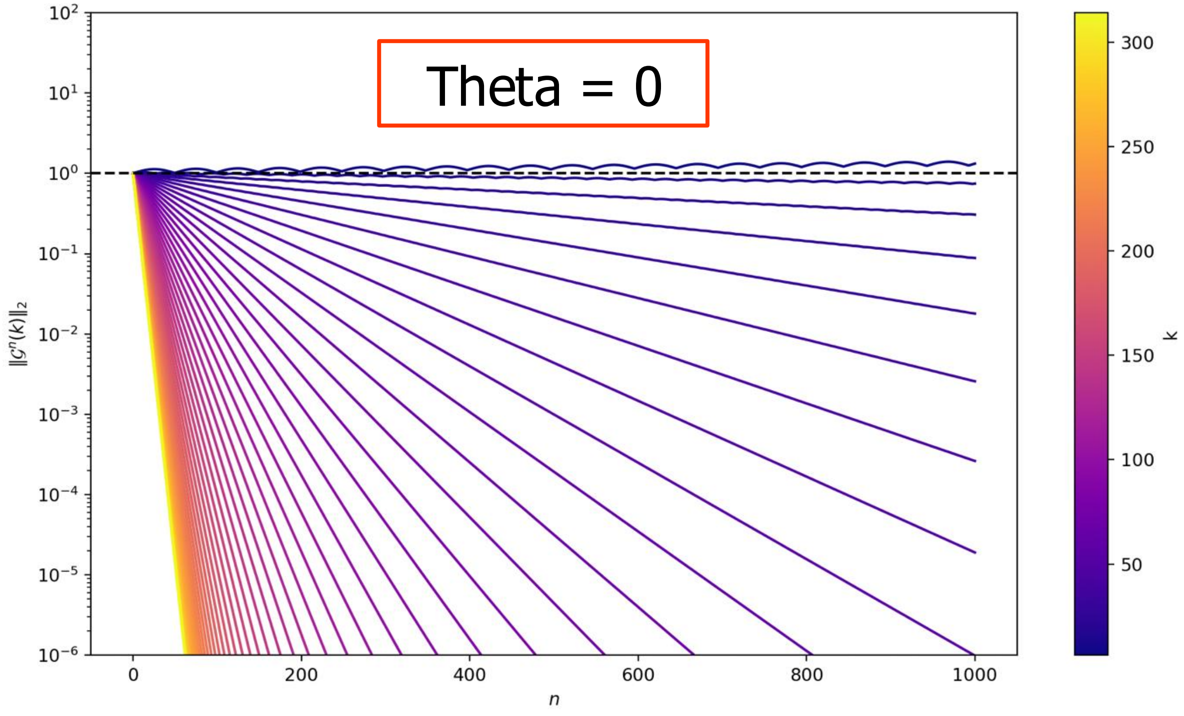
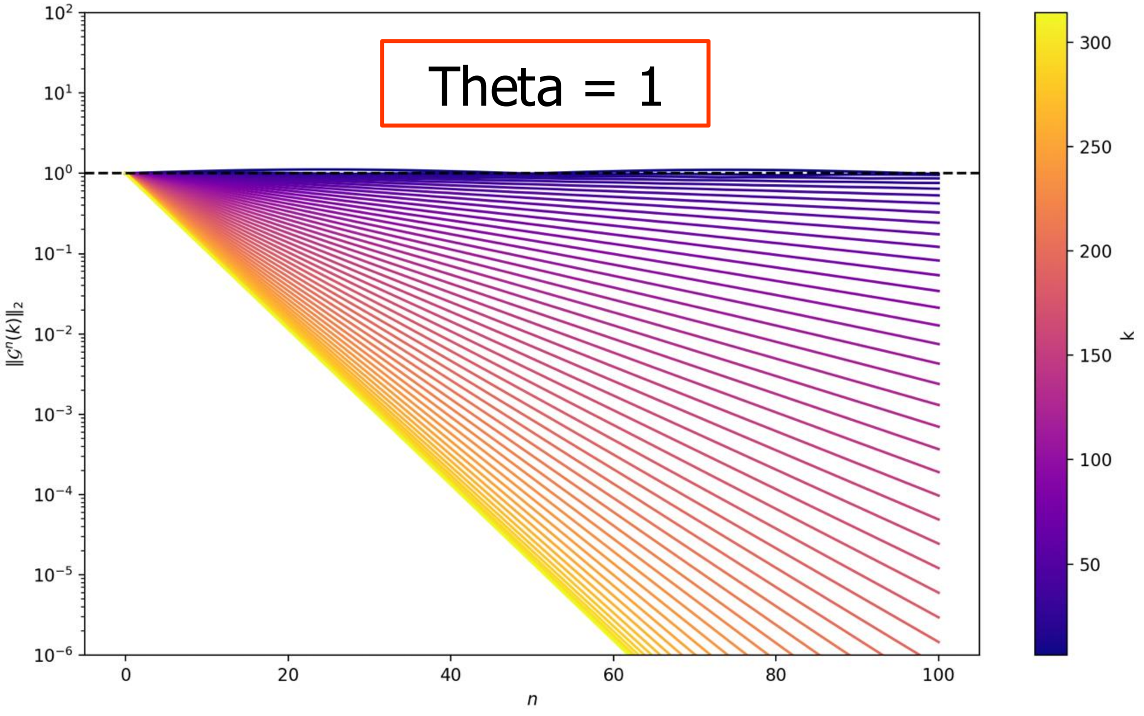
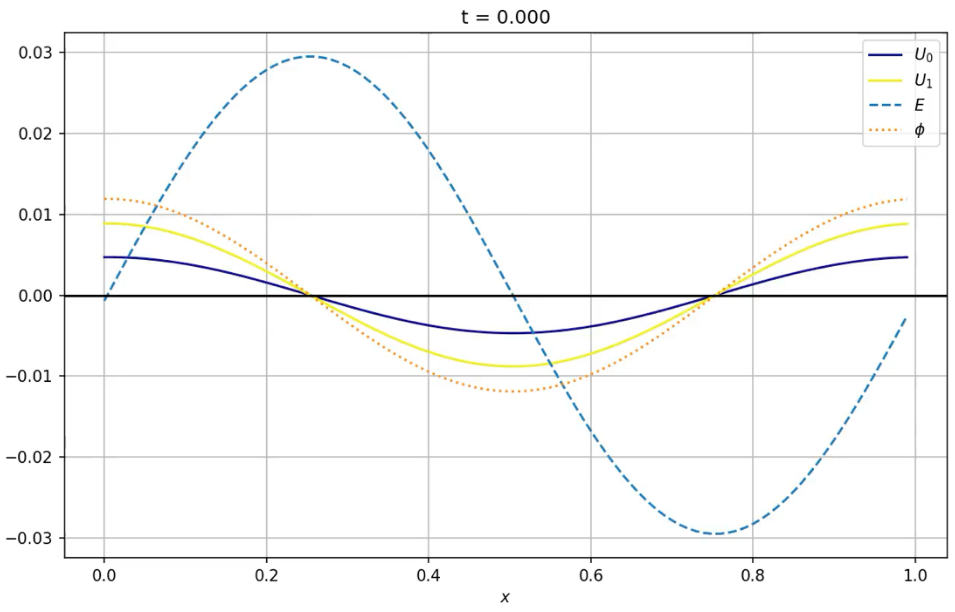


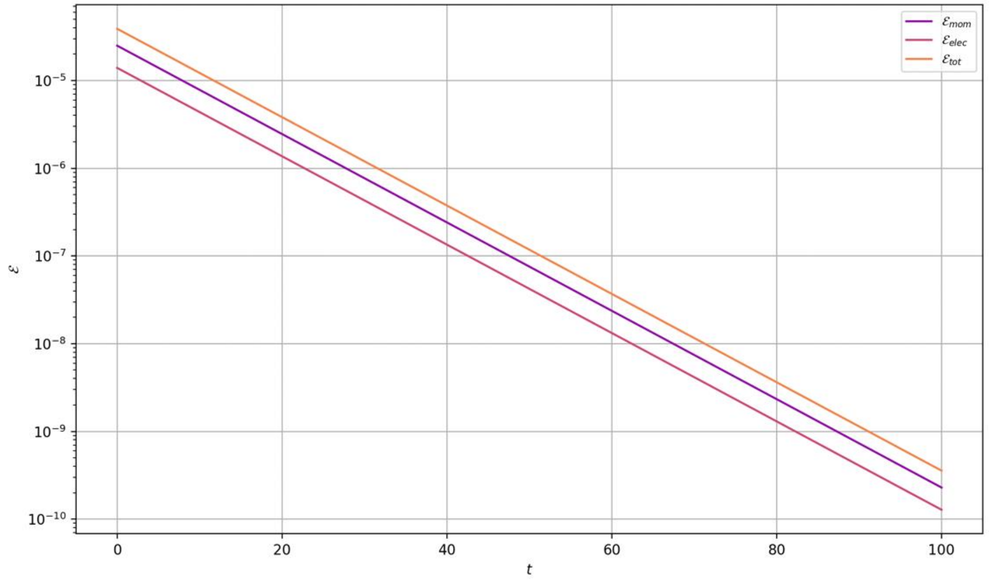


- Figures: Asymptotic boundness of IMEX amplification matrix for admissible waves.

Weakening Euler-Poisson analysis

- <u>Proposition:</u>
Restricting the stability analysis to the admissible spectrum yields <u>a relaxed CFL condition</u> that ensures numerical stability <u>for finite-resolution simulations</u> even in the regime where the continuous Von Neumann condition predicts instability.



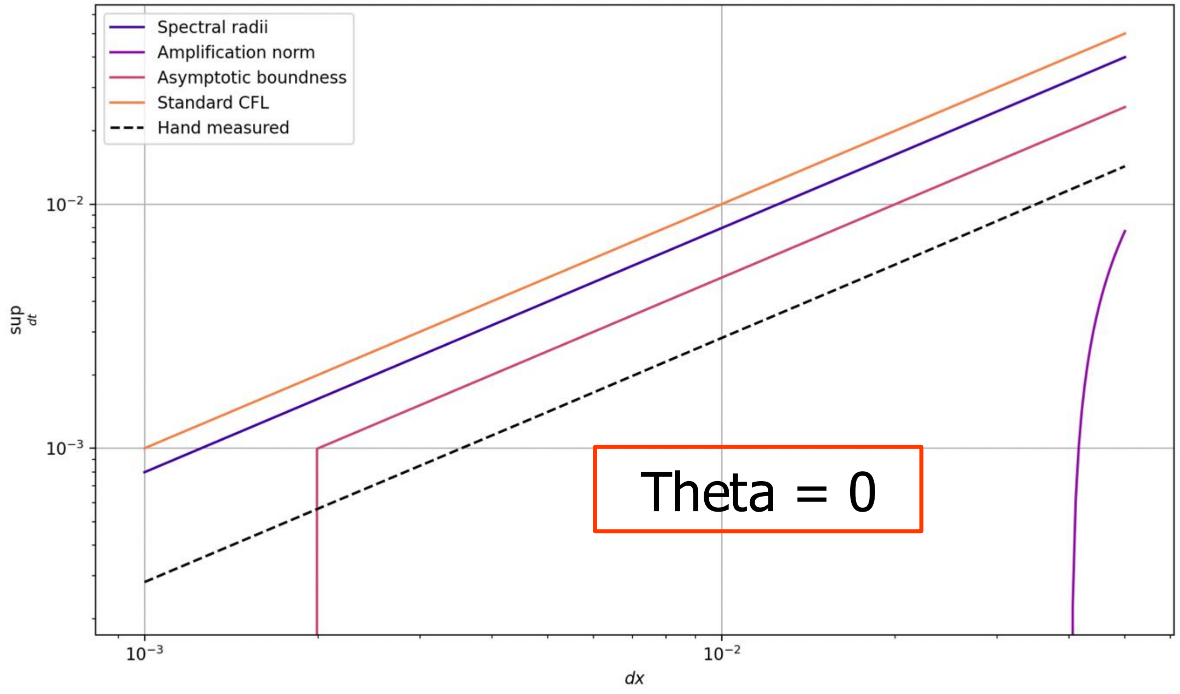


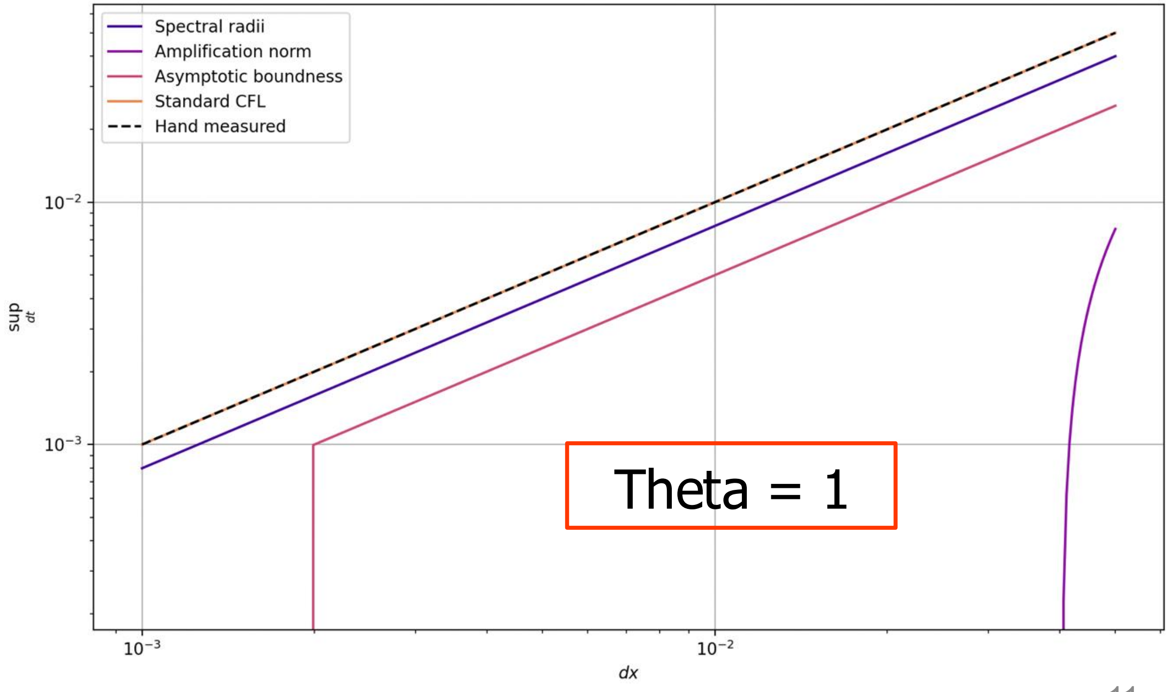
- Figures: Stable simulation of <u>worst wave number</u> with acceptable CFL: $\Delta t = 0.2\Delta x/\lambda_{max}$

Acceptability of a CFL condition ?

- We still need in practice an acceptable scaling with respect to dx!
- <u>Measures of the CFL condition:</u>
  Spectral radii, amplification norm, asymptotic boundness, dicotomic numerical tests.

$$\sup_{\Delta t}\{\rho(\mathcal{G}(k)) < 1; \forall k \in \mathcal{K}\}, \quad \sup_{\Delta t}\{\|\mathcal{G}(k)\|_2 < 1; \forall k \in \mathcal{K}\}, \quad \sup_{\Delta t}\{\sup_{n} \|\mathcal{G}^n(k)\|_2 < +\infty; \forall k \in \mathcal{K}\}.$$





Scalability to Hermite moments

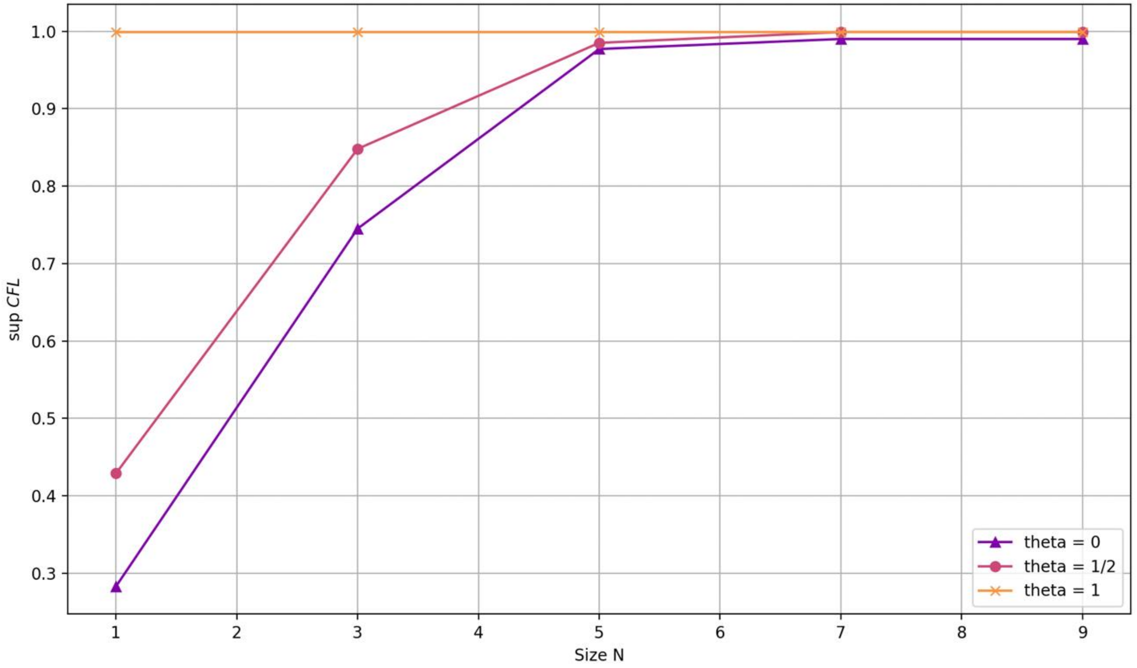
- <u>Cost of the Hermite moments:</u> The transport matrix max eigenvalue scales like $2\sqrt{N}$.

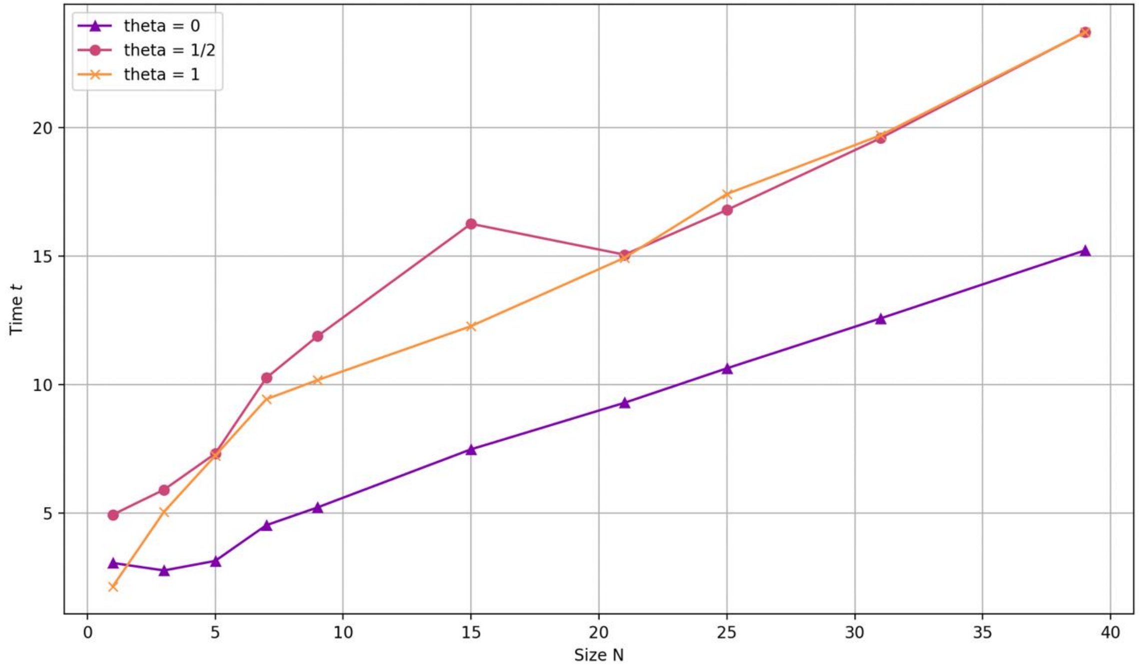
Does it exist a number of moments from which a scheme is preferred ?

- Our intuition is that the CFL cost of our relaxed Von Neumann condition vanishes as N grows.

$$\sup\{C | dt = C * dx/\lambda_{max} \text{ is stable}\}$$

+ computation time





Conclusion and perspectives

- We introduced the linearized Hermite moments hierarchy which is a generalization of Euler-Poisson in the linearize regime.
- We raised awareness on the immediate results of a naïve Von Neumann analysis: the condition is not sufficient for non-normal matrices! (which is the case for Euler-Poisson/Hermite-Poisson).
- Potential growth is governed by the norm and the asymptotic boundness rather than by the spectral radius
- We weaken the Von Neumann condition to a problem adapted condition which leads to a theoretical gain in stability for explicit schemes.
- We showed that the gain is not only theoretical and explain it partially with the CFL number.

Laboratoire de Physique des Plasmas

Thank you for any feedback!