

Flow in fractured porous media: mathematical and numerical analysis

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INRIA Paris — ENPC

47ème Congrès d'Analyse Numérique
1er au 5 juin 2026
Saint-Jacut de la Mer



Outline

- 1 Introduction and model set-up
- 2 Mixed formulation
- 3 Discrete problem – numerical analysis
- 4 Validation examples

Motivations – fractured media in geological and digital contexts

- Fractures : **discontinuities** in the subsurface medium in the form of **thin** structures.
- Present at different scales, from μm to km. In this work: from 10 cm to 100 m.
- May be conductive pathways for the flow or barriers. In this work: **conductive**.



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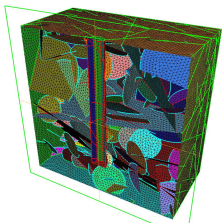
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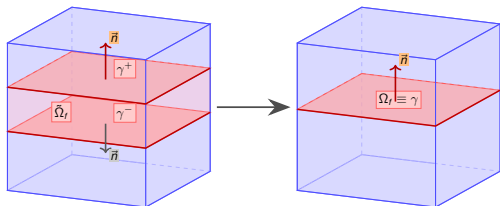
Fractures modeled as **two-dimensional** flat surfaces (e.g. polygons or disks) immersed in a three-dimensional space.



LabCom factory:
Itasca Consultants,
Géosciences Rennes

Davy, Le Goc, Darcel, Bour, de
Dreuzy, Munier, 2010

Fractured media: reduced model

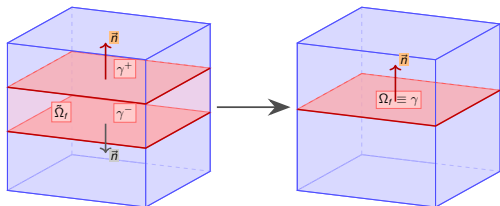


Replace real **thin** fracture by **average surface**
 In conservation law, add **exchange term** : **jump of flux** through the fracture.

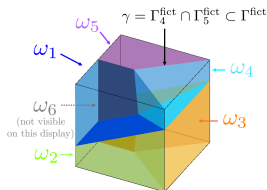
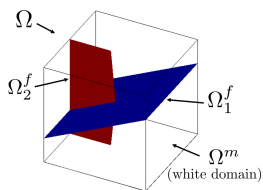
References

- Adler, Thovert (1999)
- Alboin, Jaffr , et al. (1999)
- Bastian. Helmig et al. (1999)
- Martin, Jaffr , Roberts (2005)
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Physical data and unknowns

K^m (resp. K^f) **conductivity** in the rock matrix (resp. fractures)

$p = (p^m, p^f)$: pressure

$u = (u^m, u^f)$: Darcy velocity

Mathematical model: stationary incompressible flow

Fracture model: Mass conservation + Darcy

$$\nabla_{\tau} \cdot \mathbf{u}_{\ell}^{\text{f}} = \quad + f_{\ell}^{\text{f}} \quad \text{in } \gamma_{\ell},$$

$$\mathbf{u}_{\ell}^{\text{f}} = -\mathcal{K}_{\ell}^{\text{f}} \nabla_{\tau} p_{\ell}^{\text{f}}$$

References

- Martin, Jaffré, Roberts (2005)
- Vohralík, Maryška, Severýn (2007)
- Brenner, Groza, Guichard, Lebeau, Masson (2016)
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Bulk model: Mass conservation + Darcy

$$\nabla \cdot \mathbf{u}_{\alpha}^{\text{m}} = f_{\alpha}^{\text{m}} \quad \text{in } \omega_{\alpha},$$

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Mathematical model: stationary incompressible flow

Fracture model: Mass conservation + Darcy

$$\nabla_{\tau} \cdot \mathbf{u}_{\ell}^{\text{f}} = [[\mathbf{u}^{\text{m}}]]_{|\gamma_{\ell}} + \mathbf{f}_{\ell}^{\text{f}} \quad \text{in } \gamma_{\ell},$$

$$\mathbf{u}_{\ell}^{\text{f}} = -\mathcal{K}_{\ell}^{\text{f}} \nabla_{\tau} p_{\ell}^{\text{f}}$$

Coupling conditions

$$[[\mathbf{u}^{\text{m}}]]_{|\gamma_{\ell}} = \gamma_n^{\text{m},+} \mathbf{u}_{|\gamma_{\ell}}^{\text{m}} + \gamma_n^{\text{m},-} \mathbf{u}_{|\gamma_{\ell}}^{\text{m}0},$$

$$\gamma_n^{\text{m},+} p_{|\gamma_{\ell}}^{\text{m}} = \gamma_n^{\text{m},-} p_{|\gamma_{\ell}}^{\text{m}} = p_{\ell}^{\text{f}}, \forall \ell \in \mathcal{I}.$$

Bulk model: Mass conservation + Darcy

$$\nabla \cdot \mathbf{u}_{\alpha}^{\text{m}} = \mathbf{f}_{\alpha}^{\text{m}} \quad \text{in } \omega_{\alpha},$$

$$\mathbf{u}_{\alpha}^{\text{m}} = -\mathcal{K}_{\alpha}^{\text{m}} \nabla p_{\alpha}^{\text{m}}$$

Additional conditions

- continuity of the hydraulic head at fracture-fracture intersections
- zero total flow at fractures intersections
- null flux at immersed fracture “tips”

References

- Martin, Jaffré, Roberts (2005)
 Vohralík, Maryška, Severýn (2007)
 Brenner, Groza, Guichard, Lebeau, Masson (2016)
 Boon, Nordbotten, Yotov (2018)
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Function spaces and bilinear forms

$H(\text{div})$ spaces for \mathbf{u} , $V = \tilde{V} + \text{"conditions at fracture intersections"}$

$$\tilde{V} = \{ \mathbf{v} = (\mathbf{v}^m, \mathbf{v}^f) \in L^2(\Omega^m) \times L^2(\Omega^f) \mid \mathbf{v}_\alpha^m \in H(\text{div}, \omega_\alpha), \forall \alpha \in \mathcal{C}; \\ \gamma_n^{m, \Gamma_\alpha^N} \mathbf{v}_\alpha^m = 0 \text{ on } \Gamma_\alpha^N, \forall \alpha \in \mathcal{C}; \quad (\nabla_\tau \cdot \mathbf{v}_\ell^f - [[\mathbf{v}^m]])|_{\gamma_\ell} \in L^2(\gamma_\ell), \forall \ell \in \mathcal{I} \}.$$

\tilde{V} is a Hilbert space (and V is a closed subspace) for the norm:

$$\|\mathbf{v}\|_V^2 = \sum_{\alpha \in \mathcal{C}} \|\mathbf{v}_\alpha^m\|_{H(\text{div}, \omega_\alpha)}^2 + \|\mathbf{v}^f\|_{L^2(\Omega^f)}^2 + \sum_{\ell \in \mathcal{I}} \|\nabla_\tau \cdot \mathbf{v}_\ell^f - [[\mathbf{v}^m]]\|_{L^2(\gamma_\ell)}^2.$$

L^2 space for \mathbf{p} : $S = L^2(\Omega^m) \times L^2(\Omega^f)$

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L^2 space for p : $S = L^2(\Omega^m) \times L^2(\Omega^f)$

Bilinear forms (linear forms omitted)

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega^m} (\mathcal{K}^m)^{-1} \mathbf{u}^m \cdot \mathbf{v}^m + \int_{\Omega^f} (\mathcal{K}^f)^{-1} \mathbf{u}^f \cdot \mathbf{v}^f, \quad \forall (\mathbf{u}, \mathbf{v}) \in V^2,$$

$$b(\mathbf{u}, \mathbf{s}) = \sum_{\alpha \in \mathcal{C}} \int_{\omega_\alpha} \mathbf{s}_\alpha^m \nabla \cdot \mathbf{u}_\alpha^m + \sum_{\ell \in \mathcal{I}} \int_{\gamma_\ell} \mathbf{s}_\ell^f \left(\nabla_\tau \cdot \mathbf{u}_\ell^f - [[\mathbf{u}^m]]_{|\gamma_\ell} \right), \quad \forall \mathbf{u} \in V, \forall \mathbf{s} \in S.$$

a and b are continuous

Weak mixed solution

Weak mixed formulation

Weak solution is $(u, p) \in V \times S$ st:

$$a(u, v) - b(v, p) = L_a(v), \quad \forall v \in V,$$

$$b(u, s) = L_b(s), \quad \forall s \in S.$$

Theorem

There exists a unique weak solution $(u, p) \in V \times S$

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Theorem

There exists a unique weak solution $(u, p) \in V \times S$

Prrof strategy.

Use **LBB** theory ([Boffi, Brezzi, Fortin (2013)]).

Usual uniform positivity assumption on the conductivity

Coercivity a is coercive over the space $W = \{v \in V \mid b(v, s) = 0, \forall s \in S\}$ **easy**;

Inf-sup b satisfies the inf-sup condition:

$$\exists \beta > 0 : \inf_{s \in S, s \neq 0} \sup_{v \in V, v \neq 0} \frac{b(v, s)}{\|v\|_V \|s\|_S} \geq \beta.$$

Proof of inf-sup condition

Sufficient to prove that:

There exists $C > 0$ such that, given $\mathbf{s} = (\mathbf{s}^m, \mathbf{s}^f) \in \mathbf{S}$, there exists $\mathbf{v} = (\mathbf{v}^m, \mathbf{v}^f)$ such that $b(\mathbf{v}, \mathbf{s}) = \|\mathbf{s}\|_{\mathbf{S}}^2$, and $\|\mathbf{v}\|_{\mathbf{V}} \leq C \|\mathbf{s}\|_{\mathbf{S}}$

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Solve **auxiliary problem** ([Brenner et al. (2016)]) (with zero Dirichlet BC on Γ^D):

$$\begin{aligned} -\nabla \cdot (\nabla \mathbf{z}) &= \mathbf{s}^m && \text{in } \Omega^m, \\ -\nabla_{\tau} \cdot (\nabla_{\tau} (\gamma^{m, \gamma_{\ell}} \mathbf{z})) + [[\nabla \mathbf{z}]]_{|\gamma_{\ell}} &= \mathbf{s}^f && \text{in } \gamma_{\ell}, \quad \text{for } \ell \in \mathcal{I}. \end{aligned}$$

then set $\mathbf{v} = (-\nabla \mathbf{z}, -\nabla_{\tau} \gamma^m \mathbf{z})$.

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Remark

- *Because pressure is continuous, cannot work “one piece at a time”*
- *Avoid assumption that each ω_{α} has a Dirichlet BC*

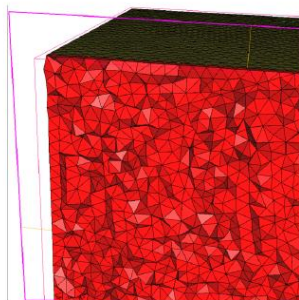
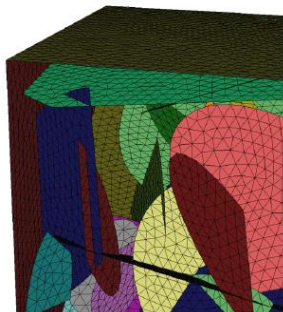
as in [Boon et al (2018)], [Amir et al. (2023)]

Meshes

- \mathcal{K}_h triangular mesh of the fracture network, \mathcal{E}_h set of edges of \mathcal{K}_h ;
- \mathcal{T}_h tetrahedral mesh of the rock matrix domain, \mathcal{F}_h set of faces of \mathcal{T}_h .

The rock matrix mesh \mathcal{T}_h is supported by the fracture mesh \mathcal{K}_h .

$$h = \max(\max_{T \in \mathcal{T}_h} h_T, \max_{K \in \mathcal{K}_h} h_K)$$



Both meshes are assumed regular

Discrete spaces

Pressure space:

$$S_h = \left\{ \mathbf{s}_h = (\mathbf{s}_h^m, \mathbf{s}_h^f) \mid \mathbf{s}_h^m|_T \in P_0(\mathbb{R}^3), \forall T \in \mathcal{T}_h; \mathbf{s}_h^f|_K \in P_0(\mathbb{R}^2), \forall K \in \mathcal{K}_h \right\}.$$

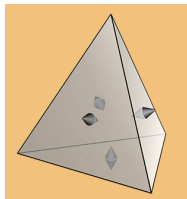
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Velocity space: **Raviart-Thomas(-Nédélec)**, $V_h = V_h^m \times V_h^f$ with $V_h \subset V$.

$$V_h^m \subset \left\{ \mathbf{v}_h^m \in L^2(\Omega^m) \mid \forall T \in \mathcal{T}_h, \mathbf{v}_h^m|_T \in RTN^0(T); \forall \alpha \in \mathcal{C}, \mathbf{v}_h^m|_{\omega_\alpha} \in H(\text{div}, \omega_\alpha) \right\}.$$



Discrete spaces

Pressure space:

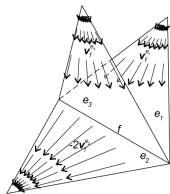
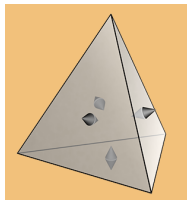
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$$V_h^f \subset \left\{ \mathbf{v}_h^f \in L^2(\Omega^f) \mid \forall K \in \mathcal{K}_h, \mathbf{v}_h^f|_K \in RT^0(K); \forall \ell \in \mathcal{I}, \mathbf{v}_h^f|_{\gamma_\ell} \in H(\text{div}, \gamma_\ell); \right.$$

$$\left. \forall \sigma \in \Sigma, \sum_{E \subset \sigma} \sum_{K, E \subset \partial K} \gamma_n^{f,E} \mathbf{v}_h^f|_K = \mathbf{0} \right\}.$$



Degrees of freedom for V_h^f have to take into account fracture–fracture intersection (remove one DOF for each intersection Vohralík et al (2007)).

Discrete mixed problem

Mixed approximation

Find $(u_h, p_h) \in V_h \times S_h$ satisfying:

$$\begin{aligned} a(u_h, v_h) - b(v_h, p_h) &= L_a(v_h), \quad \forall v_h \in V_h, \\ b(u_h, s_h) &= L_b(s_h), \quad \forall s_h \in S_h. \end{aligned}$$

Theorem (Error estimate)

- 1 The discrete problem has a unique solution $(u_h, p_h) \in V_h \times S_h$
- 2 Assuming additional regularity, the following error estimates hold:

$$\begin{aligned} \|u - u_h\|_V &\leq h \|u\|_1, \\ \|p - p_h\|_S &\leq h \left(\|u\|_1 + \|p\|_{H^1(\Omega)} \right), \end{aligned}$$

Analysis of discrete problem

Discrete kernel $W_h = \{v_h \in V_h \mid b(v_h, s_h) = 0, \forall s_h \in S_h\} \subset W$
Implies coercivity of the bilinear form a over W_h .

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Convergence requires uniform inf-sup condition.

$$\exists \tilde{\beta} > 0, \forall h > 0, \inf_{s_h \in S_h, s_h \neq 0} \sup_{v_h \in V_h, v_h \neq 0} \frac{b(v_h, s_h)}{\|v_h\|_V \|s_h\|_S} \geq \tilde{\beta} > 0.$$

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Checked via construction of Fortin operator:

$\pi_h : Z \rightarrow V_h$ ($Z \subset V_h$), such that $\forall v \in Z, \forall s_h \in S_h$;

- ① $b(v - \pi_h v, s_h) = 0$,
- ② There exists a constant $C > 0$ such that $\|\pi_h v\|_V \leq C \|v\|_V$.

Then “continuous inf-sup” \Rightarrow “discrete inf-sup”

Implies existence and uniqueness for discrete problem

The interpolation operator π_h

Interpolation operators for RTN spaces

Local operators π_T^m, π_K^f defined by the local DOFs of the FE spaces

Global operators: $Z^m \subset V^m, Z^f \subset V^f$ spaces of **smooth** functions, $Z = Z^m \times Z^f$.

Rock matrix $\pi_h^m : Z^m \rightarrow V_h^m, \quad \pi_h^m(u^m)|_T = \pi_T^m(u|_T)$

Fractures $\pi_h^f : Z^f \rightarrow V_h^f$ defined in [Vohralík (2007)] (zero total flux at intersections)

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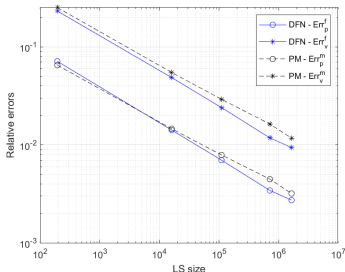
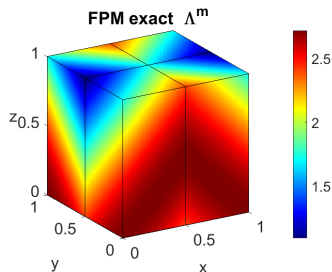
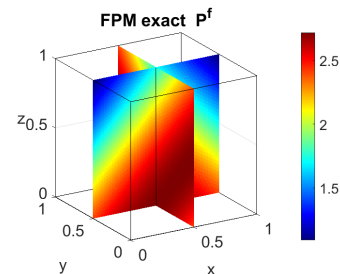
Let P_h^m (resp. P_h^f) projection operators over $L^2(\Omega^m)$ (resp. $L^2(\Omega^f)$)

Commutative diagrams for the interpolation operators:

$$\begin{array}{ccc}
 Z^m & \xrightarrow{\text{div}} & L^2(\Omega^m) \\
 \pi_h^m \downarrow & & \downarrow P_h^m \\
 V_h^m & \xrightarrow{\text{div}} & P_0(\mathbb{R}^2)
 \end{array}
 \qquad
 \begin{array}{ccc}
 Z^f & \xrightarrow{\text{div}} & L^2(\Omega^f) \\
 \pi_h^f \downarrow & & \downarrow P_h^f \\
 V_h^f & \xrightarrow{\text{div}} & P_0(\mathbb{R}^2)
 \end{array}$$

Define $\pi_h : Z \rightarrow V_h, \pi_h(v) = (\pi_h^m(v^m), \pi_h^f(v^f)), \forall v \in Z$. Then π_h is a **Fortin operator**

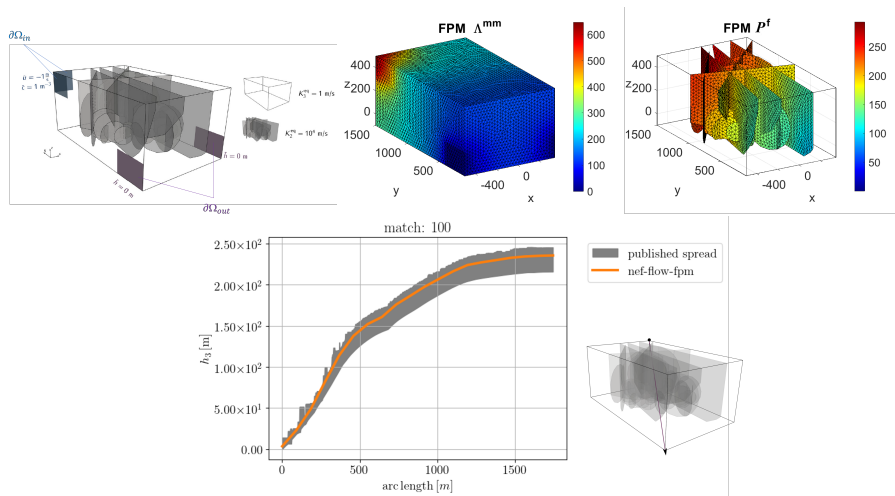
Analytical solution: numerical convergence



K. Brenner, M. Groza, C. Guichard,
G. Lebeau, and R Masson (2016.)

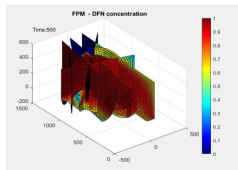
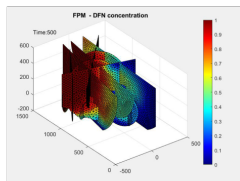
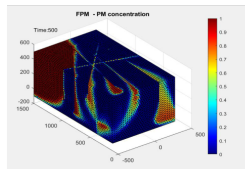
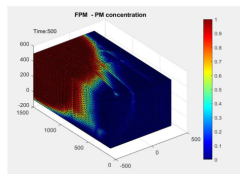
Change: one subdomain has
only **Neumann** boundary
conditions \Rightarrow no problem !

“Field” Benchmark, flow and transport



I. Berre, W. M. Boon, B. Flemisch, A. Fumagalli, D. Gläser, E. Keilegavlen, A. Scotti, I. Stefansson, A. Tatomir, K. Brenner, S. Burbullah, P. Devloo, O. Duran, M. Favino, J. Hennicker, I. Lee, K. Lipnikov, R. Masson, K. Mosthaf, M. G. C. Nestola, C. Ni, K. Nikitin, P. Schälde, D. Svyatskiy, R. Yanbarisov, P. Zulian. *Advances in Water Resources*. 2021

"Field" benchmark: simple advective transport



Permeability (rock matrix)
1.0m/s 1.0m/s

Transmissivity (fractures)
 $10^2 \text{m}^2/\text{s}$ $10^4 \text{m}^2/\text{s}$

M2 internship A. Marelli (now Univ. Nice / IFPEN)

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M. Kern, G. Pichot, D. Zegarra Vasquez. **Mathematical and numerical analysis of the mixed formulation of single phase flow in three-dimensional fractured porous media.** 2025.

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Minisymposium, Wednesday (10:30–12:30)

Le sous-sol: de l'eau, du gaz, des fractures, ... et des maths !

- 10:30 11:00 **C. Bourel**, J. Carrayrou, C. Rosier, *Écoulement de l'eau dans les aquifères peu profonds*
- 11:00 11:30 L. Beade, **P. Chassonery**, T. Guillon, M. Laaziri, S. Lopez, R. Masson, F. Smaï *Mécanique du contact dans un milieu poreux fracturé*
- 11:30 12:00 **I. Lannabi** *A posteriori error estimators and adaptivity for CO₂ sequestration*
- 12:00 12:30 P. Jolivet, M. Kern, F. Nataf, G. Pichot, R. Zanella, **D. Zegarra Vasquez** *The spectral domain decomposition method GenEO as a robust preconditioner for single-phase flow in fractured porous media*

Solution of auxiliary problem

Following ([Brenner et al. (2016)]), define $\tilde{H} = \left\{ \mathbf{s} \in H^1(\Omega) \mid \mathbf{s}^{\text{f}} := \gamma^{\Omega^{\text{f}}} \mathbf{s} \in H^1(\Omega^{\text{f}}) \right\}$ and subspace H_D with (Dirichlet BC and) traces that match at fracture-fracture intersection (also [Vohralík et al. (2007)])

$\gamma^{\text{f},\sigma} \mathbf{s}_j^{\text{f}} = \gamma^{\text{f},\sigma} \mathbf{s}_k^{\text{f}}$, for **all intersections** σ and all fractures γ_j, γ_k that intersect along σ .

Solution of auxiliary problem

Following ([Brenner et al. (2016)]), define $\tilde{H} = \left\{ \mathbf{s} \in H^1(\Omega) \mid \mathbf{s}^f := \gamma^{\Omega^f} \mathbf{s} \in H^1(\Omega^f) \right\}$ and subspace H_D with (Dirichlet BC and) traces that match at fracture-fracture intersection (also [Vohralík et al. (2007)])

$\gamma^{\mathbb{f},\sigma} \mathbf{s}_j^{\mathbb{f}} = \gamma^{\mathbb{f},\sigma} \mathbf{s}_k^{\mathbb{f}}$, for **all intersections** σ and all fractures γ_j, γ_k that intersect along σ .

Weak formulation of auxiliary problem: $\mathbf{z} \in H_D$

$$\int_{\Omega^m} \nabla \mathbf{z} \cdot \nabla \varphi + \sum_{\ell \in \mathcal{I}} \int_{\gamma^\ell} \nabla_{\tau} \gamma^{\mathbb{m},\gamma^\ell} \mathbf{z} \cdot \nabla_{\tau} \gamma^{\mathbb{m},\gamma^\ell} \varphi = \int_{\Omega^m} \mathbf{s}^m \varphi + \int_{\Omega^f} \mathbf{s}^f \gamma^{\mathbb{m},\partial\Omega^f} \varphi, \quad \forall \varphi \in H_D,$$

Proposition 4 in [Brenner et al. (2016)]: there exists unique $\mathbf{z} \in H_D$ solution of (weak) auxiliary problem, and furthermore $\mathbf{v} = (-\nabla \mathbf{z}, -\nabla_{\tau} \gamma^{\mathbb{m}} \mathbf{z}) \in V$.

Mixed-hybrid finite element

Relax the continuity of the fluxes across the interior faces and all edges.

Flux continuity enforced through Lagrange multipliers $\lambda_h \in M_{h,0,D}^{\text{em}}$ (constant on each face / edge)

Hybrid velocity space: V_h^{hyb} “like” V_h with relaxed inter-element continuity.

Bilinear form c over $V_h^{\text{hyb}} \times M_{h,0,D}^{\text{em}}$:

For $u_h \in V_h^{\text{hyb}}$, $(v_h \in V_h \iff c(v_h, \lambda_h) = 0, \forall \lambda_h \in M_{h,0,D})$.

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Mixed-hybrid discrete problem

Find $(u_h, p_h, \lambda_h) \in V_h^{\text{hyb}} \times S_h \times M_{h,0,D}$ such that:

$$\begin{aligned} a(u_h, v_h) - b_{\mathcal{T}}(v_h, p_h) + c(\lambda_h, v_h) &= L_a(v_h), & \forall v_h \in V_h^{\text{hyb}}, \\ b_{\mathcal{T}}(u_h, s_h) &= L_b(s_h), & \forall s_h \in S_h, \\ c(u_h, \lambda_h) &= 0, & \forall \lambda_h \in M_{h,0,D}. \end{aligned}$$

$b_{\mathcal{T}}$ is a **mesh dependent** version of b over $V_h^{\text{hyb}} \times S_h$.

MHFE formulation is **equivalent** to mixed approximation: unique solution (u_h, p_h, λ_h) .