A model of plasma-wall interaction : the dynamical plasma sheath

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Contents

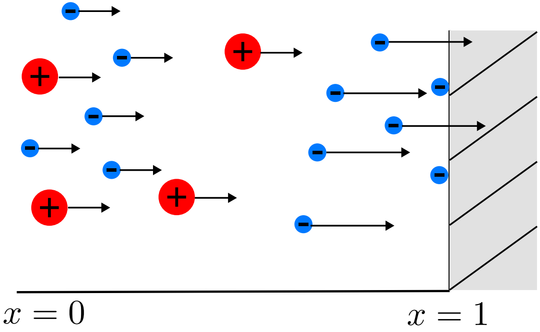
Modelling plasma near a wall

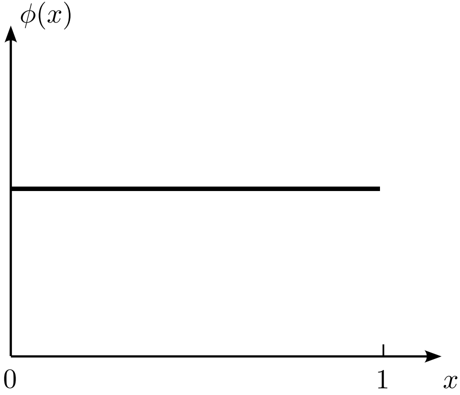
The problem at hand

A plasma is injected in the domain $[0, 1]$ at $x = 0$. A fully absorbing wall stands at $x = 1$.

The problem at hand

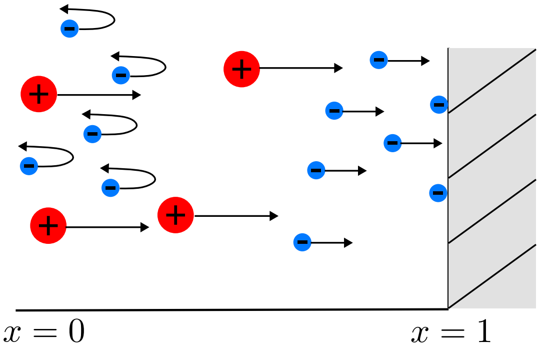
Electrons are hotter and lighter than ions. A current inbalance appears initially.

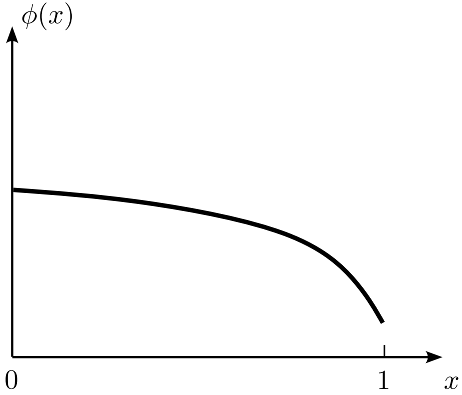




The problem at hand

In response, the wall potential drops, ions are accelerated, electrons are repelled : this is the plasma sheath.





The Vlasov-Poisson-Ampère system

Notations : $\Omega_T =]0,T[\times]0,1[\times\mathbb{R}$, $Q_T =]0,T[\times]0,1[$, $\gamma_\pm = \dfrac{q_\pm}{m_\pm}$ the charge-to-mass ratio.

$$(V^\pm)\begin{cases} \partial_t f^\pm + v\partial_x f^\pm - \gamma_\pm \partial_x\phi\partial_v f^\pm = 0 & \forall (t,x,v)\in\Omega_T \\ f^\pm(0,x,v) = \tilde{f}^\pm(x,v) & \forall (x,v)\in]0,1[\times\mathbb{R} \\ f^\pm(t,0,v) = g^\pm(t,v) & \forall (t,v)\in]0,T]\times\mathbb{R}^{+*} \\ f^\pm(t,1,v) = 0 & \forall (t,v)\in]0,T]\times\mathbb{R}^{-*} \end{cases} \tag{1}$$

Remarks

- Electric potential ϕ is self-consistant;
- Particles may leave the domain at $x=0$ or $x=1$.

The Vlasov-Poisson-Ampère system

$$(A)\begin{cases} \varepsilon_0 \partial_{tx}\phi & = & q_+ j^+ + q_- j^- \;\forall (t,x) \in Q_T \\ \phi(t,0) & = & 0 \\ \phi(t=0,.) & & \text{satisfying Poisson equation} \end{cases} \tag{2}$$

$$(P)\begin{cases} -\varepsilon_0 \partial_{xx}\phi & = & q_+ \rho^+ + q_- \rho^- \;\forall (t,x) \in Q_T \\ \phi(t,0) & = & 0 \\ \phi(t,1) & = & u(t) \end{cases} \tag{3}$$

Both Ampère and Poisson equations are satisfied only when the potential at the wall has the right expression.

The Vlasov-Poisson-Ampère system

Integration of $(V^{\pm})$ w.r.t v gives the continuity equation $\partial_t \rho^{\pm} = -\partial_x j^{\pm}$.
Consider ϕ satisfying (A). Integration of Maxwell-Ampère w.r.t t gives

$$\varepsilon_0 \partial_x \phi(t,x) - \varepsilon_0 \partial_x \phi(0,x) = \int_0^t q_+ j^+(s,x) + q_- j^-(s,x) ds \tag{4}$$

Apply $-\partial_x$ to get $-\varepsilon_0 \partial_{xx}\phi = q_+\rho^+ + q_-\rho^-$.

The floating potential

ϕ satisfies (P) with $\phi(t,1) = \phi(0,1) + \frac{1}{\varepsilon_0}\int_0^t \int_0^1 (q_+ j^+ + q_- j^-)(s,x) dx ds$

The Vlasov-Poisson-Ampère system

Consider ϕ satisfying (P). Resolution of (P) and application of ∂_{tx} gives

$$\varepsilon_0 \partial_{tx}\phi(t,x) = (q_+ j^+ + q_- j^-)(t,x) + \varepsilon_0 u'(t) - \int_0^1 (q_+ j^+ + q_- j^-)(t,s)ds \tag{5}$$

The floating potential

ϕ satisfies (A) iff $\phi(t,1) = \phi(0,1) + \frac{1}{\varepsilon_0}\int_0^t \int_0^1 (q_+ j^+ + q_- j^-)(s,x)dxds$

The Vlasov-Poisson-Ampère system

The problem we address is the Vlasov-Poisson-Ampère system.

$$(V^{\pm})\begin{cases} \partial_t f^{\pm} + v\partial_x f^{\pm} - \gamma_{\pm}\partial_x\phi\partial_v f^{\pm} = 0 & \forall(t,x,v)\in\Omega_T \\ f^{\pm}(0,x,v) = \tilde{f}^{\pm}(x,v) & \forall(x,v)\in]0,1[\times\mathbb{R} \\ f^{\pm}(t,0,v) = g^{\pm}(t,v) & \forall(t,v)\in]0,T]\times\mathbb{R}^{+*} \\ f^{\pm}(t,1,v) = 0 & \forall(t,v)\in]0,T]\times\mathbb{R}^{-*} \end{cases} \tag{6}$$

$$(P)\begin{cases} -\varepsilon_0\partial_{xx}\phi = q_+\rho^+ + q_-\rho^- & \forall(t,x)\in Q_T \\ \phi(t,0) = 0 & \forall t\in[0,T] \\ \phi(t,1) = \phi(0,1) + \dfrac{1}{\varepsilon_0}\int_0^t\int_0^1 (q_+j^+ + q_-j^-)(\tau,s)\,ds\,d\tau & \forall t\in[0,T] \end{cases} \tag{7}$$

Energy estimates

The total energy of the plasma $\mathcal{E}$ is the sum of kinetic and potential energy.

$$\mathcal{E}(t) = \frac{m_+}{2}\int_0^1\int_{\mathbb{R}} f^+ v^2 dv dx + \frac{m_-}{2}\int_0^1\int_{\mathbb{R}} f^- v^2 dv dx + \frac{\varepsilon_0}{2}\int_0^1 |\partial_x \phi|^2 dx \tag{8}$$

We can differentiate formally :

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= -\frac{m_+}{2}\int_{\mathbb{R}} v^3 f^+(t,1,v)dv + \frac{m_+}{2}\int_{\mathbb{R}} v^3 f^+(t,0,v)dv \\ &- \frac{m_-}{2}\int_{\mathbb{R}} v^3 f^-(t,1,v)dv + \frac{m_-}{2}\int_{\mathbb{R}} v^3 f^-(t,0,v)dv \\ &+ \left(\varepsilon_0 u'(t) - \int_0^1 (q_+ j^+ + q_- j^-)(t,s)ds\right) u(t) \end{aligned}$$

Energy estimates

- Integration w.r.t time and sign considerations leads to : $\forall t \leq T$,

$$\mathcal{E}(t) \leq \mathcal{E}(0) + \frac{m_+}{2}\int_0^T \int_0^{+\infty} v^3 g^+(s,v)dvds + \frac{m_-}{2}\int_0^T \int_0^{+\infty} v^3 g^-(s,v)dvds \quad (9)$$

- Solutions have bounded energy **in finite time interval** under regularity and integrability assumptions.

Literature

- The floating potential appears in physics papers [5].
- The plasma sheath (stability, steady states) and was studied by Badsi [2, 1].
- In [6], Guo used an upwind scheme to establish BV estimates for the Vlasov equation.
- Boyer and Aguillon [4] show clearly how to use the upwind scheme to get BV estimates for linear transport equations.
- The relative entropy method was used to study the long-time behavior of a Vlasov-Poisson system by Ben Abdallah and Dolbeault [3].

Existence, uniqueness

Theorem (Existence, uniqueness)

Let $p \geq 4$. If, for each species $s \in \{+,-\}$, it holds that

- $|v|^k \tilde{f}^s \in BV(\mathcal{O}), k = 0, ...p$
- $|v|^k g^s \in BV((0,T) \times \mathbb{R}^+), k = 0, ...p+1$
- $|v|^k \tilde{f}^s, |v|^k g^s \in L^\infty$, $k = 0, 1, 2, 3$
- $\sup_{t \in (0,T)} \sup_{\Delta t > 0} \frac{1}{\Delta t} |\partial_v(|v|^k g^s)|(\mathbb{R}^+ \times (t, t+\Delta t)) < +\infty,$
- $\sup_{t \in (0,T)} \sup_{\Delta t > 0} \frac{1}{\Delta t} |\partial_t(|v|^k g^s)|(\mathbb{R}^+ \times (t, t+\Delta t)) < +\infty,$
- $|v|^k(\tilde{f}^s(0,.) - g^s(0,.)) \in L^1(\mathbb{R}^+)$.

Then, the Vlasov-Poisson-Ampère system (VPA) has a unique weak solution $(f^+, f^-, \phi) \in C^0\Big([0,T]; L^1(\mathcal{O})\Big) \times W^{2,\infty}(Q_T)$ such that for all $k \in [\![0, p-1]\!]$, $|v|^k f^+, |v|^k f^- \in BV(\Omega_T)$.

Outline

We build weak solution with BV regularity. The proof is composed of 3 main steps.

1. Establish BV regularity on the solution of the Vlasov equation, with fixed electric field $E \in W^{1,\infty}(Q_T)$. It is done by establishing the estimates on an upwind finite-volume scheme and then passing to the limit.
2. Build solution for the Vlasov-Poisson system, with $\phi(t,1) := u(t) \in W^{1,\infty}(\mathbb{R}^{+*})$ prescribed. This is done by using the Schauder fixed point theorem.
3. Build solution for the Vlasov-Poisson-Ampère system, with $\phi(t,1)$ being the floating potential. This is done by using the Schauder fixed point theorem again.

BV estimates for the Vlasov equation

Choose $T_{r+1} > T_r > 0$, define $\Omega_r =]T_r, T_{r+1}[\times]0,1[\times\mathbb{R}$, $Q_r =]T_r, T_{r+1}[\times]0,1[$.
Fix an electric field $E \in W^{1,\infty}(Q_r)$. We prove BV regularity for the Vlasov equation

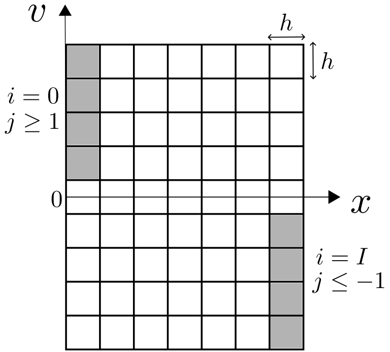
$$(V)\begin{cases} \partial_t f + v\partial_x f + \gamma E \partial_v f = 0 & \forall (t,x,v) \in \Omega_r \\ f(0,x,v) = \tilde{f}(x,v) & \forall (x,v) \in]0,1[\times\mathbb{R} \\ f(t,0,v) = g(t,v) & \forall (t,v) \in]T_r, T_{r+1}] \times \mathbb{R}^{+*} \\ f(t,1,v) = 0 & \forall (t,v) \in]T_r, T_{r+1}] \times \mathbb{R}^{-*} \end{cases} \quad (10)$$

Existence and uniqueness

Under integrability assumptions on the data, there exists a unique solution $f \in C^0([T_r, T_{r+1}], L^1(]0,1[\times\mathbb{R}))$

BV estimates for the Vlasov equation

Choose $I, N \in \mathbb{N}^*$, pose $h = \dfrac{1}{I+1}, \Delta t = \dfrac{T_{r+1} - T_r}{N}$. Partition into cubic cells of centers $(T_r + (n + \frac{1}{2})\Delta t, \frac{h}{2} + ih, jh)$.



BV estimates for the Vlasov equation

$$\begin{cases} \forall\ n \in [\![0, N-2]\!],\ (i,j) \in [\![1, I-1]\!] \times \mathbb{Z} \cup \{0\} \times \mathbb{Z}^- \cup \{I\} \times \mathbb{Z}^+ \\ \dfrac{f_{i,j}^{n+1} - f_{i,j}^n}{\Delta t} + (v_j)^+ \dfrac{f_{i,j}^n - f_{i-1,j}^n}{h} + (v_j)^- \dfrac{f_{i+1,j}^n - f_{i,j}^n}{h} + \gamma(E_i^n)^+ \dfrac{f_{i,j}^n - f_{i,j-1}^n}{h} + \gamma(E_i^n)^- \dfrac{f_{i,j+1}^n - f_{i,j}^n}{h} = 0 \\ f_{0,j}^n = g_j^n\ \forall n \geq 0, j \geq 1 \\ f_{I,j}^n = 0\ \forall n \geq 0, j \leq -1 \\ f_{i,j}^0 = \tilde{f}_{i,j}\ \forall (i,j) \in [\![1, I-1]\!] \times \mathbb{Z} \cup \{0\} \times \mathbb{Z}^- \cup \{I\} \times \mathbb{Z}^+ \end{cases} \tag{11}$$

$$f_{h,\Delta t}(t,x,v) = \sum_{n=0}^{N-1} \sum_{i=0}^{I} \sum_{j\in\mathbb{Z}} \left(\frac{f_{i,j}^{n+1}(t-t_n)}{\Delta t} + \frac{f_{i,j}^n(t_{n+1}-t)}{\Delta t} \right) \mathbb{1}_{K_{n,i,j}}(t,x,v)$$

$$v_h(v) = \sum_{j\in\mathbb{Z}} v_j \mathbb{1}_{[v_{j-1/2}, v_{j+1/2}]}(v)$$

BV estimates for the Vlasov equation

The total variation is given for $t \in [t_n, t_{n+1})$ by

$$TV(f_{h,\Delta t}(t), (0,1) \times \mathbb{R}) = TV_x(f_{h,\Delta t}(t), (0,1) \times \mathbb{R}) + TV_v(f_{h,\Delta t}(t), (0,1) \times \mathbb{R})$$

where

$$TV_x(f_{h,\Delta t}(t), (0,1) \times \mathbb{R}) := \sum_{i=0}^{I-1} \sum_{j\in\mathbb{Z}} h\Big(|d_x f_{i,j}^{n+1}| \frac{t - t_n}{\Delta t} + |d_x f_{i,j}^{n}| \frac{t_{n+1} - t}{\Delta t} \Big), \tag{12}$$

$$TV_v(f_{h,\Delta t}(t), (0,1) \times \mathbb{R}) := \sum_{i=0}^{I} \sum_{j\in\mathbb{Z}} h\Big(|d_v f_{i,j}^{n+1}| \frac{t - t_n}{\Delta t} + |d_v f_{i,j}^{n}| \frac{t_{n+1} - t}{\Delta t} \Big). \tag{13}$$

BV estimates for the Vlasov equation

$(f_{h,\Delta t})_{h,\Delta t>0}$ converges to the solution $f \in C^0([T_r, T_{r+1}], L^1((0,1)\times\mathbb{R}))$. We have BV estimates of the form

$$\sup_{t\in[T_r,T_{r+1}]} TV(|v_h(.)|^k f_{h,\Delta t}(t,.,.),(0,1)\times\mathbb{R}) \leq M_{data}$$

for every $k = 0, ...p$. Using lower semi-continuity, we discover that

$$\sup_{t\in[T_r,T_{r+1}]} TV(|v|^k f(t,.,.),(0,1)\times\mathbb{R}) \leq M_{data}$$

Integration in time proves the regularity in time : $TV(|v|^k f, \Omega_r) \leq M_{data}(T_{r+1} - T_r)$. The consistency of the scheme allows to get the weak formulation.

Simulations

FV scheme and splitting

The Vlasov equation is of the form

$$\frac{d}{dt}f^{\pm} = (A+B)f^{\pm};\ A = -v\partial_x,\ B = -\gamma_{\pm}\partial_x\phi\partial_v$$

On one time step, the semigroup can be approximated by a Strang splitting

$$e^{\Delta t(A+B)} = e^{\frac{\Delta t}{2}A}e^{\Delta tB}e^{\frac{\Delta t}{2}A}$$

Each of the factor can be approximated by a finite volume scheme (MUSCL).
The electric field and the floating potential have to be updated after each half time step.
The floating potential is approximated by using Simpson's rule.
We also choose a large finite window in velocity ($|v| \leq K$, K large) and use zero Dirichlet conditions on $v = \pm K$.

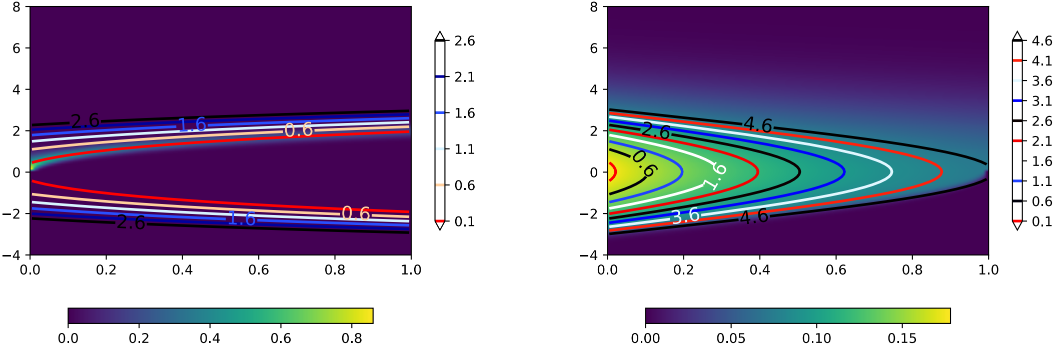
Simulation

Simulation of a plasma sheath with Maxwellian data.
Data :

- $\tilde{f}^{\pm} = 0$;
- $g^{\pm}(v) = \sqrt{\frac{m_{\pm}}{2\pi T_{\pm}}} e^{-\frac{m_{\pm} v^2}{2T_{\pm}}}$ (stationnary)
- $m_+ = 5$, $T_+ = 1$ (high mass, low temperature);
- $m_- = 1$, $T_- = 5$ (low mass, high temperature);
- $q_+ = 2;\ q_- = -1$
- $\varepsilon_0 = 0.1$;
- $\phi(0, 1) = 0$.

Simulation

Simulation



8
6
4
2
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1.6
0.6
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0.0 0.2 0.4 0.6 0.8
4.6
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Simulation

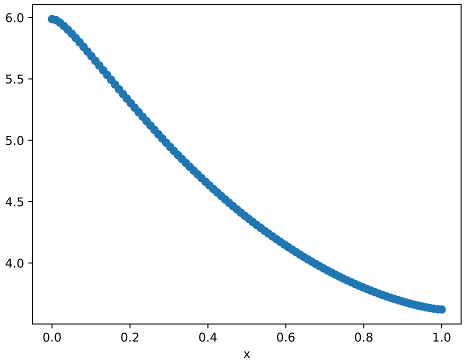


Figure: $-\partial_x \phi$

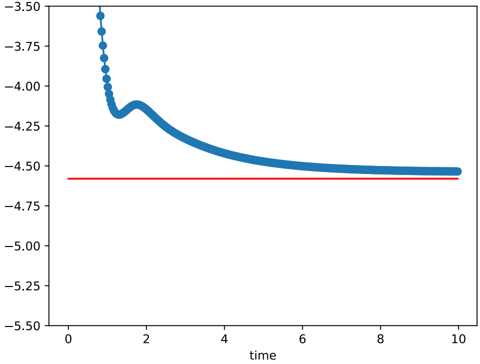


Figure: $\phi(t, 1)$

Simulation

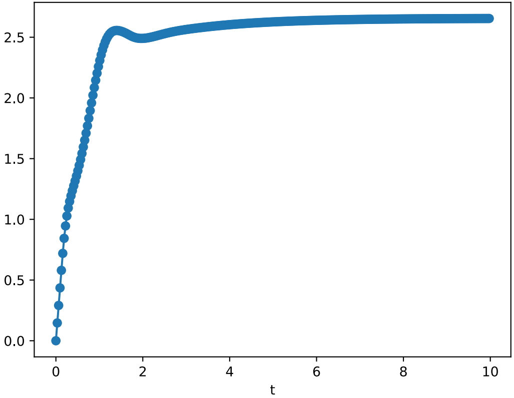


Figure: Total energy.

Stationnary solutions

Steady states

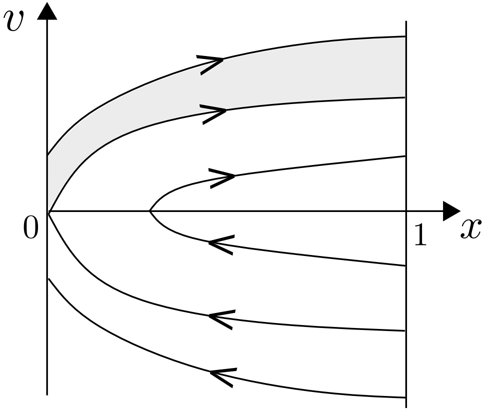
$$(V^{\pm})\begin{cases} v\partial_x f^{\pm} + \gamma_{\pm}\partial_x\phi\partial_v f^{\pm} = 0 & \forall (x,v) \in]0,1[\times\mathbb{R} \\ f^{\pm}(0,v) = g^{\pm}(\frac{m_{\pm}}{2}v^2) & \forall v \in \mathbb{R}^{+*} \\ f^{\pm}(1,v) = 0 & \forall v \in \mathbb{R}^{-*} \end{cases} \quad (14)$$

$$(P)\begin{cases} -\varepsilon_0\partial_{xx}\phi = q_+\rho^+ + q_-\rho^- \quad \forall x \in (0,1) \\ \phi(0) = 0 \\ \phi(1) = V < 0 \end{cases} \quad (15)$$

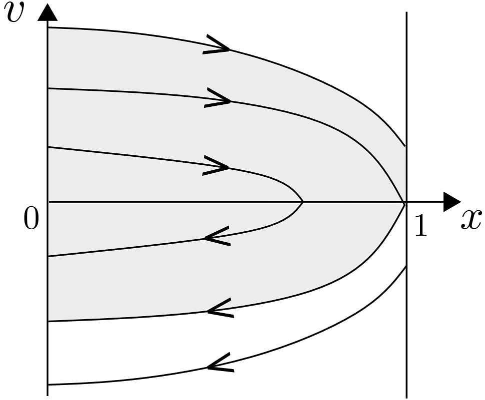
$$(A)\left\{ 0 = q_+ j^+ + q_- j^- \quad \forall x \in (0,1) \right. \quad (16)$$

We search for solutions with monotonic decreasing ϕ.

Steady states



Charges +



Charges −

Steady states

$f^{\pm}$ are locally functions of the Hamiltonians $\mathcal{H}^{\pm}(x,v) = \frac{m_{\pm}}{2}v^2 + q_{\pm}\phi(x)$:

$$f^+(x,v) = g^+(\mathcal{H}^+(x,v)) \text{ if } v > \sqrt{\frac{-2q_+}{m_+}\phi(x)};\ 0 \text{ elsewhere}$$

$$f^-(x,v) = g^-(\mathcal{H}^-(x,v)) \text{ if } v > -\sqrt{\frac{-2q_-}{m_-}(\phi(x) - V)};\ 0 \text{ elsewhere}$$

One can compute the currents $j^{\pm}$ explicitely, and use Ampère equation to compute V :

$$q_+ \int_0^{+\infty} w g^+(w) dw + q_- \int_{\sqrt{\frac{2q_- V}{m_-}}}^{+\infty} w g^-(w) dw = 0 \tag{17}$$

If $g^- > 0$, there is either one unique value for $V < 0$ or no value. With Maxwellian densities, $V = \frac{T_-}{|q_-|} \ln\left(\frac{|q_+|}{|q_-|}\sqrt{\frac{T_+ m_-}{T_- m_+}}\right) \approx -4.58$ in our setup.

Steady states

One can compute $\rho^{\pm}$ as a function of $\phi(x)$, i.e $\rho^{\pm}(x) = P^{\pm}(\phi(x))$.
Poisson equation becomes a semilinear elliptic equation

$$(P)\left\{\begin{array}{ccl} -\varepsilon_0 \partial_{xx}\phi & = & q_+ P^+(\phi) + q_- P^-(\phi)\ \forall x \in (0,1) \\ \phi(0) & = & 0 \\ \phi(1) & = & V \end{array}\right.$$

Existence of a solution $\phi \in H^1(0,1)$ by minimization of the functionnal

$$\frac{\varepsilon_0}{2}\int_0^1 \partial_x \phi^2 dx - \int_0^1 \int_0^x q_+ P^+(\phi(u)) + q_- P^-(\phi(u)) du dx. \tag{18}$$

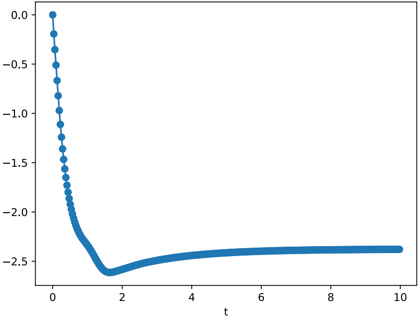
Uniqueness unclear as the functionnal lacks convexity.

Grand potential

In [3], a similar problem was tackled by considering the Grand Potential as a Lyapunov functionnal :

$$H(f^+, f^-, \phi) = \text{total energy } + \sum_{\pm} \int_\Omega \beta^\pm(f^\pm)dxdv \; ; \; \beta^\pm(u) = -\int_0^u g^{\pm -1}(z)dz$$

Problem : its time variation can't be signed.



Conclusions - Perspective

- Uniqueness of the steady states is unclear
- It seems that the steady states with nonincreasing potentials are attractors;
- A suitable entropy may be found for this system, allowing to both have uniquenes of the steady states and convergence toward it.

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