Partial approximate Riemann solvers for discrete dissipative boundary conditions of the shallow-water equations

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Contents

Shallow water equations on the half-plane

$$\begin{cases} \partial_t \begin{pmatrix} h \\ q \end{pmatrix} + \partial_x \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix} = 0, \quad \forall x, t > 0, \\ \qquad\qquad q(0,t) = q_0(t). \qquad \text{(BC)} \end{cases}$$

(*i*) Entropy inequality : $\partial_t \eta(h,q) + \partial_x G(h,q) \leq 0$,

$$\eta(h,q) = q^2/(2h) + gh^2/2, \quad G(h,q) = q\big(q^2/(2h^2) + gh\big).$$

(*ii*) *Dissipative* (BC) i.e. $q_0(t) \leq 0$:

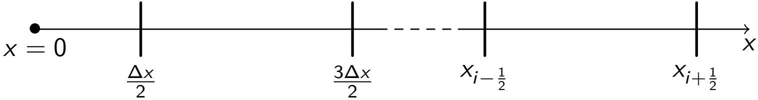
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^+} \eta(h,q)\,\mathrm{d}x \leq G(h(0,t), q_0(t)) \leq 0.$$

Contents

Finite-Volume method on the half-plane

$$w_1^n \approx \frac{1}{\Delta x}\int_{\frac{\Delta x}{2}}^{\frac{3\Delta x}{2}} w(x,t^n)\,\mathrm{d}x$$

$$w_0^n \approx (h,q)^{\mathsf{T}}(0,t^n) \qquad\qquad w_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n)\,\mathrm{d}x$$



Entropy-satisfying schemes for the interior domain (HLL, Rusanov, ...) :

$$\begin{cases} w_i^{n+1} = w_i^n - \dfrac{\Delta t}{\Delta x}\left(\mathcal{F}(w_i^n, w_{i+1}^n) - \mathcal{F}(w_{i-1}^n, w_i^n)\right), \\ \eta(w_i^{n+1}) \leq \eta(w_i^n) - \dfrac{\Delta t}{\Delta x}\left(\mathcal{G}(w_i^n, w_{i+1}^n) - \mathcal{G}(w_{i-1}^n, w_i^n)\right), \quad \forall i \geq 1. \end{cases} \tag{VF}$$

Discrete boundary conditions

$$\begin{cases} h_0^{n+1} = \frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{h}((x - x_{\frac{1}{2}})/\Delta t, w_0^n, w_1^n)\,\mathrm{d}x, \\ q_0^{n+1} = \mathcal{Q}(w_0^n, w_1^n), \end{cases} \tag{BC$_{dis}$}$$

where $\tilde{h}(\cdot, w_0^n, w_1^n)$ is a Partial Approximate Riemann Solver (PARS).

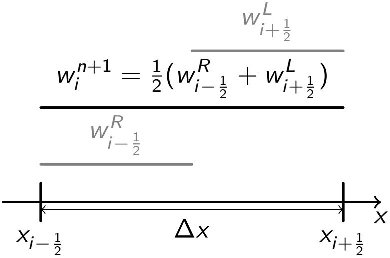
(*i*) Dubois *et al* : $w(0^+, t) \in \left\{\mathcal{W}_{\mathcal{R}}(0^+, w(0,t), w_R),\ w_R \in \mathbb{R}^2\right\}$.

(*ii*) Godunov-type schemes :

$$w^R_{i+\frac{1}{2}} = \frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\,\mathrm{d}x,$$

$$w^L_{i+\frac{1}{2}} = \frac{2}{\Delta x} \int_{-\frac{\Delta x}{2}}^{0} \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\,\mathrm{d}x,$$

$$w_i^{n+1} = \frac{1}{2}(w^R_{i-\frac{1}{2}} + w^L_{i+\frac{1}{2}}).$$



Weak-consistency of (BC_{dis})

$w^n = (h^n, q^n)^\mathsf{T} \in \mathbb{R}^2$ s.t.

$$q^n = q_0(t^n) + \mathcal{O}(\Delta t).$$

(*i*) h_0^{n+1} is consistent if $\tilde{h}(\cdot, w_0^n, w_1^n) \in L^1([0, \Delta x])$ and if

(*a*) the α-weak-consistency holds

$$\exists \alpha > 0, \quad \tilde{h}(\cdot, w^n, w^n) = h^n + \mathcal{O}(\Delta x^\alpha),$$

(*b*) the integral consistency relation holds

$$\frac{1}{\Delta x} \int_0^{\Delta x} \tilde{h}\big((x - x_{\frac{1}{2}})/\Delta t, w_0^n, w_1^n\big)\,\mathrm{d}x = \frac{h_0^n + h_1^n}{2} - \frac{\Delta t}{\Delta x}(q_1^n - q_0^n).$$

(*ii*) q_0^{n+1} is consistent if $\mathcal{Q}(\cdot, \cdot)$ ensures

$$\mathcal{Q}(w^n, w^n) = q_0(t^{n+1}) + \mathcal{O}(\Delta t).$$

Discrete stability of (BC_{dis})

Lemma :

If $h_0^n > 0, q_0^n \leq 0$ and if

$$\frac{2}{\Delta x}\int_0^{\frac{\Delta x}{2}} \eta(\tilde{h}((x - x_{\frac{1}{2}})/\Delta t, w_0^n, w_1^n), q_0^{n+1})\,\mathrm{d}x \\ \leq \eta(w_0^n) - \frac{2\Delta t}{\Delta x}\big(\mathcal{G}(w_0^n, w_1^n) - \mathcal{G}(w_0^n, w_0^n)\big),$$

then $(w_i^{n+1})_{i\geq 1} \cup w_0^{n+1}$ enforces

$$\sum_{i\geq 0} \frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t}|C_i| \leq 0,$$

where $|C_i| > 0$ denotes the measure of the cell i.

Sketch of the proof

(*i*) Entropy-satisfying (VF) : $\displaystyle\sum_{i\geq 1} \frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t}\, \Delta x \leq \mathcal{G}(w_0^n, w_1^n).$

(*ii*) $h_0^n > 0$, $q_0^n \leq 0$ involve $-G(w_0^n) = -\mathcal{G}(w_0^n, w_0^n) \geq 0$:

$$\sum_{i\geq 0} \frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t}\, |C_i|$$
$$\leq \frac{\eta(w_0^{n+1}) - \eta(w_0^n)}{\Delta t} \frac{\Delta x}{2} + \mathcal{G}(w_0^n, w_1^n) - \mathcal{G}(w_0^n, w_0^n).$$

(*iii*) Jensen inequality :

$$\eta(w_0^{n+1}) \leq \frac{2}{\Delta x} \int_0^{\frac{\Delta x}{2}} \eta\big(\tilde{h}((x - x_{\frac{1}{2}})/\Delta t, w_0^n, w_1^n), q_0^{n+1}\big)\, \mathrm{d}x.$$

Application with HLL for a practical design

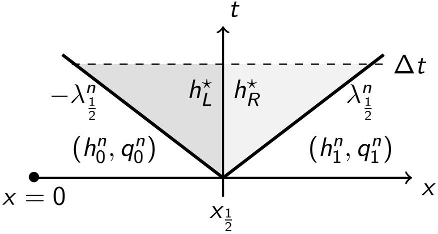
Interior domain : symmetric HLL scheme defined by

$$\mathcal{F}(w_i^n, w_{i+1}^n) = \frac{1}{2}(f(w_i^n) + f(w_{i+1}^n)) - \frac{1}{2}\lambda_{i+\frac{1}{2}}^n (w_{i+1}^n - w_i^n), \quad \forall i \geq 0,$$

where $f(w) = (q, q^2/h + gh^2/2)^{\mathsf{T}}$, $\lambda_{i+\frac{1}{2}}^n > 0$.

PARS :

$$\tilde{h}((x - x_{\frac{1}{2}})/\Delta t, w_0^n, w_1^n) = \begin{cases} h_0^n, & \text{if } x - x_{\frac{1}{2}} \leq -\lambda_{\frac{1}{2}}^n \Delta t, \\ h_L^\star, & \text{if } -\lambda_{\frac{1}{2}}^n \Delta t < x - x_{\frac{1}{2}} \leq 0, \\ h_R^\star, & \text{if } 0 < x - x_{\frac{1}{2}} \leq \lambda_{\frac{1}{2}}^n \Delta t, \\ h_1^n, & \text{if } \lambda_{\frac{1}{2}}^n \Delta t < x - x_{\frac{1}{2}}, \end{cases}$$



Practical consistency design for PARS

(*i*) Integral consistency relation $\frac{1}{2}(h_L^\star + h_R^\star) = h_{\frac{1}{2}}^{\mathrm{HLL}}$ where

$$w_{\frac{1}{2}}^{\mathrm{HLL}} := (h_{\frac{1}{2}}^{\mathrm{HLL}}, q_{\frac{1}{2}}^{\mathrm{HLL}})^{\mathsf{T}} = \frac{1}{2}(w_0^n + w_1^n) - \frac{1}{2\lambda_{\frac{1}{2}}^n}\left(f(w_1^n) - f(w_0^n)\right).$$

(*ii*) Formulation in (BC_{dis}) :

$$h_0^{n+1} = h_0^n - \frac{\Delta t}{\Delta x}(q_1^n - q_0^n) + \lambda_{\frac{1}{2}}^n \frac{\Delta t}{\Delta x}\left((h_1^n - h_0^n) - (h_R^\star - h_L^\star)\right).$$

(*iii*) Formal strong consistency :

$$(\partial_t h + \partial_x q)(0,t) = \lambda_{\frac{1}{2}}^n(\partial_x h - \partial_x h^\star)(0,t) = \mathcal{O}(\Delta x).$$

Practical entropy stability design for PARS

(i) Entropy condition :

$$\eta(h_L^\star, q_0^{n+1}) \leq \tfrac{1}{2\nu^n_{\frac{1}{2}}}(1-2\nu^n_{\frac{1}{2}})\big(\eta(h_0^n,q_0^n)-\eta(h_0^n,q_0^{n+1})\big)+\eta^{\mathrm{HLL}}_{\frac{1}{2}},$$

$$\eta^{\mathrm{HLL}}_{\frac{1}{2}} := \tfrac{1}{2}(\eta(w_0^n)+\eta(w_1^n)) - \tfrac{1}{2\lambda^n_{\frac{1}{2}}}\big(G(w_1^n)-G(w_0^n)\big),$$

with $\nu^n_{\frac{1}{2}} = \lambda^n_{\frac{1}{2}}\Delta t/\Delta x$.

(ii) Reformulation with $\frac{1}{2}(h_L^\star + h_R^\star) = h^{\mathrm{HLL}}_{\frac{1}{2}}$ and $\delta h^\star = (h_R^\star - h_L^\star)/(2h^{\mathrm{HLL}}_{\frac{1}{2}})$:

$$\frac{\delta h^\star}{1-\delta h^\star}\left((\mathrm{Fr}^{n+1})^2 - 2 + 3\delta h^\star - (\delta h^\star)^2\right) \leq (1-2\nu^n_{\frac{1}{2}})\frac{h^{\mathrm{HLL}}_{\frac{1}{2}}}{2h_0^n\nu^n_{\frac{1}{2}}}\left((\mathrm{Fr}^n)^2 - (\mathrm{Fr}^{n+1})^2\right)$$

$$+(\mathrm{Fr}^{\mathrm{HLL}})^2 - (\mathrm{Fr}^{n+1})^2 + 2\frac{\eta^{\mathrm{HLL}}_{\frac{1}{2}} - \eta(w^{\mathrm{HLL}}_{\frac{1}{2}})}{g(h^{\mathrm{HLL}}_{\frac{1}{2}})^2},$$

where $(\mathrm{Fr}^k)^2 = (q_0^k)^2/(g(h^{\mathrm{HLL}}_{\frac{1}{2}})^3)$, $(\mathrm{Fr}^{\mathrm{HLL}})^2 = (q^{\mathrm{HLL}}_{\frac{1}{2}})^2/(g(h^{\mathrm{HLL}}_{\frac{1}{2}})^3)$.

Practical entropy stability design for PARS

(*i*) Entropy condition :

$$\eta(h_L^\star, q_0^{n+1}) \leq \tfrac{1}{2\nu_{\frac12}^n}(1-2\nu_{\frac12}^n)\big(\eta(h_0^n,q_0^n)-\eta(h_0^n,q_0^{n+1})\big)+\eta_{\frac12}^{\mathrm{HLL}},$$
$$\eta_{\frac12}^{\mathrm{HLL}} := \tfrac12(\eta(w_0^n)+\eta(w_1^n)) - \tfrac{1}{2\lambda_{\frac12}^n}\big(G(w_1^n)-G(w_0^n)\big),$$

with $\nu_{\frac12}^n = \lambda_{\frac12}^n \Delta t/\Delta x$.

(*ii*) Reformulation with $\frac12(h_L^\star + h_R^\star) = h_{\frac12}^{\mathrm{HLL}}$ and $\delta h^\star = (h_R^\star - h_L^\star)/(2h_{\frac12}^{\mathrm{HLL}})$:

$$\frac{\delta h^\star}{1-\delta h^\star}\left((\mathrm{Fr}^{n+1})^2 - 2 + 3\delta h^\star - (\delta h^\star)^2\right) = (1-2\nu_{\frac12}^n)\frac{h_{\frac12}^{\mathrm{HLL}}}{2h_0^n\nu_{\frac12}^n}\left((\mathrm{Fr}^n)^2 - (\mathrm{Fr}^{n+1})^2\right)$$
$$+ (\mathrm{Fr}^{\mathrm{HLL}})^2 - (\mathrm{Fr}^{n+1})^2 + 2\frac{\eta_{\frac12}^{\mathrm{HLL}} - \eta(w_{\frac12}^{\mathrm{HLL}})}{g(h_{\frac12}^{\mathrm{HLL}})^2}, \qquad (\star_{\text{cubique}})$$

<u>*and*</u> $\delta h^\star \in (-\infty, 1)$, $\delta h^\star|_{w_0^n = w_1^n = w^n} = \mathcal{O}(\Delta x^\alpha)$.

Entropy stability enforcement on the boundary

Lemma : If $q_0^{n+1} = \mathcal{Q}(w_0^n, w_1^n)$ is weak-consistent with $q_0(t) \in C^1(\mathbb{R}^+, \mathbb{R}^-)$ and if

$$3\sqrt[3]{(\mathrm{Fr}^{n+1})^4/4} \leq 1 + (1 - 2\nu^n_{\frac{1}{2}})\frac{h^{\mathrm{HLL}}_{\frac{1}{2}}}{2h_0^n \nu^n_{\frac{1}{2}}}\left((\mathrm{Fr}^n)^2 - (\mathrm{Fr}^{n+1})^2\right) + (\mathrm{Fr}^{\mathrm{HLL}})^2 + 2\frac{\eta^{\mathrm{HLL}}_{\frac{1}{2}} - \eta(w^{\mathrm{HLL}}_{\frac{1}{2}})}{g(h^{\mathrm{HLL}}_{\frac{1}{2}})^2}, \qquad (\star\star)$$

then $(\star_{\mathrm{cubique}})$ admits at least one solution in $(-\infty, 1)$ that also satisfies $\delta h^\star|_{w_0^n = w_1^n = w^n} = \mathcal{O}(\Delta x^\alpha)$.

Sketch of the proof

$$\Phi(\delta h^\star) := \delta h^\star \left((\mathrm{Fr}^{n+1})^2 - 2 + 3\delta h^\star - (\delta h^\star)^2 \right) / (1 - \delta h^\star).$$

(*i*) Convex in $(-\infty, 1)$ and minimum for $\delta h^\star_m = 1 - \sqrt[3]{(\mathrm{Fr}^{n+1})^2/2}$.

(*ii*) Taylor-Lagrange expansion around $\delta h^\star_m$:

$$\Phi(\delta h^\star) = -(\mathrm{Fr}^{n+1})^2 + 3\sqrt[3]{(\mathrm{Fr}^{n+1})^4/4} - 1 + \tfrac{1}{2}\Phi''(\xi)(\delta h^\star - \delta h^\star_m)^2.$$

(*iii*) Reformulation of $(\star_{\mathrm{cubique}})$:

$$\tfrac{1}{2}\Phi''(\xi)(\delta h^\star - \delta h^\star_m)^2 = -3\sqrt[3]{(\mathrm{Fr}^{n+1})^4/4} + 1 + (1 - 2\nu^n_{\frac{1}{2}}) \frac{h^{\mathrm{HLL}}_{\frac{1}{2}}}{2h^n_0 \nu^n_{\frac{1}{2}}} \left((\mathrm{Fr}^n)^2 - (\mathrm{Fr}^{n+1})^2 \right) + (\mathrm{Fr}^{\mathrm{HLL}})^2 + 2\frac{\eta^{\mathrm{HLL}}_{\frac{1}{2}} - \eta(w^{\mathrm{HLL}}_{\frac{1}{2}})}{g(h^{\mathrm{HLL}}_{\frac{1}{2}})^2}.$$

Definition for q_0^{n+1}

Lemma : Let $q_0(t) \in C^1(\mathbb{R}^+, \mathbb{R}^-)$ and

$$h_b^n = (1 - 2\nu_{\frac{1}{2}}^n) h_{\frac{1}{2}}^{\mathrm{HLL}} + 2\nu_{\frac{1}{2}}^n h_0^n > 0,$$
$$\beta^n = (1 - 2\nu_{\frac{1}{2}}^n) h_{\frac{1}{2}}^{\mathrm{HLL}} / h_b^n \in (0, 1].$$

The following :

$$q_0^{n+1} = -|\beta^n q_0^n + (1 - \beta^n) q_{\frac{1}{2}}^{\mathrm{HLL}} + q_0(t^{n+1}) - q_0(t^n)| \leq 0,$$

is weak consistency and if $|\mathrm{Fr}^{n+1}| \neq \sqrt{2}$ then there exists $\lambda_{\frac{1}{2}}^n > 0$ large enough such that $(\star\star)$ holds.

Sketch of the proof

(i) Expansion of $(\star\star)$ with q_0^{n+1} :

$$3\sqrt[3]{(\mathrm{Fr}^{n+1})^4/4} \leq \; 1 + (\mathrm{Fr}^{n+1})^2 + \tfrac{h_b^n}{2h_0^n\nu^n_{\frac{1}{2}}}\left(-(\mathrm{Fr}^{n+1})^2 + (\beta^n\mathrm{Fr}^n + (1-\beta^n)\mathrm{Fr}^{\mathrm{HLL}})^2\right)$$
$$+ \tfrac{h_b^n}{2h_0^n\nu^n_{\frac{1}{2}}}\left(\beta^n(\mathrm{Fr}^n)^2 + (1-\beta^n)(\mathrm{Fr}^{\mathrm{HLL}})^2 - (\beta^n\mathrm{Fr}^n + (1-\beta^n)\mathrm{Fr}^{\mathrm{HLL}})^2\right)$$
$$+ \tfrac{2}{g(h^{\mathrm{HLL}}_{\frac{1}{2}})^2}(\eta^{\mathrm{HLL}}_{\frac{1}{2}} - \eta(w^{\mathrm{HLL}}_{\frac{1}{2}})).$$

(ii) HLL stability, convexity of $s \mapsto s^2$:

$$0 \leq 1 - 3\sqrt[3]{(\mathrm{Fr}^{n+1})^4/4} + (\mathrm{Fr}^{n+1})^2 + \tfrac{h_b^n}{2h_0^n\nu^n_{\frac{1}{2}}}\left(-(\mathrm{Fr}^{n+1})^2 + (\beta^n\mathrm{Fr}^n + (1-\beta^n)\mathrm{Fr}^{\mathrm{HLL}})^2\right).$$

(iii) Expansion of $(\mathrm{Fr}^{n+1})^2 = (q_0^{n+1})^2/(g(h^{\mathrm{HLL}}_{\frac{1}{2}})^3)$ and $\nu^n_{\frac{1}{2}} = \lambda^n_{\frac{1}{2}}\Delta t/\Delta x$:

$$0 \leq 1 - 3\left(\sqrt[3]{(\mathrm{Fr}^{n+1})^2/2}\right)^2 + (\mathrm{Fr}^{n+1})^2 + \frac{\Delta x}{\lambda^n_{\frac{1}{2}}}\frac{q_0(t^{n+1}) - q_0(t^n)}{\Delta t}\mathcal{O}(1).$$

Weak-consistent entropy-stable (BC_{dis})

Theorem : Let $q_0(t) \in C^1(\mathbb{R}^+, \mathbb{R}^-)$ and $(w_i^{n+1})_{i\geq 1}$ given by the (VF)-HLL scheme. If

$$q_0^{n+1} = -|\beta^n q_0^n + (1-\beta^n) q_{\frac{1}{2}}^{\text{HLL}} + q_0(t^{n+1}) - q_0(t^n)| \leq 0,$$
$$h_0^{n+1} = h_0^n - \tfrac{\Delta t}{\Delta x}(q_1^n - q_0^n) + \lambda_{\frac{1}{2}}^n \tfrac{\Delta t}{\Delta x}\big((h_1^n - h_0^n) - (h_R^\star - h_L^\star)\big),$$

with $h_R^\star - h_L^\star = 2h_{\frac{1}{2}}^{\text{HLL}} \delta h^\star$ defined with $(\star_{\text{cubique}})$ under $(\star\star)$ then w_0^{n+1}

(*i*) is weak-consistent,

(*ii*) is robust : $h_0^{n+1} > 0$, $q_0^{n+1} \leq 0$,

(*iii*) enforces

$$\sum_{i\geq 0} \frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} |C_i| \leq 0.$$

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Implementation

w_0^n, w_1^n, $\lambda_{\frac{1}{2}}^n$, $\Delta t, \Delta x, \Delta t^{\mathrm{HLL}}$ `# Data`

while $(\star\star)$ *is false* **do**

  increase $\lambda_{\frac{1}{2}}^n$

  update $w_{\frac{1}{2}}^{\mathrm{HLL}}, q_0^{n+1}$, $(\star\star)$

  `# possibly additional procedure to check and correct` $|\mathrm{Fr}^{n+1}| = \sqrt{2}$.

solve $(\star_{\mathrm{cubique}})$ to get $\delta h^\star$ `# explicit cubic solver`

update h_0^{n+1}

(re)-restrict $\Delta t = \min\left(\Delta x/(2\lambda_{\frac{1}{2}}^n), \Delta t^{\mathrm{HLL}}\right)$

return w_0^{n+1}, Δt

Numerical protocol

(i) Reference solution :

(a) Godunov scheme for the interior domain.
(b) Iterative solver for $\{\mathcal{W}_{\mathcal{R}}(0^+, w(0,t^n), w_1^n)\}$.

(ii) Comparative with discrete Riemann invariants $\varphi(w) = q/h + 2\sqrt{gh}$, $\lambda^-(w) = q/h - \sqrt{gh}$:

$$\varphi(w_0^{n+1}) = \varphi(w_0^n) - \lambda^-(w_0^n)\tfrac{\Delta t}{\Delta x}\left(\varphi(w_1^n) - \varphi(w_0^n)\right).$$

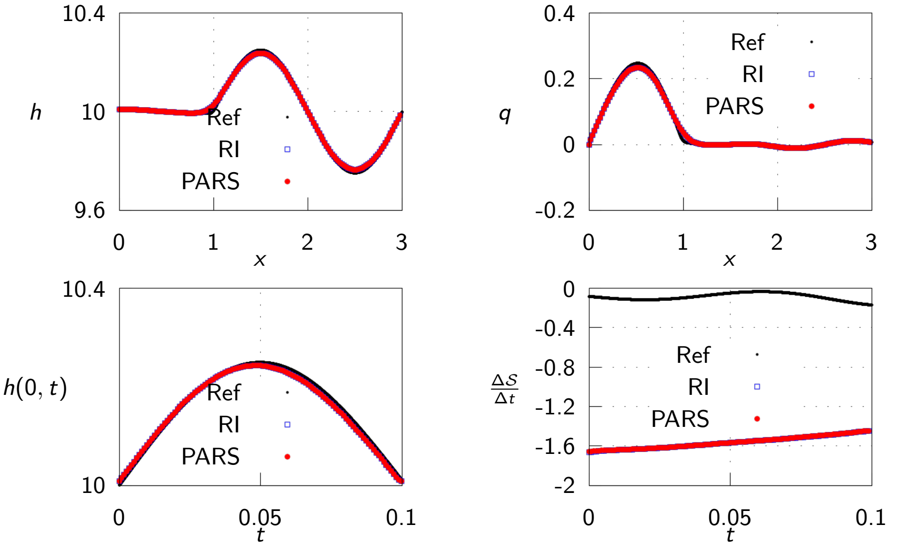
(iii) Data : $w^0(x) = (10 + 0.25\sin(\pi x), 0)^\mathsf{T}$, $q_0(t) = -q_{\text{left}}\sin^2(t/\tau)$.

(iv) Entropy stability

$$\frac{\Delta\mathcal{S}}{\Delta t} := \sum_{i\geq 0}\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t}\,|C_i|.$$

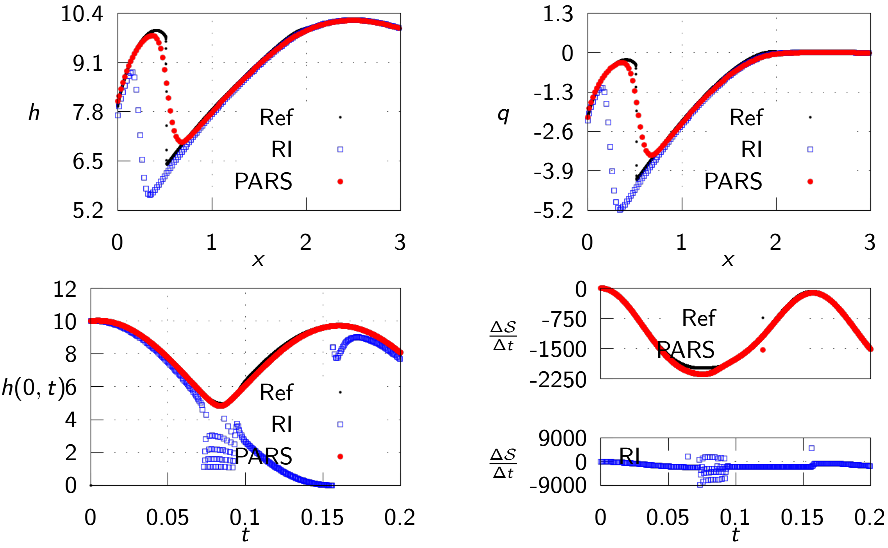
Numerical results at rest

$q_{\text{left}} = 0$, $x \in [0, 10]$, 500 cells, $t_{final} = 0.1$, CFL$= 0.9$.



Numerical results for generic dissipative (BC)

$q_{\text{left}} = 30$, $\tau = 0.05$, $x \in [0, 10]$, 500 cells, $t_{final} = 0.2$, CFL$= 0.9$.



Contents

Conclusion

Formulation of (BC_{dis}) with PARS

(i) generic framework for the weak-consistency,
usually : Taylor expansions.

(ii) enforces weak-consistency, robustness, entropy-stability,
usually : discrete Riemann invariants.

Investigate extensions for other (dispersive) systems.

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