

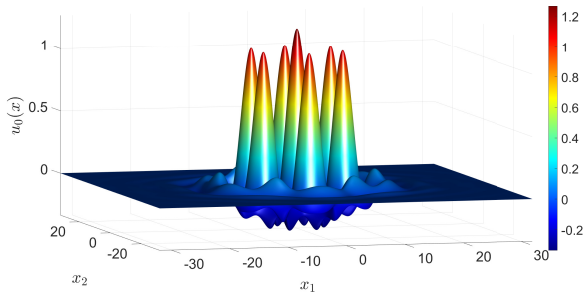
Computer-assisted existence proofs of localized solutions on \mathbb{R}^m

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How do mathematicians study differential equations?

Pen and Paper Analysis
&
Computations

- Partial Differential Eqs.
- Nonlinear Analysis
- Functional Analysis
- Fixed Point Theory
- Picard Iteration
- Fourier Analysis
- Calculus of Variations
- Bifurcation Analysis
- Compact Resolvent
- Sobolev Embeddings

- Scientific Computing
- LLMs
- Finite Elements
- Spectral Methods
- Chebyshev Series
- Newton-Kantorovich
- Galerkin Approx.
- Continuation Methods
- Fast Fourier Transform
- Interval Arithmetic



Computer-Assisted Proofs

In this talk, computer-assisted proofs is referred to as a numerical tool for completing an analytical proof, in the form of a rigorous enclosure of :

- A matrix norm
- A discrete convolution
- An upper bound for a series
- An upper bound for an integral
- ...

The rigorous numerics are achieved thanks to *interval arithmetic*. Interval arithmetic is a mathematical technique used to mitigate rounding and measurement errors in mathematical computation by computing function bounds.

Example :

The number π can be enclosed in the interval $[3, 4]$ where 3 and 4 are exactly represented on the computer. Then, an enclosure for π^2 is given as

$$\pi^2 \in [3^2, 4^2] = [9, 16]. \quad (1)$$

Suppose that we are interested in finding a zero $\tilde{u} \in H$ of

$$\mathbb{F} : H \rightarrow H_0, \quad \text{where } H, H_0 \text{ are Banach spaces.} \quad (2)$$

Example : $\mathbb{F}(u) = (I + \Delta)^2 u + \mu u + u^3 - \nu u^2$, $H = H^4(\mathbb{R})$ and $H_0 = L^2(\mathbb{R})$.

Suppose that we have access to a numerical solution $\bar{u} \in H$ (via a series representation for instance).

Question : Can we prove that there exists $\tilde{u} \in H$ such that $\mathbb{F}(\tilde{u}) = 0$ and $\|\tilde{u} - \bar{u}\|_H \leq r$, where r is explicit ?

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A Newton-Kantorovich approach

Suppose that $\mathbb{A} \approx D\mathbb{F}(\bar{u})^{-1}$ with $\mathbb{A} : H_0 \rightarrow H$. We want to prove that the Newton-like operator

$$\mathbb{T}(u) = u - \mathbb{A}\mathbb{F}(u)$$

has a fixed point in $B_r(\bar{u}) \subset H$.

Newton-Kantorovich

Suppose that $\mathbb{A} : H_0 \rightarrow H$ is a bounded injective linear operator and that Y , Z_1 and Z_2 satisfy

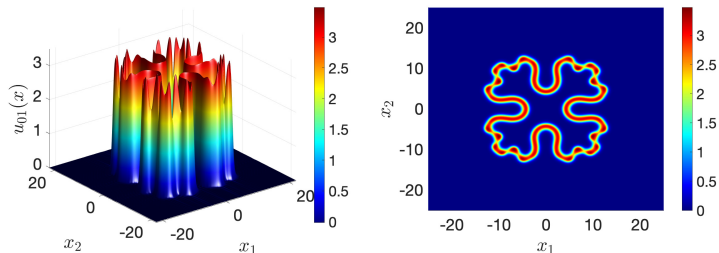
- $\|\mathbb{A}\mathbb{F}(\bar{u})\|_H \leq Y$
- $\|I - \mathbb{A}D\mathbb{F}(\bar{u})\|_H \leq Z_1$
- $\|\mathbb{A}(D\mathbb{F}(\bar{u} + h) - D\mathbb{F}(\bar{u}))\|_H \leq Z_2(r)r$ for all $h \in B_r(0)$.

If there exists $r > 0$ such that

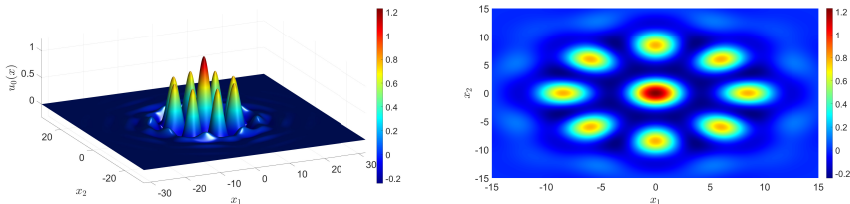
$$\frac{1}{2}Z_2(r)r^2 - (1 - Z_1)r + Y < 0 \text{ and } Z_1 + Z_2(r)r < 1, \quad (3)$$

then \mathbb{F} has a unique zero in $B_r(\bar{u}) \subset H$.

Localized stationary solutions



Gray-Scott Model : $\partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbf{u} + \mathbb{G}(\mathbf{u})$



Swift-Hohenberg PDE : $\partial_t u = -\mu u - (I + \Delta)^2 u + \mathbb{G}(u)$

An example : Stationary localized solutions in the 1D Swift-Hohenberg equation

The Swift-Hohenberg PDE is a classical model for pattern formation which reads as follows

$$\partial_t u + \mathbb{F}(u) = 0, \text{ where } \mathbb{F}(u) \stackrel{\text{def}}{=} (\Delta + I)^2 u + \mu u + u^3 - \nu u^2 \text{ and } \mu, \nu > 0. \quad (4)$$

In particular, it possesses *localized stationary solutions* solving

$$\mathbb{F}(u) = 0 \text{ where } u = u(x) \text{ with } x \in \mathbb{R}^m \text{ and } \lim_{|x| \rightarrow \infty} u = 0. \quad (5)$$

We will consider $m = 1$ as an example.

We denote

$$\mathbb{L} = (\Delta + I)^2 + \mu I \quad (6)$$

$$\mathbb{G}(u) = u^3 - \nu u^2 \quad (7)$$

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Banach space H

We define the space H as

$$H \stackrel{\text{def}}{=} \overline{\{u \in L^2(\mathbb{R}), u \text{ even and } \|u\|_H < \infty\}}^{\|\cdot\|_H} \text{ where } \|u\|_H = \|\mathbb{L}u\|_2. \quad (8)$$

$(H, \|\cdot\|_H)$ is a Banach space whenever $\mu > 0$ and $H \cong H^4(\mathbb{R})$.

Isometry : $\mathbb{L} : H \rightarrow L^2$ is an isometric isomorphism.

Bounded product : It allows to define bounded products $H \times H \rightarrow L^2$ (Sobolev embedding) :

$$\|uv\|_2 \leq \kappa \|u\|_H \|v\|_H \quad \text{with } \kappa \leq \frac{\|1/l\|_2}{\mu} \text{ and } l = \mathcal{F}(\mathbb{L}). \quad (9)$$

Actually, we have that the product $H \times H \rightarrow H$ is also a bounded bilinear map. Consequently $\mathbb{F}(u) = \mathbb{L}u + \mathbb{G}(u) : H \rightarrow L^2$ is a smooth map (since its is polynomial).

Regularity : Let $u \in H$ such that $\mathbb{F}(u) = 0$, then

$$u = -\mathbb{L}^{-1}\mathbb{G}(u). \quad (10)$$

Using the above $\mathbb{G}(u) \in H \cong H^4(\mathbb{R})$, implying that $u = -\mathbb{L}^{-1}\mathbb{G}(u) \in H^8(\mathbb{R})$. Using a bootstrapping argument

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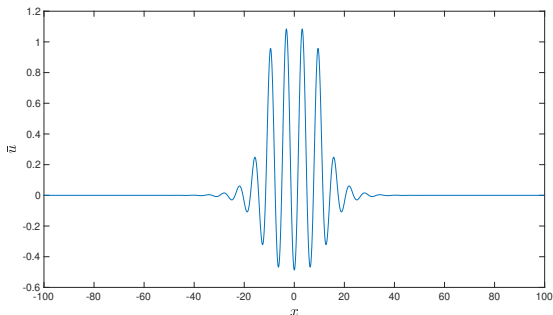
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Approximate solution \bar{u}



- $\text{supp}(\bar{u}) \subset \Omega_d = (-d; d)^m$.
- Numerically, we can define \bar{u} on Ω_d by a Fourier series $\bar{U} = (\bar{U}_n)_{n \in \mathbb{Z}^m}$ where $\bar{U}_n = 0$ if $|n| > N$ for some $N \in \mathbb{N}$.
- \bar{u} is fully determined by \bar{U} :
$$\bar{u}(x) = \mathbf{1}_{\Omega_d}(x) \left(\bar{U}_0 + 2 \sum_{n=0}^N \bar{U}_n \cos\left(\frac{n\pi}{d}x\right) \right).$$

The numerical approximation

We define operators on the Fourier coefficients. Recall

$$\partial_x \iff \text{diag} \left(i \frac{n\pi}{d} \right) \text{ and } uv \iff (U * V)_n = \sum_{k \in \mathbb{Z}} U_{n-k} V_k \quad (12)$$

Define

$$L_d = \text{diag} \left(l \left(\frac{n}{2d} \right) \right) \text{ with } l(\xi) = ((2\pi\xi)^2 - 1)^2 + \mu \text{ and } G_d(U) = U * U * U - \nu U * U. \quad (13)$$

Note that l is the Fourier transform of \mathbb{L} .

Numerically, our problem is

$$F_d(U) = 0 \text{ where } F_d(U) = L_d U + G_d(U). \quad (14)$$

We look for solutions that decay quickly as $|x| \rightarrow d$.

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Do we have $\bar{u} \in H$?

Recall $\bar{u}(x) = \mathbf{1}_{\Omega_d}(x)(\bar{U}_0 + 2 \sum_{n=0}^N \bar{U}_n \cos(\frac{n\pi}{d}x))$. So u is smooth everywhere except at $\pm d$
 $\implies u \in L^2$ **but not in H !!**

Trace theorem : In order to have $\bar{u} \in H \cong H^4(\mathbb{R})$, we need

$$\frac{d^k}{dx^k} \bar{u}(d) = 0 \text{ for all } k = 0, \dots, 3 \quad (\text{the case } x = -d \text{ is satisfied by periodicity}). \quad (15)$$

For $k = 1, 3$ this is already satisfied since $\sin(n\pi) = 0$.

Equivalently

$$U_0 \delta_k + 2 \sum_{n=1}^N U_n \left(\frac{n\pi}{d}\right)^k = 0 \text{ for all } k = 0, 2 \quad (16)$$

that is $U = (U_n)_{n=0}^N$ is in the kernel of a matrix $\mathcal{T} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^2$.

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$$|U_n| \approx \rho^n \text{ with } 0 < \rho < 1. \quad (17)$$

We want to construct a projection that preserves this decay.

Let S be the diagonal matrix defined as

$$(SU)_n = \rho^{-n}, \quad (18)$$

we want to solve

$$\mathcal{P}_S(U) = \arg \min_{V \in \text{Ker}(\mathcal{T})} \|S(U - V)\|_{\ell^2}. \quad (19)$$

The associated orthogonal projection $\mathcal{P}_S : \mathbb{R}^{N+1} \rightarrow \text{Ker}(\mathcal{T})$ is given explicitly by

$$\mathcal{P}_S(U) = U - S^{-2} \mathcal{T}^* (\mathcal{T} S^{-2} \mathcal{T}^*)^{-1} \mathcal{T} U \quad (20)$$

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Complete construction of \bar{u}

- We use a numerical method to compute U such that $F_d(U)$ is machine precision small (usually a Newton method).
- We build the matrix $\mathcal{P}_S = I - S^{-2}\mathcal{T}^*(\mathcal{T}S^{-2}\mathcal{T}^*)^{-1}\mathcal{T}$ **using interval arithmetic**.
- We compute $\bar{U} \stackrel{\text{def}}{=} \mathcal{P}_S U$ **using interval arithmetic**.

Then we define (abstractly)

$$\bar{u}(x) = \mathbf{1}_{\Omega_d}(x) \left(\bar{U}_0 + 2 \sum_{n=1}^N \bar{U}_n \cos\left(\frac{n\pi}{d}x\right) \right) \quad (21)$$

and obtain that

- $\bar{u} \in H \cong H^4(\mathbb{R})$
- $\text{supp}(\bar{u}) \subset \Omega_d = (-d, d)$
- \bar{u} is represented exactly thanks to $N + 1$ Fourier coefficients given as numerical intervals
- \bar{U} preserves the geometric decay ρ^n of the original U .

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Approximate inverse

A Fourier analysis problem : Can we construct $D\mathbb{F}(\bar{u})^{-1}$ thanks to $DF(\bar{U})^{-1}$?

$$\mathbb{A} : L^2 \rightarrow H \iff \mathbb{B} = \mathbb{L}\mathbb{A} : L^2 \rightarrow L^2 \quad (22)$$

and \mathbb{B} approximates $D\mathbb{F}(\bar{u})\mathbb{L}^{-1} = I + DG(\bar{u})\mathbb{L}^{-1} : L^2 \rightarrow L^2$ with $DG(\bar{u}) = 3\bar{u}^2 - 2\nu\bar{u}$.

Lemma

$DG(\bar{u})\mathbb{L}^{-1} : L^2 \rightarrow L^2$ is compact $\implies DG(\bar{u})\mathbb{L}^{-1}$ is the limit of a family of matrices.

In fact, because \bar{u} is compactly supported, we can prove that there exists $C, a > 0$ such that

Lemma

$$\|DG_d(\bar{U})L_d^{-1} - DG(\bar{u})\mathbb{L}^{-1}\|_2 \leq C\|e^{ax}\bar{v}\|_2 e^{-ad} \quad \text{where } \bar{v} = 3\bar{u}^2 - 2\nu\bar{u}$$

Let Π^N be the Galerkin projection of order N , there exists C independent of N such that

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A Fourier analysis problem : Can we construct $DF(\bar{u})^{-1}$ thanks to $DF(\bar{U})^{-1}$?

$$\mathbb{A} : L^2 \rightarrow H \iff \mathbb{B} = \mathbb{L}\mathbb{A} : L^2 \rightarrow L^2 \quad (22)$$

and \mathbb{B} approximates $DF(\bar{u})\mathbb{L}^{-1} = I + DG(\bar{u})\mathbb{L}^{-1} : L^2 \rightarrow L^2$ with $DG(\bar{u}) = 3\bar{u}^2 - 2\nu\bar{u}$.

Lemma

$DG(\bar{u})\mathbb{L}^{-1} : L^2 \rightarrow L^2$ is compact $\implies DG(\bar{u})\mathbb{L}^{-1}$ is the limit of a family of matrices.

In fact, because \bar{u} is compactly supported, we can prove that there exists $C, a > 0$ such that

Lemma

$\|DG_d(\bar{U})L_d^{-1} - DG(\bar{u})\mathbb{L}^{-1}\|_2 \leq C\|e^{ax}\bar{v}\|_2 e^{-ad}$ where $\bar{v} = 3\bar{u}^2 - 2\nu\bar{u}$

Let Π^N be the Galerkin projection of order N , there exists C independent of N such that

Lemma

$\|DG_d(\bar{U})L_d^{-1} - \Pi^N DG_d(\bar{U})L_d^{-1} \Pi^N\|_2 \leq C \frac{d^4}{N^4}$

We have shown that

$$I + DG(\bar{u})\mathbb{L}^{-1} = I + \Pi^N DG_d(\bar{U})L_d^{-1}\Pi^N + \mathcal{O}(e^{-ad}) + \mathcal{O}\left(\frac{d^4}{N^4}\right). \quad (23)$$

Numerically, we construct the matrix $\Pi^N + \Pi^N DG_d(\bar{U})L_d^{-1}\Pi^N$ and we construct an approximate inverse B^N for it. Then, our approximate inverse is

$$\mathbb{A} \stackrel{\text{def}}{=} \mathbb{L}^{-1}\mathbb{B} = \mathbb{L}^{-1}(I - \Pi^N + B^N). \quad (24)$$

Moreover, we have that

$$\|I - \mathbb{A}DF(\bar{u})\|_H = \|I - \mathbb{B}DF(\bar{u})\mathbb{L}^{-1}\|_2 \quad (25)$$

$$= \|\Pi^N - B^N(\Pi^N + \Pi^N DG_d(\bar{U})L_d^{-1}\Pi^N)\|_2 + \mathcal{O}(e^{-ad}) + \mathcal{O}\left(\frac{d^4}{N^4}\right). \quad (26)$$

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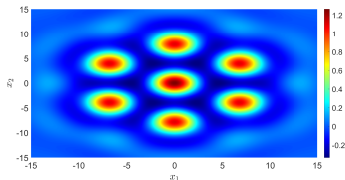
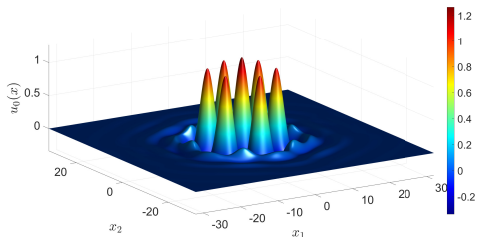
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Example of a proof in the Swift-Hohenberg PDE

We look for a solution \tilde{u} of

$$-\mu u - (I + \Delta)^2 u - 1.6u^2 + u^3 = 0, \text{ where } u = u(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ and } \lim_{|(x,y)| \rightarrow \infty} u(x, y) = 0 \quad (27)$$



On the computer, the approximate solution is written as

$$\bar{u}(x, y) = \begin{cases} \sum_{n_1=0}^{100} \sum_{n_2=0}^{100} u_{n_1, n_2} \cos\left(\frac{n_1 \pi}{d} x\right) \cos\left(\frac{n_2 \pi}{d} y\right) & \text{if } (x, y) \in [-d, d]^2 \\ 0 & \text{if } (x, y) \notin [-d, d]^2 \end{cases} \quad (28)$$

Theorem

Let $r = 10^{-5}$, then there exists a unique solution \tilde{u} to (27) such that $\|\bar{u} - \tilde{u}\|_{H^4(\mathbb{R}^2)}$.

Summary of the constructive existence proof

We are able to construct $\bar{u} \in H$, using Fourier coefficients on $\Omega_d = (-d, d)^m$, such that $\text{supp}(\bar{u}) \subset \Omega_d$ and prove that there exists $\tilde{u} \in H$ such that

$$\mathbb{F}(\tilde{u}) = 0 \text{ and } \|\tilde{u} - \bar{u}\|_H \leq r$$

for some explicit (small) $r > 0$.

We are interested in parabolic PDEs of the form

$$\partial_t u = \mathbb{F}(u) \text{ where } \mathbb{F}(u) \stackrel{\text{def}}{=} \mathbb{L}u + \mathbb{G}(u)$$

and $u : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}$, $\lim_{|x| \rightarrow \infty} u(t, x) = 0$.

Given \tilde{u} such that $\mathbb{F}(\tilde{u}) = 0$, we are interested in controlling the spectrum of $D\mathbb{F}(\tilde{u})$.

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Given \tilde{u} such that $\mathbb{F}(\tilde{u}) = 0$, we are interested in controlling the spectrum of $D\mathbb{F}(\tilde{u})$.

The essential spectrum of $D\mathbb{F}(\tilde{u})$ can be computed explicitly thanks to the Fourier transform of \mathbb{L} : $\sigma_{ess}(D\mathbb{F}(\tilde{u})) = \text{Range}(\mathcal{F}(\mathbb{L}))$ (using Weyl's perturbation theory).

In practice, σ_{ess} is on the left part of the complex plane (strictly negative real part) : it does not influence stability.

We want to control the eigenvalues of $D\mathbb{F}(\tilde{u})$ away from σ_{ess} .

Lemma

Let $\delta > 0$ and σ_δ be defined as

$$\sigma_\delta \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C}, |l(\xi) - \lambda| > \delta \text{ for all } \xi \in \mathbb{R}^m\}.$$

Then,

$$\lambda \in \sigma_\delta \cap \sigma(D\mathbb{F}(\tilde{u})) \implies \lambda \in \sigma_\delta \cap \text{Eig}(D\mathbb{F}(\tilde{u})). \quad (29)$$

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Enclosure of the spectrum of $D\mathbb{F}(\tilde{u})$

We prove that the map $M : [d, \infty] \rightarrow \mathcal{B}(L^2)$, where $\mathcal{B}(L^2)$ is the set of bounded linear operators on L^2 , given as follows

$$M(q) = (DF_d(\bar{U}))^{-1} \quad (30)$$

is smooth and $M(\infty) = (D\mathbb{F}(\bar{u}))^{-1}$ in the strong L^2 sense.

Proposition

There exists $C, a_\delta > 0$ such that for all $q \in [d, \infty]$, given $\lambda \in \text{Eig}(DF_q(\bar{U})) \cap \sigma_\delta$, there exists $\mu \in \text{Eig}(DF_d(\bar{U})) \cap \sigma_\delta$ such that

$$|\lambda - \mu| \leq Ce^{-a_\delta d}. \quad (31)$$

We can build disks $B_{r_j}(\mu_j)$ around the eigenvalues μ_j of $DF_d(\bar{U})$, and prove that $\text{Eig}(DF_q(\bar{U})) \cap \sigma_\delta \subset \cup_{j \in \mathbb{N}} B_{r_j}(\mu_j)$.

Using Kato *Perturbation Theory for Linear Operators*, we prove the following

Theorem

Let $\mathcal{I} \subset \mathbb{N}$ such that $|\mathcal{I}| = k$. If $\cup_{k \in \mathcal{I}} B_{r_j}(\mu_j)$ is disjoint from $\cup_{j \in \mathbb{N} \setminus \mathcal{I}} B_{r_j}(\mu_j)$, then $\cup_{k \in \mathcal{I}} B_{r_j}(\mu_j) \cap \sigma_\delta$ contains exactly k eigenvalues of $D\mathbb{F}(\bar{u})$ counted with multiplicity.

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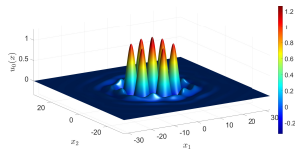
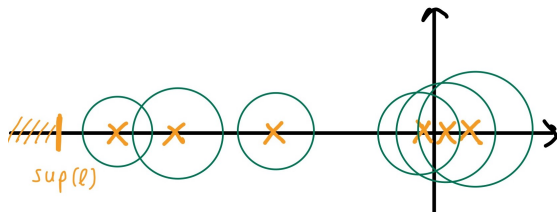
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$$\partial_t u = -\mu u - (I + \Delta)^2 u - 1.6u^2 + u^3.$$



We prove that 0 is an eigenvalue of multiplicity 3, corresponding to the invariances under rotation and translation (in x and y). The rest of the spectrum is strictly negative.

Lemma

The hexagonal solution is stable (quotienting out the invariances).

Extensions of the method : recent publications and preprints

- Nonlocal equations : solitary waves in the Whitham equation
- Existence proofs of branches of solutions
- Non-polynomial non-linearities : traveling wave solutions in the suspension bridge equation (with L. van der Aalst)
- Dihedral-symmetric solutions in the planar Swift-Hohenberg PDE (with D. Blanco)
- Bifurcations of localized solutions : saddle node in activator-inhibitor type models (with D. Blanco and D. Fassler)

Developing analytical techniques for

- Periodic operators : Schrodinger type $\Delta + b(x)$ where b is periodic.
- Solutions oscillating at infinity (instead of vanishing).
- Self-similar solutions decaying like $\frac{1}{|x|^k}$

Developing numerical techniques for

- Estimating computations with minimal numerical memory and computational time.
- Finding non-trivial solutions : blow-up profiles.

Supplementary material (lecture notes and code) is available at

<https://www.matthieucadiot.com/teaching>

- [1] (with JP. Lessard and JC. Nave) *Rigorous computation of solutions of semi-linear PDEs on unbounded domains via spectral methods*
- [2] (with JP. Lessard and JC. Nave) *Stationary non-radial localized patterns in the planar Swift-Hohenberg PDE: constructive proofs of existence*
- [3] (with D. Blanco) *The 2D Gray-Scott system of equations: constructive proofs of existence of localized stationary patterns*
- [4] *Constructive proofs of existence and stability of solitary waves in the Whitham and capillary-gravity Whitham equations*
- [5] *Stability analysis for localized solutions in PDEs and nonlocal equations on \mathbb{R}^m*