Numerical homogenization of a linear hyperbolic conservation law

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This work deals with the prototypical multiscale hyperbolic conservation law

$$\partial_t u^\varepsilon(x,t) + \operatorname{div} F\left(\frac{x}{\varepsilon}, u^\varepsilon(x,t)\right) = 0 \tag{1}$$

where u^ε is a scalar-valued function and the flux F is a vector-valued function, highly oscillatory at the small scale ε. Accurate numerical approximation typically requires a spatial discretization finer than ε, leading to prohibitive computational costs. From the viewpoint of asymptotic analysis, some classes of equations, for example linear ergodic transport [2], admit a homogenized equation that preserves the form of a conservation law, while in other cases such as *shear* transport, the oscillatory problem converges to a non-local diffusion equation with memory terms [3]. In general, the limit obtained remains multiscale in nature [2]. It is also well known that, as soon as the original hyperbolic equation is slightly perturbed by a small viscous (parabolic) term, then the homogenized equation is a standard conservation law, with effective parameters defined in terms of a corrector function, solution to an elliptic cell problem encoding the fine-scale structure of the flux [1].

Based on that observation, we have considered a strategy aiming at efficiently computing an approximation of the numerical solution (given by a fine-scale finite volume scheme) of (1). The idea is to homogenize the numerical scheme discretizing the PDE, rather than the PDE itself, by exploiting the numerical viscosity naturally present in the numerical scheme. This strategy has been first put in practice for the classical Lax–Friedrichs scheme. We assume the flux $F(y, v)$ to be Y-periodic with respect to $y \in Y = [0,1]^d$. We derive the equivalent equation associated to the scheme and, by means of a two-scale Ansatz, we obtain a well-posed cell problem in Y, that allows to compute a well-posed cell problem in Y and the associated effective equation.

The strategy has been implemented in a linear setting, with *well-prepared* initial conditions, following [1]. We will discuss numerical results showing that the resulting homogenized model provides an accurate and computationally efficient approximation of the reference solution (i.e. computed with the fine-scale numerical scheme) in one and two space dimensions for a broad class of multiscale conservation laws.

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[3] L. Tartar. *Nonlocal effects induced by homogenization.* In *Partial Differential Equations and the Calculus of Variations : Essays in Honor of Ennio De Giorgi Volume 2*, pp. 925–938. Springer, 1989.

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