Convergence Rates For Non-degenerate Elliptic PDEs On Junctions With Kirchhoff Conditions

Phi Hoang TRAN, IRMAR - INSA Rennes - France

Olivier LEY, IRMAR - INSA Rennes - France
Othmane JERHAOUI, IRMAR - INSA Rennes - France
Hélène HIVERT, IRMAR - INRIA - France

We present monotone finite-difference schemes for second-order nonlinear elliptic equations on a junction, with Dirichlet conditions at the boundary vertices and a Kirchhoff condition at the interior vertex, see [1] for a more general problem. In contrast with fully coupled discretizations on the whole junction studied by Morfe [4], we propose a decoupling strategy : the network problem is reduced to a family of Dirichlet problems posed on the individual branches and parametrized by the unknown junction value. Each branch problem is solved by a monotone scheme inspired by Crandall and Lions [2], while the junction value is recovered from a scalar nonlinear flux-balance equation by a Newton-type method. This approach is simple to implement, preserves sparsity, and is well suited to extensions to more general networks. On each branch, the numerical analysis yields first-order convergence for the solution and order 1/2 for the discrete derivative. At the junction level, the reconstruction recovers the classical 1/2 convergence rate obtained for coupled schemes such as the one in [4]. We illustrate the method for Hamiltonians of absolute-value type, using Lax–Friedrichs and upwind numerical approximations. The resulting nonlinear algebraic systems are solved by a semi-smooth Newton method, in particular Howard's algorithm, together with recent techniques for nonlinear absolute value equations proposed in [3].

[1] G. Barles, O. Ley, E. Topp. *Degenerate Elliptic PDEs on a Network with Kirchhoff Conditions.* Preprint, arXiv :2509.12848 [math.AP] (2025), 2025.

[2] M. G. Crandall, P.-L. Lions. *Two approximations of solutions of Hamilton-Jacobi equations.* Math. Comput., **43**, 1–19, 1984. doi :10.2307/2007396.

[3] A. Daniilidis, M. Haddou, T. M. Le, O. Ley, P. H. Tran. *Solving Nonlinear Absolute Value Equations.* Preprint, arXiv :2402.16439 [math.OC] (2026), 2026.

[4] P. S. Morfe. *Convergence & rates for Hamilton-Jacobi equations with Kirchoff junction conditions.* NoDEA Nonlinear Differential Equations Appl., **27(1)**, Paper No. 10, 69, 2020. doi : 10.1007/s00030-020-0615-1.

Contact : hoang-phi.tran@insa-rennes.fr