CANUM 2026

Energy minimization for rotating Bose–Einstein condensates via a finite volume scheme

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Contents

Gross–Pitaevskii equation

Finite volume discretization of the Gross–Pitaevskii energy

Convergence of discrete minimizers

Numerical results

Contents

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Gross–Pitaevskii equation

Bose–Einstein condensates:

- State of matter observed in bosonic gases at low temperature.
- $T \to 0$: described by a single wave function Ψ solution of the **Gross–Pitaevskii equation**

$$i\partial_t \Psi = H\Psi + \gamma|\Psi|^2\Psi. \tag{GP}$$

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- $\gamma > 0$ (defocusing): nonlinear interaction parameter between particles.
- H: linear Hamiltonian of the system. In our case

$$H\Psi = -\underbrace{\Delta\Psi}_{\text{Kinetic}} + \underbrace{V\Psi}_{\text{Potential}} - \underbrace{\Omega L_z \Psi}_{\text{Rotation}}.$$

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- Nonlinear properties similar to those of fluids (e.g. vortices, turbulence).

Stationary solutions

If $\Psi(t, x) = e^{-i\mu t} u(x)$, then u satisfies

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Critical points of the energy $E : H_0^1(\mathcal{D}) \to \mathbb{R}$

$$E(u) = \frac{1}{2}\langle Hu, u\rangle + \frac{\gamma}{4}\|u\|_{L^4}^4,$$

under the constraint $\|u\|_{L^2} = 1$. Associated minimization problem:

$$\mathcal{U} = \operatorname*{argmin}_{\|u\|_{L^2}=1} E(u).$$

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Theoretical results since the 2000s (Aftalion, Ignat, Millot):

- Existence of minimizers iff $|\Omega| < \Omega_{\mathrm{crit}}(V)$.
- Presence of vortices for nonzero Ω.
- Thomas–Fermi limit $\gamma \to +\infty$.

Spatial discretization of Gross–Pitaevskii

In the literature:

- Finite differences and pseudospectral methods (Bao et al.).
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Our work:

- Develop a finite volume discretization.
- Numerical analysis of the discretization and convergence of minimizers in 2D and 3D.
- Analysis of a normalized gradient flow.
- Python implementation in 2D.

Contents

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Convergence of discrete minimizers

Numerical results

Admissible meshes

We work with admissible finite volume meshes $\mathcal{M} = (\mathcal{F}, \mathcal{E})$.

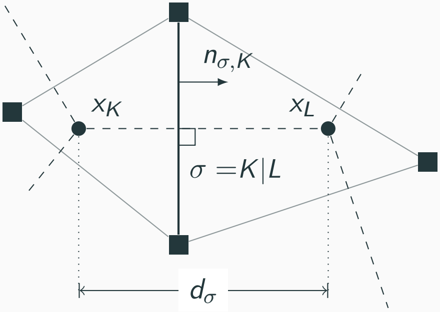

Figure 1: Cells of a mesh with associated notations.

Discrete functional spaces

We define X_h as the set of complex-valued functions that are piecewise constant on $\mathcal{F}$.

For $u \in X_h$

$$\|u\|_{L^p}^p = \sum_{K \in \mathcal{F}} |u_K|^p |K|.$$

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We also define

$$D_{\sigma,K} u = \begin{cases} u_L - u_K & \sigma = K|L, \\ -u_K & \sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}, \end{cases}$$

and the discrete H_0^1 seminorm

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Discrete functional inequalities: Poincaré, Sobolev, Gagliardo–Nirenberg (Bessemoulin-Chatard et al. 2014)

Definition of the scheme

We wish to discretize the Hamiltonian $H = -\Delta + V - \Omega L_z$.

- $\Delta \to A_h$: classical TPFA Laplacian.
- $L_z \to L_h$: TPFA-like discretization of the angular momentum.
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Discrete GP energy

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Discrete GP energy

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Questions:

- If $\varphi \in H_0^1(\mathcal{D})$ is sufficiently regular, does $E_h(P_h\varphi) \to E(\varphi)$?
- Convergence of normalised minimizers of E_h towards continuous ground states?

Contents

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Convergence of discrete minimizers

Numerical results

Convergence of discrete minimizers

We denote

$$\mathcal{U}_h = \underset{\substack{u \in X_h \\ \|u\|_{L^2}=1}}{\operatorname{argmin}} E_h(u) \text{ and } \mathcal{U} = \underset{\substack{u \in H_0^1(\mathcal{D}) \\ \|u\|_{L^2}=1}}{\operatorname{argmin}} E(u).$$

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Theorem (Chauleur, Dujardin, R. (in preparation))
Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of regular meshes with $h_n \to 0$ and $u_n \in \mathcal{U}_{h_n}$. Then, up to a subsequence, $(u_n)_{n\in\mathbb{N}}$ converges in $L^2(\mathcal{D})$ to some $u \in \mathcal{U}$. Moreover,

$$\lim_{n\to\infty} E_{h_n}(u_n) = E(u).$$

Elements of proof

Existence of a limit:

- Coercivity of the energy for $|\Omega|$ small enough: $E_h(u) \geq C|u|_{1,2}^2$.
- Compactness lemma (Eymard, Gallouet, Herbin):

$$|u_n|_{1,2} \leq C \implies \text{subsequence such that } u_n \xrightarrow{L^2} u, \; u \in H_0^1(\mathcal{D}).$$

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Convergence to a critical point:

- u_n satisfies

$$H_{h_n} u_n + \gamma |u_n|^2 u_n = \mu_n u_n.$$

- Passing to the limit, u satisfies

$$Hu + \gamma |u|^2 u = \mu u.$$

- From $\mu_n \to \mu$, we deduce $E_{h_n}(u_n) \to E(u)$.

Minimality of the limit:

- From the minimality of u_n, we have for all φ:

$$E_{h_n}(u_n) \leq E_{h_n}(P_{h_n}\varphi).$$

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Remarks:

- Compactness-based proof: no convergence rate is obtained!
- Can rates be obtained under Hessian coercivity assumptions? (method used by Henning and Yadav for finite elements).

Contents

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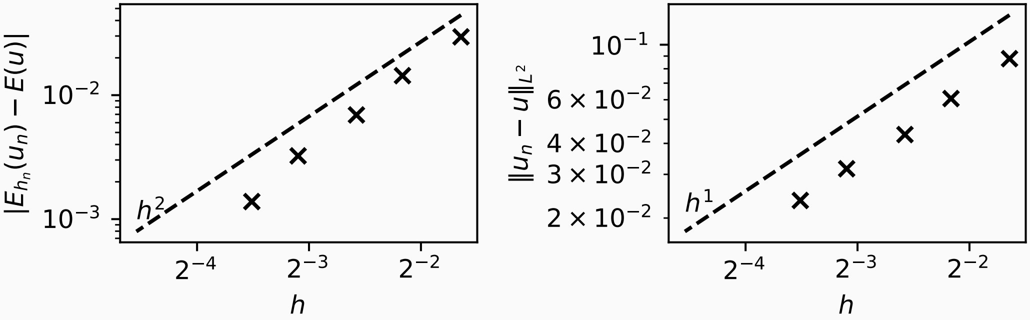

Figure 2: Convergence of discrete minimizers towards a reference solution.

Numerical results

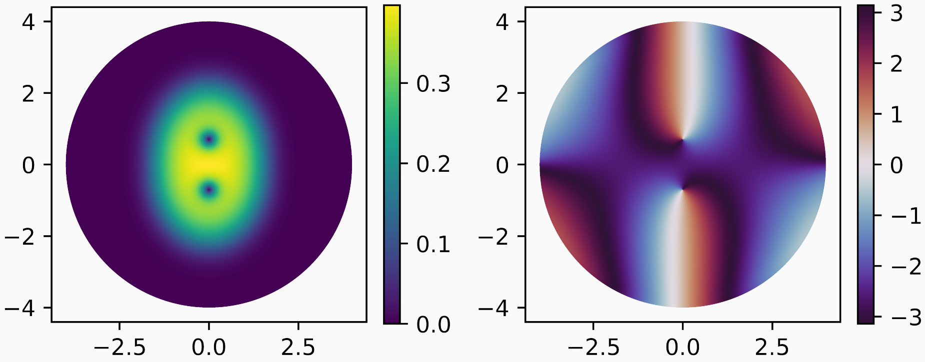

Figure 3: Reference solution.

Thank you for your attention

Numerical computation of ground states

We use a normalized gradient descent algorithm (see Bao):

$$\frac{\tilde{u}_{n+1} - u_n}{\tau} = -H_h \tilde{u}_{n+1} + \gamma |u_n| \tilde{u}_{n+1},$$
$$u_{n+1} = \frac{\tilde{u}_{n+1}}{\|\tilde{u}_{n+1}\|_{L^2}}.$$

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Theoretical analysis of a continuous projected gradient flow:

$$\partial_t u = -H_h u - \gamma |u|^2 u + \mu(u) u,$$

with local exponential convergence towards local minimizers.