Modelling internal tides using plane waves: a Plane Wave Discontinuous Galerkin method for the (linearized) rotating shallow water equations.

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INRIA, Odyssey

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What do internal tides look like [2]

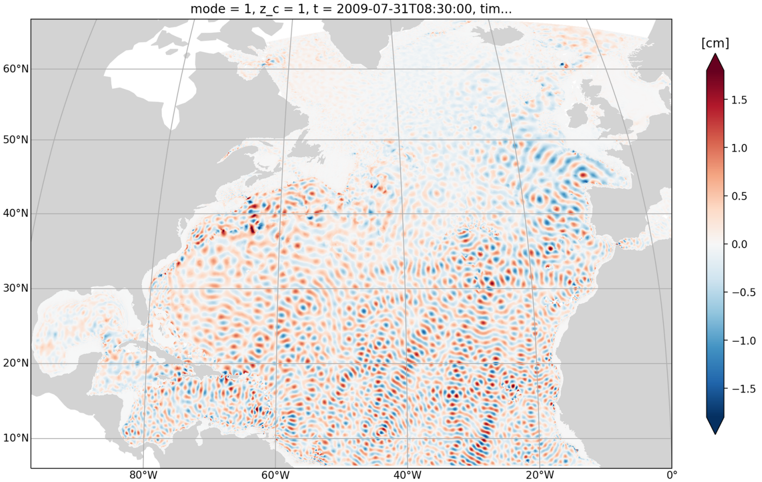

Figure: Contribution to sea level anomaly of mode 1 of the barotropic tide filtered at the semi-diurnal tide frequency (simulation).

Numerical methods

Finite Difference methods (FDM):

- needs lots of dofs (>3 points/wavelength),
- boundary conditions,
- does not use the information that the problem is pseudo-harmonic.

Spectral methods:

- less dofs than FDM,
- boundary conditions,
- when few dofs, poorly resolves subscale features like sources,
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Our approach

The solution seems to be locally a superposition of $\int_0^{2\pi} D(\theta)e^{i(\mathbf{k}(\theta)\cdot\mathbf{x}-\omega t)}d\theta$.

Existing for Helmholtz equation: Plane Wave Discontinuous Galerkin (PWDG) [1]

- good results with low resolution (<0.3 points/wavelength),
- Can implement b.c., source, background flow...

Plane waves and DG.

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The equation

$$\frac{\partial u}{\partial t} - fv + \mathbf{v}\cdot\nabla u + \frac{\partial p}{\partial x} = s_u$$
$$\frac{\partial v}{\partial t} + fu + \mathbf{v}\cdot\nabla v + \frac{\partial p}{\partial y} = s_v$$
$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + c^2(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = s_p$$

c, $\mathbf{v}$, f and s_* depend on the location.

The equation

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + CU = S$$

With

$$A = \begin{pmatrix} u_0 & 0 & 1 \\ 0 & u_0 & 0 \\ c^2 & 0 & u_0 \end{pmatrix}, B = \begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & 1 \\ 0 & c^2 & v_0 \end{pmatrix}, C = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S = \begin{pmatrix} s_u \\ s_v \\ s_p \end{pmatrix} \text{ and } U = \begin{pmatrix} u \\ v \\ p \end{pmatrix}$$

Agenda

1. Semi-discrete (time-harmonic) method: basis and test functions
2. Semi-discrete (time-harmonic) method: fluxes and boundary conditions
3. Space-time method
4. Test cases

Semi-discrete (time-harmonic) method: basis and test functions

Time-harmonicity

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + (C - i\omega I)U = S$$

We consider A, B and C locally constant and look for solutions $U|_K = \int_0^{2\pi} D_K(\theta)e^{i\mathbf{k}_K(\theta)\cdot\mathbf{x}}$, K is a cell of a mesh $\mathcal{T}_h$, D_K is a column.

With the variational formulation:

$$\sum_{K\in\mathcal{T}_h}\int_K (W^T(C-i\omega I) - \partial_x(W^T A) - \partial_y(W^T B))U + \sum_{K\cap K'\neq\varnothing}\int_{K\cap K'}(W^T FU)_K + (W^T FU)_{K'}$$
$$+ \sum_{K\cap\partial\Omega\neq\varnothing}\int_{K\cap\partial\Omega}(W^T FU)_K = \sum_{K\in\mathcal{T}_h}\int_K W^T S$$

where $F = An_x + Bn_y$.

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Basis functions

Looking for basis functions means looking for D_K and $\mathbf{k}_K$. They are solutions of the homogeneous problem

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + (C - i\omega I)U = 0.$$

$$|\mathbf{k}_1(\theta)| = \frac{\omega}{\mathbf{v}\cdot\theta},\ |\mathbf{k}_2^-| = \frac{\sqrt{\delta_2} - \omega\mathbf{v}\cdot\theta}{c^2 - (\mathbf{v}\cdot\theta)^2} \text{ and } |\mathbf{k}_2^+| = \frac{\sqrt{\delta_2} + \omega\mathbf{v}\cdot\theta}{(\mathbf{v}\cdot\theta)^2 - c^2}$$

$\delta_2 = f^2(\mathbf{v}\cdot\theta)^2 + (\omega^2 - f^2)c^2$.

Associated with the vectors:

$$R_1 = \begin{pmatrix} -i\theta\cdot e_y \\ i\theta\cdot e_x \\ f/|\mathbf{k}_1| \end{pmatrix} \text{ and } R_2^{\pm} = \begin{pmatrix} \omega_{\pm}^*\theta\cdot e_x + if\theta\cdot e_y \\ \omega_{\pm}^*\theta\cdot e_y - if\theta\cdot e_x \\ |\mathbf{k}_2^{\pm}|c^2 \end{pmatrix}$$

with $\omega_{\pm}^* = \sqrt{f^2 + \mathbf{k}_2^{\pm\,2}c^2}$.

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with $\omega_\pm^* = \sqrt{f^2 + \mathbf{k}_2^{\pm\,2}c^2}$.

Basis functions: direction

The solutions will be built in $V_h = \bigcup_{K \in \mathcal{T}_h} V_h^K$ where

$$V_h^K = \text{span}(\{\exp(i\,\mathbf{k}_1(\theta_1^j)\cdot(x,y))R_1(\theta_1^j) \mid j \in [\![1, N_1]\!]\} \cup \{\exp(i\,\mathbf{k}_2^-(\theta_{2-}^j)\cdot(x,y))R_2^-(\theta_{2-}^j) \mid j \in [\![1, N_2^-]\!]\} \cup \{\exp(i\,\mathbf{k}_2^+(\theta_{2+}^j)\cdot(x,y))R_2^+(\theta_{2+}^j) \mid j \in [\![1, N_2^+]\!]\}).$$

$$|\mathbf{k}_1(\theta)| = \frac{\omega}{\mathbf{v}\cdot\theta}.$$

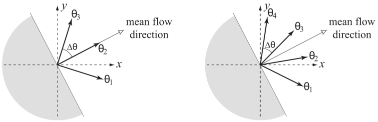

Figure: Galerkin Projection for $\mathbf{k}_1$ [1].

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$$
\begin{aligned}
V_h^K = \text{span}\big(&\{\exp\left(i\,\mathbf{k}_1(\theta_1^j)\cdot(x,y)\right)R_1(\theta_1^j) \mid j \in [\![1, N_1]\!]\} \\
&\cup \{\exp\left(i\,\mathbf{k}_2^-(\theta_{2-}^j)\cdot(x,y)\right)R_2^-(\theta_{2-}^j) \mid j \in [\![1, N_2^-]\!]\} \\
&\cup \{\exp\left(i\,\mathbf{k}_2^+(\theta_{2+}^j)\cdot(x,y)\right)R_2^+(\theta_{2+}^j) \mid j \in [\![1, N_2^+]\!]\}\big).
\end{aligned}
$$

$$
|\mathbf{k}_2^-| = \frac{\sqrt{\delta_2} - \omega \mathbf{v}\cdot\theta}{c^2 - (\mathbf{v}\cdot\theta)^2}.
$$

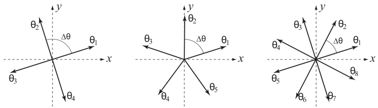

Figure: Galerkin Projection for $\mathbf{k}_2$ with a subsonic flow [1].

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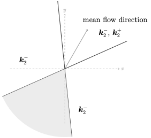


Figure: Galerkin Projection for $\mathbf{k}_2$ with a supersonic flow.

Test functions

$$\sum_{K\in\mathcal{T}_h}\int_K (W^T(C-i\omega I)-\partial_x(W^TA)-\partial_y(W^TB))U+\sum_{K\cap K'\neq\varnothing}\int_{K\cap K'}(W^TFU)_K+(W^TFU)_{K'}$$
$$+\sum_{K\cap\partial\Omega\neq\varnothing}\int_{K\cap\partial\Omega}(W^TFU)_K=\sum_{K\in\mathcal{T}_h}\int_K W^TS$$

For simplicity of the implementation, the test functions are chosen as solutions of the homogeneous problem

$$(C-i\omega I)^TW-A^T\partial_xW-B^T\partial_yW=0.$$

Test functions

$$(C - i\omega I)^T W - A^T \partial_x W - B^T \partial_y W = 0.$$

$$|\mathbf{k}_1(\theta)| = \frac{\omega}{\mathbf{v}\cdot\theta},\ |\mathbf{k}_2^-| = \frac{\sqrt{\delta_2} - \omega\mathbf{v}\cdot\theta}{c^2 - (\mathbf{v}\cdot\theta)^2} \text{ and } |\mathbf{k}_2^+| = \frac{\sqrt{\delta_2} + \omega\mathbf{v}\cdot\theta}{(\mathbf{v}\cdot\theta)^2 - c^2}.$$

Associated with the vectors:

$$Q_1 = \begin{pmatrix} -i\theta\cdot e_y \\ i\theta\cdot e_x \\ f/(|\mathbf{k}_1|c^2) \end{pmatrix} \text{ and } Q_2^{\pm} = \begin{pmatrix} \omega_\pm^*\theta\cdot e_x + if\theta\cdot e_y \\ \omega_\pm^*\theta\cdot e_y - if\theta\cdot e_x \\ |\mathbf{k}_2^{\pm}| \end{pmatrix}$$

with $\omega_\pm^* = \sqrt{f^2 + \mathbf{k}_2^{\pm\,2} c^2}$.

Test functions

The test functions are

$$\bigcup_{K\in\mathcal{T}_h} \{\{\exp\left(-i\,\mathbf{k}_1(\theta_1^j)\cdot(x,y)\right)Q_1(\theta_1^j) \mid j\in[\![1,N_1]\!]\} \cup \{\exp\left(-i\,\mathbf{k}_2^-(\theta_{2-}^j)\cdot(x,y)\right)Q_2^-(\theta_{2-}^j) \mid j\in[\![1,N_2^-]\!]\} \cup \{\exp\left(-i\,\mathbf{k}_2^+(\theta_{2+}^j)\cdot(x,y)\right)Q_2^+(\theta_{2+}^j) \mid j\in[\![1,N_2^+]\!]\}\}.$$

And we call W_h the span of the test functions.

Semi-discrete (time-harmonic) method: fluxes and boundary conditions

Fluxes, boundary conditions and source term

$$\underbrace{\sum_{K\cap K'\neq\varnothing}\int_{K\cap K'}(W^T FU)_K + (W^T FU)_{K'}}_{(1)} + \underbrace{\sum_{K\cap\partial\Omega\neq\varnothing}\int_{K\cap\partial\Omega}(W^T FU)_K}_{(2)}$$

$$= \underbrace{\sum_{K\in\mathcal{T}_h}\int_K W^T S}_{(3)}$$

Fluxes, boundary conditions and source term

$$\underbrace{\sum_{K\cap K'\neq\varnothing}\int_{K\cap K'}(W|_K - W|_{K'})^T(F^+_{K,K'}U|_K + F^-_{K,K'}U|_{K'})}_{\text{upwind-splitting flux}} + \underbrace{\sum_{K\cap\partial\Omega\neq\varnothing}\int_{K\cap\partial\Omega}(W^TFU)_K}_{(2)}$$

$$= \underbrace{\sum_{K\in\mathcal{T}_h}\int_K W^TS}_{(3)}$$

Fluxes, boundary conditions and source term

$$\underbrace{\sum_{K\cap K'\neq\varnothing}\int_{K\cap K'}(W|_K - W|_{K'})^T(F^+_{K,K'}U|_K + F^-_{K,K'}U|_{K'})}_{\text{upwind-splitting flux}} + \underbrace{\sum_{K\cap\partial\Omega\neq\varnothing}\int_{K\cap\partial\Omega} W^T F_\partial U_K}_{(2)}$$

$$= \underbrace{\sum_{K\in\mathcal{T}_h}\int_K W^T S}_{(3)} - \underbrace{\sum_{K\cap\partial\Omega\neq\varnothing}\int_{K\cap\partial\Omega} W^T G}_{(2)}$$

Space-time method

Space-time problem

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + CU = S$$

where $S(t,\mathbf{x}) = e^{-i\omega t}S^*(\mathbf{x})$. We look for solutions of the form $U = e^{-i\omega t}V$ where

$$\frac{\partial V}{\partial t} + A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y} + (C - i\omega I)V = S^*.$$

Thus we seek $V \in \mathcal{C}^1([0,T], V_h)$ s.t.

$$\forall t,\ \forall W \in W_h,\ \sum_{K\in\mathcal{T}_h}\int_K W^T\frac{\partial V}{\partial t} + W^T A_h V(t) = \text{RHS(W)}.$$

RHS is linear and represents boundary conditions and source terms.

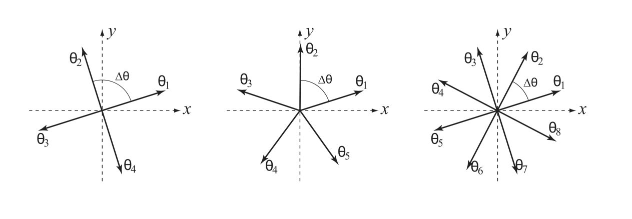
Space-time problem

Thus we look for $V : [0,T] \longrightarrow V_h$ and

$$M \frac{\partial V}{\partial t} + A_h V = \text{RHS}$$

where $M = (\int_\Omega W_i^T U_j)_{ij}$ is block diagonal, $\text{RHS} = (\text{RHS}(W_i))_i$ is a column.

Implementation



When $\Delta\theta$ is too small, M is (numerically) singular and the method is not stable.

Test cases

Convergence: plane wave, constant features

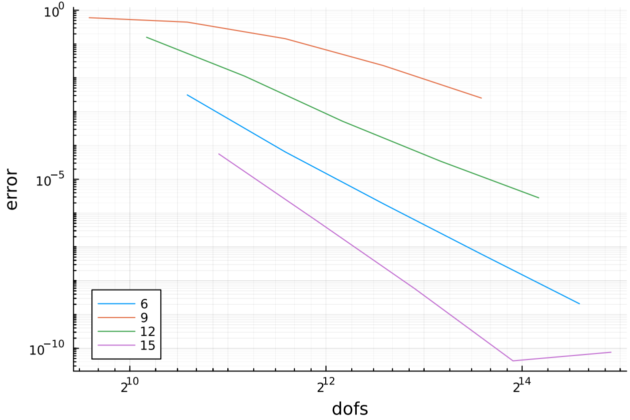


Figure: Convergence in the worst case with no background flow for a domain where the features remain constant.

Convergence: plane wave with background flow, constant features

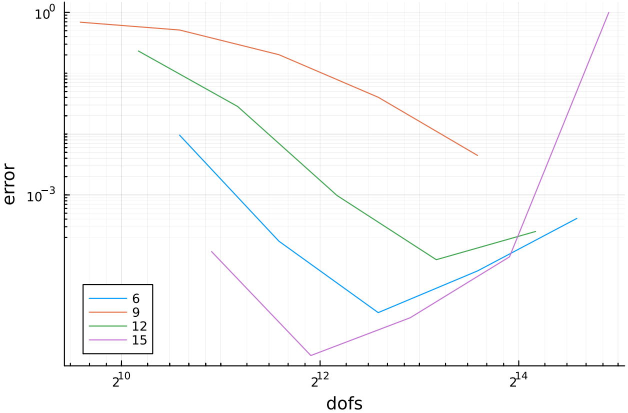


Figure: Convergence in the worst case with background flow for a domain where the features remain constant.

Convergence: varying speed of sound.

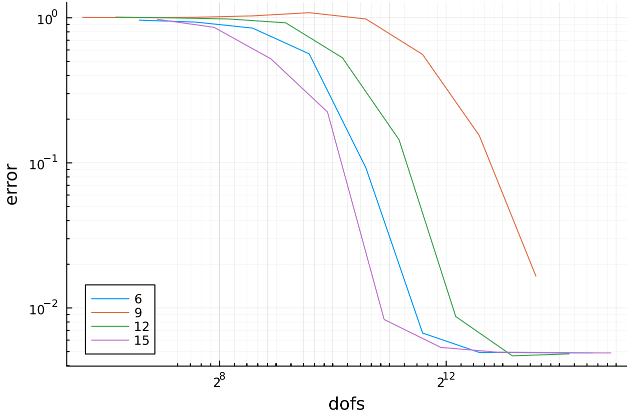


Figure: Convergence in the worst case with no background flow for a domain where the speed of sound changes.

Conclusion

- It does not need many dofs to converge.
- Easy to implement source terms.
- It is possible to implement many features (background flow, supersonic/subsonic).

Questions ?

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Questions ?

References

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