Postprocessed frozen-flow methods for the long time sampling of ergodic dynamics on Riemannian manifolds

Sébastien Macé
Joint work with A. Busnot Laurent

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Equation: $dX(t) = -\nabla V(X(t))dt + \sqrt{2}dB_{\mathcal{M}}(t)$ on $\mathcal{M}$

The Euclidian space $\mathbb{R}^d$

Consider the differential equation: $X'(t) = F(X(t))$. The **explicit Euler** method relies on the finite differences approximation:

$$X'(t) \sim \frac{X(t+\delta t) - X(t)}{\delta t} \text{ becomes } X_{n+1} = X_n + hF(X_n).$$

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Generalization to manifolds

The vector $X_n + hF(X_n)$ does not belong on $\mathcal{M}$ but on the tangent space $T_{X_n}\mathcal{M}$.

Question

How to sample the long time behaviour of **ergodic dynamic** on a **Riemannian manifold**?

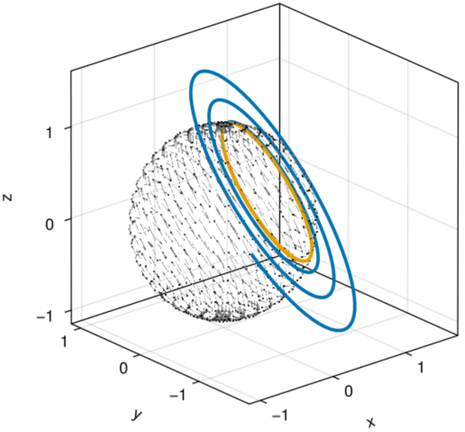

Figure 1: The trajectory does not stay on the sphere.

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Reference of this talk:

- A. Busnot Laurent, S. Macé, Postprocessed frozen-flow methods for the long time sampling of ergodic dynamics on Riemannian manifolds, *In preparation*.

Butcher series for ODEs on $\mathbb{R}^d$ (A. Cayley: 1857; J. Butcher: 1963)

We consider the problem $\boxed{X' = F(X), \quad X(0) = x}$ with $X : \mathbb{R}_+ \to \mathbb{R}^d$ and $F : \mathbb{R}^d \to \mathbb{R}^d$. A Taylor expansion at the order p yields,

$$X(t) = x + tX'(0) + \frac{t^2}{2}X''(0) + \frac{t^3}{6}X^{(3)}(0) + \cdots + \mathcal{O}(t^{p+1}).$$

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However, the ODE allows us to replace the derivates of X by the ones of F evaluated on x:

$$X'(0) = F(x) \quad \Bigg| \quad \begin{aligned} X''(0) &= (F \circ X)'(0) \\ &= dF(x) \cdot F(x) \end{aligned} \quad \Bigg| \quad \begin{aligned} X^{(3)}(0) &= (F \circ X)^{(2)}(0) \\ &= dF(x) \cdot dF(x) \cdot F(x) + d^2F(x) \cdot (F(x), F(x)) \end{aligned}$$

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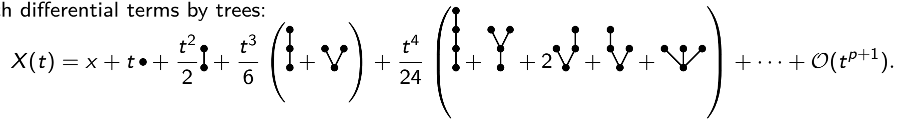
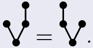
Express each differential terms by trees:



Remark

Note that on $\mathbb{R}^d$, the differential operators commute and 

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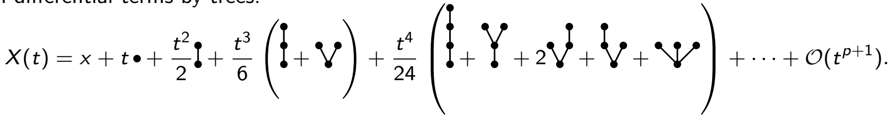
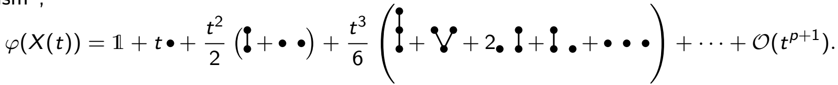


We focus on the weak convergence, that is, on the approximation of $\varphi(X)$ for all test function $\varphi \in C_P^\infty(\mathcal{M})$, we apply the same formalism[1],

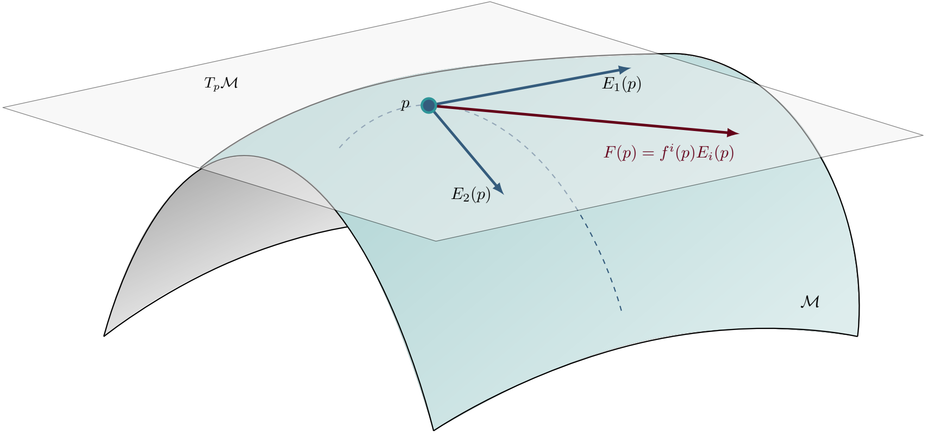


[1]The unique node associated to φ being at the root, it is implicit.

S-series for ODEs on $(\mathcal{M}, g)$ (H. Munthe-Kaas: 1999)

Let $E_1, \ldots, E_D$ be an **orthonormal frame** of a Riemannian manifold $(\mathcal{M}, g)$. General ODEs are of the form:

$$\boxed{X'(t) = F(X(t)), \quad X(0) = x \in \mathcal{M}.} \tag{1}$$



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For all φ, we denote by $v[\varphi]$ **the differential** $d\varphi \cdot v$.
The **Weitzenböck affine connection** $\rhd$ is defined on vector fields by

$$X \rhd Y = (X \rhd y^d)E_d, \quad X \rhd \varphi = X[\varphi].$$

Differential operators act on functions φ by:

$$(X_1 \cdots X_n) \rhd \varphi = x_1^{i_1} \cdots x_n^{i_n} E_{i_1}[\ldots E_{i_n}[\varphi] \ldots], \quad X_j = \sum_{i_j=1}^{D} x_j^{i_j} E_{i_j}.$$

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A Taylor expansion at the order p yields,

$$\varphi(X(t)) = \varphi(x) + tf^i(x)E_i \triangleright \varphi(x) + \frac{t^2}{2}\left(f^j(x)E_j[f^i(x)]E_i + f^i(x)f^j(x)(E_i \cdot E_j)\right) \triangleright \varphi(x) + \cdots + \mathcal{O}(t^{p+1}).$$

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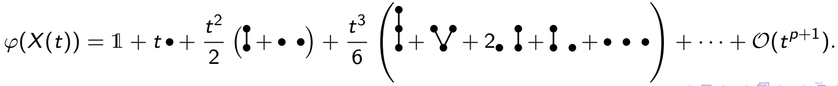

S-series for SDEs on $(\mathcal{M}, g)$ (A. B. Laurent and G. Vilmart: 2022)

General stochastic differential equations with an additive noices are of the form:

$$\boxed{dX(t) = F(X(t))dt + \sqrt{2}\sum_{d=1}^{D} E_d(X(t)) \circ dW_d(t), \quad X(0) = x \in \mathcal{M}.} \tag{2}$$

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We are not interested in $\varphi(X)$ but in $u(t,x) = \mathbb{E}\left[\varphi(X(t))\right]$. A Taylor expansion at the order p yields,

$$u(t,x) = \varphi(x) + t\left(f^i(x)E_i + (E_{d_1} \cdot E_{d_1})\right) \rhd \varphi(x) + \cdots + \mathcal{O}(t^{p+1}).$$

The generator $\mathcal{L}$ of the SDE (2) is defined by

$$\mathcal{L}\varphi = \left(f^i(x)E_i + (E_{d_1} \cdot E_{d_1})\right) \rhd \varphi. \tag{3}$$

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$$\boxed{u(t,x) = \mathbb{1} + t\,(\bullet + \textcircled{1}\textcircled{1}) + \frac{t^2}{2}\left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix} + \begin{smallmatrix}\textcircled{1}\ \textcircled{1}\\ \diagdown\ \diagup\\ \bullet\end{smallmatrix} + \bullet\ \bullet + 2\,\textcircled{1}\begin{smallmatrix}\textcircled{1}\\ |\\ \bullet\end{smallmatrix} + \bullet\,\textcircled{1}\textcircled{1} + \textcircled{1}\textcircled{1}\,\bullet + \textcircled{2}\textcircled{2}\textcircled{1}\textcircled{1}\right) + \cdots + \mathcal{O}(t^{p+1}).}$$

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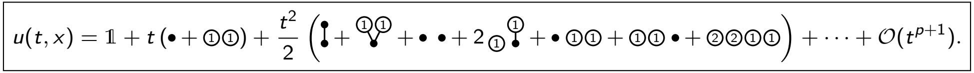


We are, in particular, interested in the case where $F = -\nabla V - \nabla_{E_p} E_p, \quad f^i = -E_i[V] - \left\langle \nabla_{E_p} E_p | E_i \right\rangle$ for a certain potential V. Hence equation (2) becomes, by denoting $B_{\mathcal{M}}(t)$ a Brownian motion on $\mathcal{M}$, the **Riemannian Langevin equation**

$$dX(t) = -\nabla V(X(t))dt + \sqrt{2}dB_{\mathcal{M}}(t), \quad X(0) = X_0 \in \mathcal{M}. \tag{3}$$

Exotic forests algebra

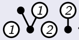
Definition (Exotic forest and order)

An ***exotic forest*** *is a planar decorated forest with the decorations* $D = \{\bullet\} \cup \mathbb{N}$, $\mathbb{N} = \{1, 2, \dots\}$ *satisfying* ***if an integer is used as decoration then it must decorate exactly two leaves****. Two pairly decorated nodes are call lianas. The set of exotic forests is* EF*, the associated vector space is* $\mathcal{E}\mathcal{F} = \mathrm{Span}_{\mathbb{R}}(\mathrm{EF})$*.*
The ***order of a forest*** $\pi \in \mathrm{EF}$ *is defined by the order of the associated differential operators.*

Examples

Consider these examples.

$$f^i(E_i \rhd \varphi) \mapsto \bullet, \quad (E_{d_1} \cdot E_{d_1}) \rhd \varphi \mapsto ①①,$$

$$f^k(E_{d_2} \rhd f^j)((E_k \cdot E_{d_1}) \rhd f^i)((E_{d_1} \cdot E_i \cdot E_{d_2} \cdot E_j) \rhd \varphi) \mapsto$$  .

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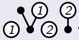
$$f^k(E_{d_2} \rhd f^j)((E_k \cdot E_{d_1}) \rhd f^i)((E_{d_1} \cdot E_i \cdot E_{d_2} \cdot E_j) \rhd \varphi) \mapsto$$ .

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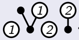
$$f^k(E_{d_2} \rhd f^j)((E_k \cdot E_{d_1}) \rhd f^i)((E_{d_1} \cdot E_i \cdot E_{d_2} \cdot E_j) \rhd \varphi) \mapsto$$ .

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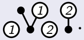
$$f^k(E_{d_2} \rhd f^j)((E_k \cdot E_{d_1}) \rhd f^i)((E_{d_1} \cdot E_i \cdot E_{d_2} \cdot E_j) \rhd \varphi) \mapsto$$ .

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Frozen flow of the differential equation $X'(t) = F(X(t))$ on $\mathcal{M}$

Definition (Frozen flow method)

We freeze the vector field, using an approximation α_n^d *of* f^d *in the segment of* $[X_n, X_{n+1}]$,

$$Y'(t) = \alpha_n^d E_d(Y(t)), \quad Y(0) = X_n.$$

Then let $\exp\left(h\alpha_n^d E_d\right)$ *be the flow of the solution and we denote* $X_{n+1} = \exp\left(h\alpha_n^d E_d\right) X_n$.

Frozen flow on $\mathbb{R}^D$

With $\alpha_n^d = f^d(X_n)$, we recover the **explicit Euler method**

$$X_{n+1} = X_n + hf^d(X_n)e_d,$$

and with

$$\alpha_n^d = \frac{1}{2}\left(f^d(X_n) + f^d(X_n + hf^d(X_n)e_d)\right),$$

we recover the **Heun method**,

$$H_n = X_n + hf^d(X_n)e_d,$$

$$X_{n+1} = X_n + \frac{h}{2}\left(f^d(X_n) + f^d(H_n)\right)e_d.$$

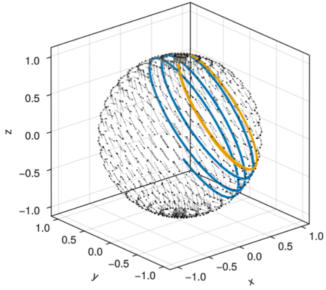

Figure 2: The trajectory does stay on the sphere.

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The easiest method to solve The SDE (2) is called **frozen flow Euler method**

$$\boxed{X_{n+1} = \exp\left(\left(hf^d(X_n) + \sqrt{2h}\xi_n^d\right) E_d\right) X_n, \quad \xi_n^d \sim \mathcal{N}(0,1).}$$

In the case of the **Riemannian Langevin equation** (3), K. Bharath, A. Lewis, A. Sharma, and M. V. Tretyakov, 2023; proposes an alternate approach, based on the Levi-Civita connection called Riemannian Langevin method:

$$\boxed{X_{n+1} = \exp^{Riem}\left(-h\nabla V(X_n) + \sqrt{2h}\xi_n^d E_d\right) X_n, \quad \xi_n^d \sim \mathcal{N}(0,1).}$$

Invariant measure

Definition (Processus ergodique)

A process X is ergodic if there exists $d\mu_\infty$ a unique invariant measure with density function ρ_∞ with respect to the mesure $d\,\mathrm{vol}_{\mathcal{M}}$,such that for all test function φ and all initial data $X(0) = x$, the time average of $\varphi(X)$ converge, that is,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi(X(s))ds = \int_{\mathcal{M}} \varphi(y)d\mu_\infty(y) \text{ almost surely.}$$

A method is of ***order $p \geq 1$ for the invariant measure*** *if $|e(\varphi, h)| \leq Ch^r$, where*

$$e(\varphi, h) := \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \varphi(X_n) - \int_{\mathcal{M}} \varphi(y)d\mu_\infty(y).$$

In the case of the **Riemannian Langevin dynamics**,

$$dX(t) = -\nabla V(X(t))dt + \sqrt{2}dB_{\mathcal{M}}(t), \quad X(0) = X_0 \in \mathcal{M},$$

we recover the **Gibbs measure:** $\rho_\infty \propto \exp(-V)$.

Characterization of the higher order for the invariant measure

Theorem (A. Busnot Laurent & S. Macé)

We consider the SDE (3) on $\mathcal{M}$. Suppose some technical assumption and that the SDE is solved by a consistent ergodic numerical method satisfying, for all test function φ,

$$\mathbb{E}[\![\varphi(X_1)|X_0 = x]\!] = \varphi(x) + \sum_{j=1}^{p} h^j A_j \varphi(x) + h^{p+1} R_p^h(\varphi, x).$$

In that case, if the method satisfies

$$A_j^* d\mu_\infty = 0, \quad j = 2 \dots p,$$

then the method is of ***order p for the invariant measure****. More precisely, the error of the invariant measure $e(\varphi, h)$ satisfies for all $\varphi \in C_P^\infty$ and all $h \in [0, h_0]$ for a small h_0:*

$$e(\varphi, h) = h^p \int_0^\infty \int_{\mathcal{M}} u(x,t) A_{p+1}^* d\mu_\infty(x) + \underset{h \to 0}{\mathcal{O}}(h^{p+1}).$$

In the spirit of A. Debussche & E. Faou, 2012; A. Abdulle & G. Vilmart & K. C. Zygalakis, 2014.

New post processed second order frozen flow method

General method (extending B. Owren & A. Marthinsen, 1999; A. Rössler, 2006)

$$H_n = \exp\left(\left(h\alpha f^d(X_n) + \sqrt{h}\beta\xi_n^d\right) E_d\right) X_n,$$
$$X_{n+1} = \exp\left(\left(h\alpha_2 f^d(X_n) + h\gamma_2 f^d(H_n) + \sqrt{h}\beta_2\xi_n^d\right) E_d\right) \exp\left(\left(h\alpha_1 f^d(X_n) + h\gamma_1 f^d(H_n) + \sqrt{h}\beta_1\xi_n^d\right) E_d\right) X_n.$$

Consider post processed methods, where the postprocessor $X \mapsto \overline{X}$ is buld as a perturbation of the identity of $\mathcal{M}$,

$$X_0 \in \mathcal{M}, \quad X_{n+1} = \Psi_h(X_n), \quad 0 \leq n \leq N-1, \quad \overline{X}_N = \overline{\Psi}_h(X_N). \tag{4}$$

General post processed method (extending G. Vilmart, 2015)

$$H_n = \exp\left(\left(\sqrt{h}\beta\xi_n^d\right) E_d\right) X_n,$$
$$X_{n+1} = \exp\left(\left(h\gamma_2 f^d(H_n) + \sqrt{h}\beta_2\xi_n^d\right) E_d\right) \exp\left(\left(h\gamma_1 f^d(H_n) + \sqrt{h}\beta_1\xi_n^d\right) E_d\right) X_n.$$
$$\overline{X_N} = \exp\left(\sqrt{h\overline{\beta}}\overline{\xi_N^d} E_d\right) X_N.$$

New post processed second order frozen flow method

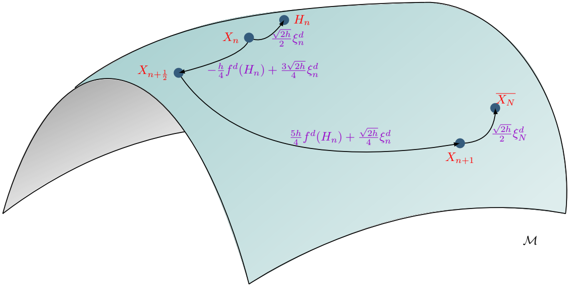
Theorem (A. Busnot Laurent & S. Macé)

Assume some technical assumptions from A. Abdulle, G. Vilmart, & K. C. Zygalakis, *2015. Assume that the SDE is solved by a consistent numerical method with postprocessor satisfying, for all test function,*

$$A_{j+1}^* d\mu_\infty = \overline{A_j}^* d\mu_\infty = 0 \text{ for } j = 1 \dots p-1, \quad \left(A_{p+1} + [\mathcal{L}, \overline{A_p}]\right)^* d\mu_\infty = 0.$$

Then the method is of ***order $p+1$ for the invariant measure****.*

This theorem is a generalization of A. Abdulle, G. Vilmart, & K. C. Zygalakis, 2015; to Riemannian manifolds.



New post processed second order frozen flow method

Method 1 Post processed frozen-flow method for equation (3)

$$H_n = \exp\left(\frac{\sqrt{2h}}{2}\xi_n^d E_d\right) X_n$$

$$X_{n+1} = \exp\left(\left(\frac{5h}{4} f^d(H_n) + \frac{\sqrt{2h}}{4}\xi_n^d\right) E_d\right) \exp\left(\left(-\frac{h}{4} f^d(H_n) + \frac{3\sqrt{2h}}{4}\xi_n^d\right) E_d\right) X_n$$

$$\overline{X_N} = \exp\left(\frac{\sqrt{2h}}{2}\xi_N^d E_d\right) X_N.$$

Corollary (A. Busnot Laurent & S. Macé)

*The method 1 is of **weak order 1** and of **second order for the invariant measure**.*

Remark

In the Euclidian case $\mathcal{M} = \mathbb{R}^D$, we recover the Leimkuhler-Matthews method, B. Leimkuhler & C. Matthews, *2013; with the post processed formalism from* G. Vilmart, *2015;.*

Second order frozen flow methods

We propose two alternative scheme, without postprocessor, for the sake of comparison.

Method 2 Stochastic frozen-flow Heun method for equation (3)

$$H_n = \exp\left(\left(hf^d(X_n) + \sqrt{2h}\xi_n^d\right) E_d\right) X_n$$
$$X_{n+1} = \exp\left(\tfrac{h}{2} f^d(H_n) E_d\right) \exp\left(\left(\tfrac{h}{2} f^d(X_n) + \sqrt{2h}\xi_n^d\right) E_d\right) X_n$$

Method 3 2-steps frozen-flow Runge-Kutta method for equation (3)

$$H_n = \exp\left(\left(\frac{h}{4} f^d(X_n) + \frac{\sqrt{2h}}{2}\xi_n^d\right) E_d\right) X_n$$
$$X_{n+1} = \exp\left(\left(\frac{h}{6} f^d(X_n) + \frac{2h}{3} f^d(H_n) + \frac{\sqrt{2h}}{2}\xi_n^d\right) E_d\right) \exp\left(\left(\frac{h}{6} f^d(X_n) + \frac{\sqrt{2h}}{2}\xi_n^d\right) E_d\right) X_n$$

Corollary (A. Busnot Laurent & S. Macé)

*Methods 2 and 3 are of **weak order 1** and of **second order for the invariant measure**.*

Two potentials on $\mathcal{M} = SO_q$

Let $\mathcal{M} = SO_q$ be the compact Lie group of special orthogonal matrices and $(A_d)_{d=1,\dots,D}$ be a basis of the skew-symmetric matrices. As the manifold is smooth and compact, the assumptions of our theorems are satisfied and our analysis applied.

Proposition

Let $\mathcal{M}$ be a matrix Lie group and let $(A_d)_{d=1,\dots,D}$ be an orthonormal basis of its Lie algebra $\mathfrak{m} = T_e\mathcal{M}$. The associated orthonormal frame on $\mathcal{M}$ is

$$E_d(y) = A_d y, \quad d = 1, \dots, D.$$

Method 1 Postprocessed frozen-flow method for equation (3)

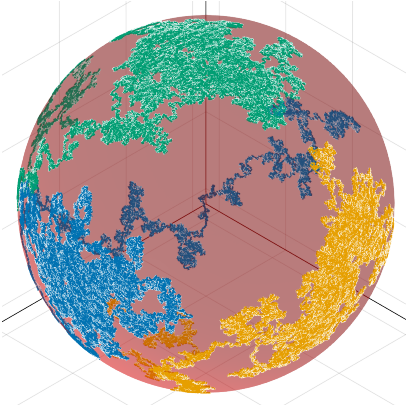
$$H_n = \mathrm{Exp}\left(\sqrt{\tfrac{h}{2}}\xi_n^d A_d\right) X_n,$$

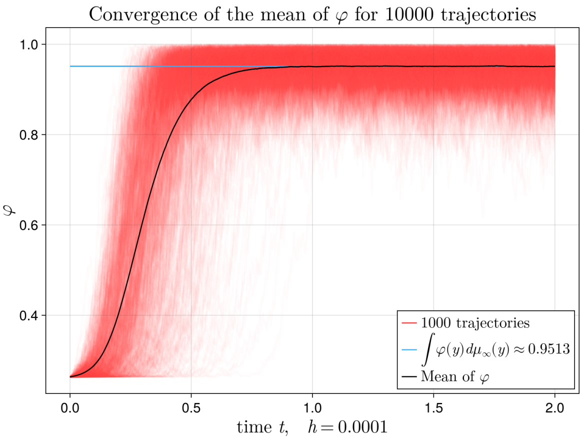
$$X_{n+1} = \mathrm{Exp}\left(\left(\tfrac{5h}{4} f^d(H_n) + \tfrac{\sqrt{2h}}{4}\xi_n^d\right) A_d\right) \mathrm{Exp}\left(\left(-\tfrac{h}{4} f^d(H_n) + \tfrac{3\sqrt{2h}}{4}\xi_n^d\right) A_d\right) X_n,$$

$$\overline{X_N} = \mathrm{Exp}\left(\sqrt{\tfrac{h}{2}}\xi_N^d A_d\right) X_N.$$

We compare a quadratic potential well and a sextic potential barrier on $\mathcal{M} = SO_3$ to confirm our analysis.

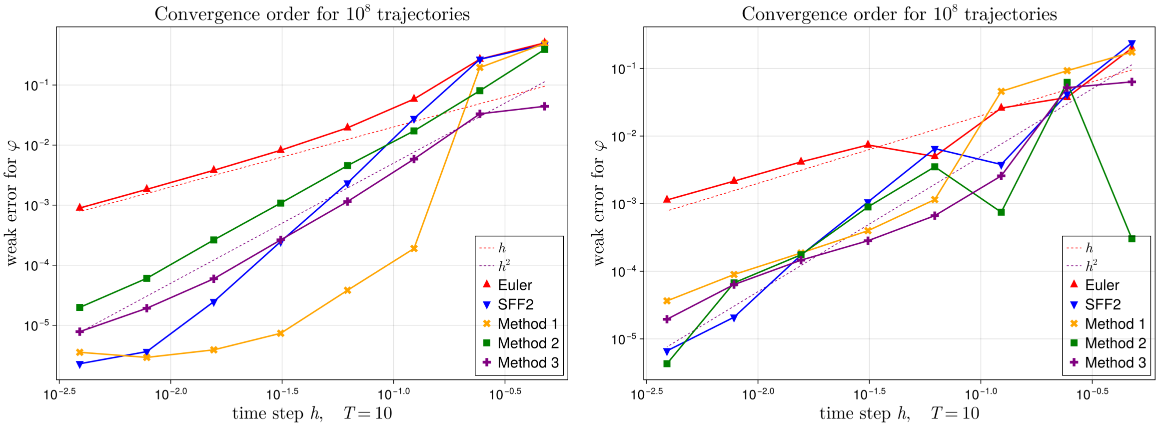
Two potentials on $\mathcal{M} = SO_q$





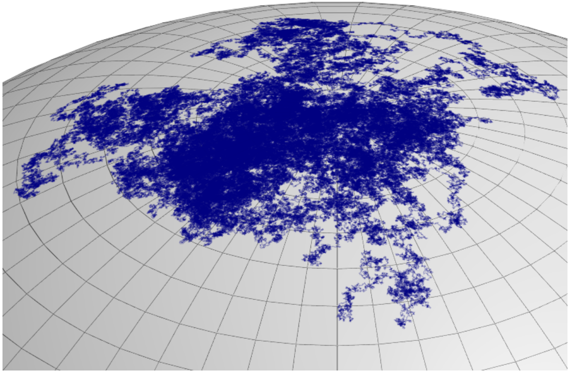
Trajectory on SO_3 for the quadratic potential with $T = 2$ and $h = 10^{-5}$ (Left). Ergodic convergence computed with 10^4 trajectories and $h = 10^{-4}$ for the quadratic potential. (Right).

Two potentials on $\mathcal{M} = SO_q$



Order of convergence for both potentials. The mean is at time $T = 10$ on 10^8 trajectories.

Von Mises Fisher dynamic on the sphere $\mathbb{S}^2$



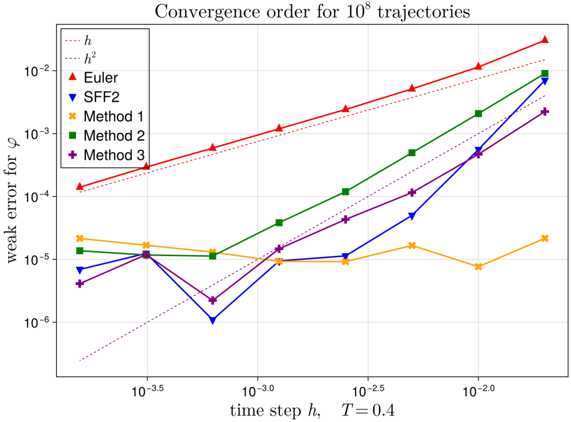


Figure 3: Order of convergence for the potential $(x, y, z) \mapsto -25z$ on the sphere $\mathbb{S}^2$. The mean is at time $T = 0.4$ on 10^8 trajectories and the test function is $\varphi : (x, y, z) \mapsto z^2$.

Integration by parts

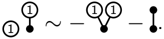
Goal: Express A_n^* and derive conditions for higher orders of $d\mu_\infty$.

Proposition

For $\varphi, \psi \in C_P^\infty(\mathcal{M})$ the integration by parts with respect to the Gibbs measure associated to the equation (3) ensures that

$$\boxed{\int_{\mathcal{M}} E_i[\varphi]\psi d\mu_\infty = -\int_{\mathcal{M}} \varphi E_i[\psi] d\mu_\infty - \int_{\mathcal{M}} \varphi\psi f^i d\mu_\infty},$$

where $F = f^i E_i$ is given by $f^i = -E_i[V] - \left\langle \nabla_{E_p} E_p | E_i \right\rangle$.

Example: $\displaystyle\int_{\mathcal{M}} E_{d_1}[f^i]E_{d_1}[E_i[\varphi]]d\mu_\infty = -\int_{\mathcal{M}} E_{d_1}[E_{d_1}[f^i]]E_i[\varphi]d\mu_\infty - \int_{\mathcal{M}} E_{d_1}[f^i]f^{d_1}[E_i[\varphi]]d\mu_\infty$, ie 

Remark

Note that the application of Proposition 9 does not always rely on exotic forests. If we replace E_i by the vector field F, ie $\psi = f^i$, then a $E_i[f^i]$ appears and cannot be solely expressed by exotic forests,

$$\int_{\mathcal{M}} f^i(E_i \triangleright \varphi)d\mu_\infty = -\int_{\mathcal{M}} (E_i \triangleright f^i)\varphi d\mu_\infty - \int_{\mathcal{M}} f^i f^i \varphi d\mu_\infty.$$

Integration by parts

Definition (**Irreducible forests**)

The set of irreducible forests I_R *is the set of exotic forest whose first tree is not a numbered node. We denote* $\mathcal{I}_\mathcal{R} = \mathrm{Span}_\mathbb{R}(\mathsf{I}_\mathsf{R})$.

Definition (IBP and RED)

Define the linear operator $\mathrm{IBP} : \mathcal{EF} \to \mathcal{EF}$ *by* $\mathrm{IBP}\,|_{\mathsf{I}_\mathsf{R}} \equiv 0$ *and else*

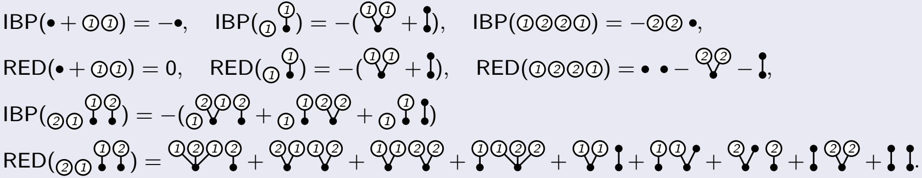
$$\mathrm{IBP}(ⓝ\tilde{\pi}) = -\tilde{\pi}|_{ⓝ\to\bullet} - \sum_{v \in V_\bullet} \tilde{\pi}|_{ⓝ\curvearrowright v}.$$

where $\tilde{\pi}|_{ⓝ\to\bullet}$ *is the exotic forest obtained by substituting the unique* ⓝ *of* $\tilde{\pi}$ *by* $\bullet$ *and* $\tilde{\pi}|_{ⓝ\curvearrowright v}$ *is the exotic forest obtained by grafting a node* ⓝ *on the left of the node* v *of* $\tilde{\pi}$.
Define the linear operator $\mathrm{RED} : \mathcal{EF} \to \mathcal{I}_\mathcal{R}$ *by applying IBP on each reducible forest until it remains only irreducible forests, that is*

$$\mathrm{RED} = \lim_{n\to\infty} \left(\mathrm{id}_{\mathcal{I}_\mathcal{R}} + \mathrm{IBP}\right)^n,$$

where RED *is well defined as the limit of a stationary sequence, as* $\left(\mathrm{id}_{\mathcal{I}_\mathcal{R}} + \mathrm{IBP}\right)^{2|\pi|}\pi \in \mathcal{I}_\mathcal{R}$.

Examples



Then we can express the adjoint.

Proposition

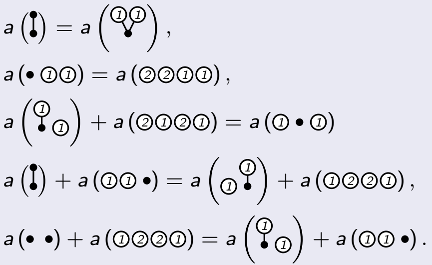
For all $\pi \in \mathcal{E}\mathcal{F}$, π *and* $\mathrm{RED}(\pi)$ *represent the same adjoint operator, that is,*

$$\boxed{\pi^* d\mu_\infty = \mathrm{RED}(\pi)^* d\mu_\infty}.$$

Second order conditions for $A_2 = \sum_{\pi \in \mathrm{EF}} a(\pi)\pi$

Theorem (A. Busnot Laurent & S. Macé)

A numerical method is of ***order 2 for the invariant measure*** *of equation (3) if it satisfies*



For the third order, there are **40 equations and with 95 variables**. We recall from E. Bronasco & A. Busnot Laurent & B. Huguet, 2025; that for the weak error, there are **8 equations for the order 2** and **73 for the order 3**.

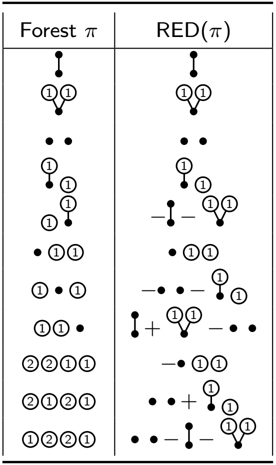


Table 1: Second order reductions.

Second order conditions for $A_2 = \sum_{\pi \in \mathrm{EF}} a(\pi)\pi$

Applying RED to $A_2\varphi = \sum_{\pi \in \mathrm{EF}_2} a(\pi)\, \mathbb{F}^F(\pi) \triangleright \varphi$ gives us

$$
\begin{aligned}
\int_{\mathcal{M}} A_2 \varphi d\mu_\infty = & \sum_{\pi \in \mathrm{EF}_2} a(\pi) \int_{\mathcal{M}} \mathbb{F}^F(\mathrm{RED}(\pi)) \triangleright \varphi d\mu_\infty \\
= & a\left(\overset{\bullet}{\underset{\bullet}{|}} - \textcircled{1}\overset{\textcircled{1}}{\underset{\bullet}{|}} + \textcircled{1}\textcircled{1}\bullet - \textcircled{1}\textcircled{2}\textcircled{2}\textcircled{1} \right) \int_{\mathcal{M}} \mathbb{F}^F\left(\overset{\bullet}{\underset{\bullet}{|}} \right) \triangleright \varphi d\mu_\infty \\
+ & a\left(\overset{\textcircled{1}\textcircled{1}}{\underset{\bullet}{\vee}} - \textcircled{1}\overset{\textcircled{1}}{\underset{\bullet}{|}} + \textcircled{1}\textcircled{1}\bullet - \textcircled{1}\textcircled{2}\textcircled{2}\textcircled{1} \right) \int_{\mathcal{M}} \mathbb{F}^F\left(\overset{\textcircled{1}\textcircled{1}}{\underset{\bullet}{\vee}} \right) \triangleright \varphi d\mu_\infty \\
+ & a\left(\bullet\ \bullet - \textcircled{1}\bullet\textcircled{1} - \textcircled{1}\textcircled{1}\bullet + \textcircled{2}\textcircled{1}\textcircled{2}\textcircled{1} + \textcircled{1}\textcircled{2}\textcircled{2}\textcircled{1} \right) \int_{\mathcal{M}} \mathbb{F}^F(\bullet\ \bullet) \triangleright \varphi d\mu_\infty \\
+ & a\left(\overset{\textcircled{1}}{\underset{\bullet}{|}}\textcircled{1} - \textcircled{1}\bullet\textcircled{1} + \textcircled{2}\textcircled{1}\textcircled{2}\textcircled{1} \right) \int_{\mathcal{M}} \mathbb{F}^F\left(\overset{\textcircled{1}}{\underset{\bullet}{|}}\textcircled{1} \right) \triangleright \varphi d\mu_\infty \\
+ & a\left(\bullet\textcircled{1}\textcircled{1} - \textcircled{2}\textcircled{2}\textcircled{1}\textcircled{1} \right) \int_{\mathcal{M}} \mathbb{F}^F(\bullet\textcircled{1}\textcircled{1}) \triangleright \varphi d\mu_\infty .
\end{aligned}
$$

Future works

- Application of our construction to Runge-Kutta-Munthe-Kaas methods on Lie groups
- Intrinsic stochastic backward error analysis to modify the equation to reach higher order
- Application to multiscale context
- Extension to fully Riemannian method: $\exp^{Riem}$
- Extension to general ergodic SDEs on manifolds (work in progress)