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The Structural Method for Partial Differential Equations

Virgile BERTRAND, Inria, IRMA - Université de Strasbourg - CNRS UMR 7501 - Strasbourg

Raphaël CÔTE, IRMA - Université de Strasbourg - CNRS UMR 7501 - Strasbourg
Emmanuel FRANCK, Inria, IRMA - Université de Strasbourg - CNRS UMR 7501 - Strasbourg
Victor MICHEL-DANSAC, Inria, IRMA - Université de Strasbourg - CNRS UMR 7501 - Strasbourg

The order of convergence of Finite Difference methods is limited by the stencil size. For exemple, a 3-point scheme cannot be more than 2nd order accurate. Taking a larger stencil allows for higher order schemes, at the cost of lower spectral resolution and some special care for boundary points.

The Structural Method [2] overcomes those limitations by introducing state variables for the differential operators involved in the PDE. For example, if we consider a simple partial differential equation

$$\nu\partial_x u - \kappa\partial_{xx}u = f \tag{1}$$

The Structural Method introduces state variables $Z_i := u(x_i)$, $D_i := \partial_x u\big|_{x_i}$ and $S_i := \partial_{xx}u\big|_{x_i}$ and rewrite the (discretized) PDE as

$$\begin{cases} \nu D_i - \kappa S_i = f_i & \text{(PE)} \\ D_i = \dfrac{Z_{i+1} - Z_{i-1}}{2\Delta x} & \text{(SE1)} \\ S_i = \dfrac{Z_{i-1} - 2Z_i + Z_{i+1}}{\Delta x^2} & \text{(SE2)} \end{cases}$$

Written as such, the physics of the problem (PE) is local and does not depend on the underlying discretization. It also allows us to change the so-called "structural equations" (SE1) and (SE2), for example one could use the high order Combined Compact Difference Scheme [1], that reads

$$\begin{cases} 0 = -15\dfrac{Z_{i+1} - Z_{i-1}}{(\Delta x)^2} + \dfrac{7D_{i+1} + 16D_i + 7D_{i-1}}{\Delta x} - (S_{i+1} - S_{i-1}) & \text{(SE1)} \\ 0 = 24\dfrac{Z_{i+1} - 2Z_i + Z_{i-1}}{\Delta x^2} - 9\dfrac{D_{i+1} - D_{i-1}}{\Delta x} + (S_{i+1} - 8S_i + S_{i-1}) & \text{(SE2)} \end{cases}$$

Thanks to the local information of the state variables, such a scheme is of order 6th while keeping a compact 3-point stencil, thus having nice spectral properties.

We are able to compute the coefficients of such equations as the kernel of a well chosen matrix. The extra degrees of freedom can also be used to enforce some physical constraint on the solution (e.g. div(u) = 0).

In the presentation, we will first introduce the structural method, comparing it to classical finite differences. We will then explain how to automatically compute the structural equations given a stencil and state variables. The properties of the structural method will finally be shown on a few simulations of hyperbolic PDEs (e.g. Advection, Burger's, Shallow water ...).

[1] P. C. Chu, C. Fan. *A three-point combined compact difference scheme*. Journal of Computational Physics, **140(2)**, 370–399, 1998. doi :https ://doi.org/10.1006/jcph.1998.5899.

[2] S. Clain, R. M. Pereira, P. A. Pereira, D. Lopes. *Structural schemes for one dimension stationary equations*. Applied Mathematics and Computation, **457**, 128207, 2023. doi : https ://doi.org/10.1016/j.amc.2023.128207.

Contact : `virgile.bertrand@inria.fr`