Convergence analysis of a penalization method for compressible bubbly flows

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Introduction

Bubble effects on the surrounding fluid

- Resonant scattering of acoustic waves by gas bubbles
- Strong modification of sound speed and compressibility
- Nonlinear regimes: cavitation

Applications

- Medical ultrasonics
- Sonar and underwater acoustics
- Aerospace propulsion systems

Two-phase flow models

- *Sharp-interface*: explicit moving boundary
- *Diffuse-interface*: regularized transition layer

Goal: Analysis and simulation of compressible fluid–bubble flows.

Derivation of the fluid–bubble system

Diphasic flow:

Bubble phases: $(\rho_i, u_i, p_i)_{1\le i\le m}$ in $\bigcup_{i=1}^m \mathcal{B}_i$

Fluid phase: (ρ_f, u_f, p_f) in $\Omega \setminus \bigcup_{i=1}^m \overline{\mathcal{B}_i} := \mathcal{F}$

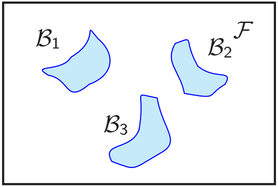
Viscous barotropic compressible Navier–Stokes system:

Mass conservation: $\partial_t \rho_j + \nabla \cdot (\rho_j u_j) = 0$

Momentum balance: $\partial_t(\rho_j u_j) + \nabla \cdot (\rho_j u_j \otimes u_j) = \nabla \cdot \mathbb{T}(u_j, p_j) - \rho_j g$

Stress tensor: $\mathbb{T}(u_j, p_j) = 2\mu_j \left(\mathbb{D}(u_j) - \frac{1}{3}(\nabla \cdot u_j)\mathbb{I}_3 \right) + (\nu_j \nabla \cdot u_j - p_j)\mathbb{I}_3$

Barotropic law: $p_j = p_j(\rho_j) = a_j \rho_j^{\gamma_j}$



Constants:

$a_j > 0,\ \gamma_j > \frac{3}{2},\ \mu_j > 0,\ \nu_j \ge 0$

Interface conditions on $\partial\mathcal{B}_i$:

$u_f = u_i$

$\big(\mathbb{T}(u_f, p_f) - \mathbb{T}(u_i, p_i)\big)\boldsymbol{n} = \tilde{\kappa}_i \boldsymbol{n}$

Derivation of the fluid-bubble system

Definition of the composite variables:

Ind. func.: $\partial_t \mathbb{1}_{\mathcal{F}} + u_f \cdot \nabla \mathbb{1}_{\mathcal{F}} = 0, \quad \partial_t \mathbb{1}_{\mathcal{B}_i} + u_i \cdot \nabla \mathbb{1}_{\mathcal{B}_i} = 0,$

Velocity/Density: $u = \mathbb{1}_{\mathcal{F}} u_f + \sum_{i=1}^{m} \mathbb{1}_{\mathcal{B}_i} u_i, \quad \rho = \mathbb{1}_{\mathcal{F}} \rho_f + \sum_{i=1}^{m} \mathbb{1}_{\mathcal{B}_i} \rho_i$

Presure law: $p = p(\rho, \{\mathcal{B}_i\}_{i=1}^m) = \mathbb{1}_{\mathcal{F}} a_f \rho_f^{\gamma_f} + \sum_{i=1}^{m} \mathbb{1}_{\mathcal{B}_i} a_i \rho_i^{\gamma_i}$

Visc. coeff.: $\mu = \mathbb{1}_{\mathcal{F}} \mu_f + \sum_{i=1}^{m} \mathbb{1}_{\mathcal{B}_i} \mu_i, \ \nu = \mathbb{1}_{\mathcal{F}} \nu_f + \sum_{i=1}^{m} \mathbb{1}_{\mathcal{B}_i} \nu_i$

The triplet (ρ, u, p) satisfies in Ω:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) = \nabla \cdot (\mathbb{T}(u, p)) + \nabla \tilde{\kappa} - \rho g \end{cases}$$

Derivation of the fluid–bubble system

Energy equation

$$\partial_t e+\nabla\cdot\Big(\rho u\tfrac{|u|^2}{2}-(\Sigma+\kappa\mathbb{I}_3)\Big)+2\mu\Big|\mathbb{D}(u)-\tfrac{1}{3}(\nabla\cdot u)\mathbb{I}_3\Big|^2+\nu|\nabla\cdot u|^2+(\tilde{\kappa}\cdot\nabla)u=-\rho g$$

Internal energy

$$e=\tfrac{1}{2}\rho|u|^2+P(\rho,\{\mathbb{1}_{\mathcal{B}_i}\}),\qquad P(\rho,\{\mathbb{1}_{\mathcal{B}_i}\}_{i=1}^m)=\mathbb{1}_{\mathcal{F}}a_f\frac{\rho_f^{\gamma_f}}{\gamma_f-1}+\sum_{i=1}^m\mathbb{1}_{\mathcal{B}_i}a_i\frac{\rho_i^{\gamma_i}}{\gamma_i-1}$$

Taking formally $\mu_i=\infty$, we obtain

$$\mathbb{D}(u_i)=\tfrac{1}{3}(\nabla\cdot u_i)\mathbb{I}_3,.$$

The bubble velocity then satisfies

$$u_i(x)=V_i+\omega_i\times(x-x_i)+\frac{\Lambda_i}{3}(x-x_i)+a_i\left((x-x_i)\otimes(x-x_i)-\frac{|x-x_i|^2}{2}\right)$$

Derivation of the fluid–bubble system

Energy equation

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Internal energy

$$e=\tfrac{1}{2}\rho|u|^2+P(\rho,\{\mathbb{1}_{\mathcal{B}_i}\}),\qquad P(\rho,\{\mathbb{1}_{\mathcal{B}_i}\}_{i=1}^m)=\mathbb{1}_{\mathcal{F}}a_f\frac{\rho_f^{\gamma_f}}{\gamma_f-1}+\sum_{i=1}^m\mathbb{1}_{\mathcal{B}_i}a_i\frac{\rho_i^{\gamma_i}}{\gamma_i-1}$$

Taking formally $\mu_i=\infty$, we obtain

$$\mathbb{D}(u_i)=\tfrac{1}{3}(\nabla\cdot u_i)\mathbb{I}_3,.$$

Assuming further that $\mathcal{B}_i=\mathcal{B}(x_i,R_i)$, the bubble velocity then satisfies

$$u_i(x)=V_i+\omega_i\times(x-x_i)+\frac{\Lambda_i}{3}(x-x_i).$$

Derivation of the fluid–bubble system

We construct w smooth vanishing on $\partial\Omega$ such that

$$w(x) = \tilde{V}_i + \tilde{\omega}_i \times (x - x_i) + \frac{\tilde{\Lambda}_i}{3}(x - x_i) \text{ in } \mathcal{B}_i.$$

Multiplying

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = \nabla \cdot (\mathbb{T}(u, p)) + \nabla \tilde{\kappa} - \rho g$$

by w and integrating by parts, we derive the following *ODEs* for $(V_i, \omega_i, \Lambda_i)$:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(m_i V_i) = -\int_{\partial\mathcal{B}_i} \mathbb{T}(u_f, p_f) n \mathrm{d}s - \int_{\mathcal{B}_i} \rho_i g \mathrm{d}y, \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{J}_i \omega_i) = -\int_{\partial\mathcal{B}_i} (s - x_i) \times \mathbb{T}(u_f, p_f) n \mathrm{d}s \\
&\frac{\mathrm{d}}{\mathrm{d}t}(K_i \Lambda_i) - \frac{1}{3}\left(\mathbb{J}_i \omega_i \cdot \omega_i + K_i |\Lambda_i|^2\right) = -\frac{1}{3}\int_{\partial\mathcal{B}_i} (s - x_i) \cdot \mathbb{T}_f(u_f, p_f) n \mathrm{d}s \\
&+ \left(\nu_i \Lambda_i - a_i \rho_i^{\gamma_i} + \frac{\kappa_i}{R_i}\right) |\mathcal{B}_i|
\end{aligned}$$

The fluid-bubble system

Fluid equations

$$\partial_t \rho_f + \nabla \cdot (\rho_f u_f) = 0$$
$$\partial_t(\rho_f u_f) + \nabla \cdot (\rho_f u_f \otimes u_f) = \nabla \cdot \mathbb{T}(u_f, p_f) - \rho_f g$$
$$\mathbb{T}(u_f, p_f) = 2\mu_f \left(\mathbb{D}(u_j) - \tfrac{1}{3}(\nabla \cdot u_j)\mathbb{I}_3\right) + (\nu_f \nabla \cdot u_f - a_f \rho_f^{\gamma_f})\mathbb{I}_3$$
$$p_f = p_f(\rho_f) = a_f \rho_f^{\gamma_f}$$

Bubbles equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_i V_i) = -\int_{\partial \mathcal{B}_i} \mathbb{T}(u_f, p_f) n \mathrm{d}s - \int_{\mathcal{B}_i} \rho_i g \mathrm{d}y,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{J}_i \omega_i) = -\int_{\partial \mathcal{B}_i} (s - x_i) \times \mathbb{T}(u_f, p_f) n \mathrm{d}s$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(K_i \Lambda_i) - \frac{1}{3}\left(\mathbb{J}_i \omega_i \cdot \omega_i + K_i |\Lambda_i|^2\right) = -\frac{1}{3}\int_{\partial \mathcal{B}_i} (s - x_i) \cdot \mathbb{T}_f(u_f, p_f) n \mathrm{d}s$$
$$+ \left(\nu_i \Lambda_i - a_i \rho_i^{\gamma_i} + \frac{\kappa_i}{R_i}\right) |\mathcal{B}_i|$$

Interface conditions

$$u_f = V_i + \omega_i \times (x - x_i) + \frac{\Lambda_i}{3}(x - x_i) \text{ on } \partial \mathcal{B}_i$$

Weak formulation of the fluid-bubble problem

Continuity equation

$$\int_0^T \int_\Omega \rho\, \partial_t \varphi + (\rho u) \cdot \nabla \varphi \, dx\, dt = 0 \quad \forall \varphi \in \mathcal{D}((0,T) \times \Omega).$$

Test functions

$$\mathcal{T}(Q_b) = \left\{ \varphi \in \mathcal{D}((0,T) \times \Omega) \ : \ \varphi = \varphi_i \text{ in } \mathcal{V}(\overline{Q_i}),\ \varphi_i \in \mathcal{D}(0,T;\mathcal{S}) \right\}.$$

$$\mathcal{S} = \left\{ \varphi_b \ : \ \exists\, V, \omega \in \mathbb{R}^3,\ \Lambda \in \mathbb{R}, \quad \varphi_b(x) = V + \omega \times x + \frac{\Lambda}{3} x \right\}.$$

Momentum equation

For all $\varphi \in \mathcal{T}(Q_b)$,

$$\int_0^T \int_\Omega (\rho u) \cdot \partial_t \varphi + (\rho u \otimes u) : \nabla \varphi + p\, \nabla \cdot \varphi$$
$$= \int_0^T \int_\Omega 2\mu D(u) : D(\varphi) + \nu\, \nabla \cdot u\, \nabla \cdot \varphi + \tilde{\kappa}_b\, \nabla \cdot \varphi - \rho g \cdot \varphi$$

Definition of the fluid-bubble problem

Initial conditions

$$\rho_0 \in L^{\gamma_f}(\Omega), \quad \rho_0 \geq 0 \text{ a.e. in } \Omega, \quad \rho_0 = \rho_{i,0} \text{ a.e. in } \mathcal{B}_{i,0},$$

$$q_0 \in L^{\frac{2\gamma_f}{\gamma_f+1}}(\Omega), \quad q_0 \mathbb{1}_{\rho_0=0} = 0 \text{ a.e. in } \Omega, \quad \frac{|q_0|^2}{\rho_0}\mathbb{1}_{\rho_0>0} \in L^1(\Omega),$$

$$u_0 = V_{i,0} + \omega_{i,0} \times (x - x_{i,0}) + \frac{\Lambda_{i,0}}{3}(x - x_{i,0}) \text{ a.e. in } \mathcal{B}_{i,0},$$

$$q_0 = \rho_0 u_0 \text{ a.e. in } \mathcal{B}_{i,0}.$$

Compatibility conditions

$$u \in L^2(0, T; H^1(\Omega)), \ \rho \in L^\infty(0, T; L^{\gamma_f}(\Omega))$$

$$\rho(t, x) = (R_{i,0}/R_i(t))^3 \rho_{i,0} \text{ for a.e. } (t, x) \in Q_i$$

$$\begin{cases} \mathcal{B}_i(t) = \eta_i[t](\mathcal{B}_{i,0}) \text{ for all } t \in [0, T] \\ u(t, x) = (\partial_t \eta_i[t]) \left((\eta_i[t])^{-1}(x) \right) \text{ for a.e. } (t, x) \in Q_i \\ \eta_i[t](y) = x_i(t) + \dfrac{R_i(t)}{R_{i,0}} \mathbb{O}_i(y - x_{i,0}) \end{cases}$$

Approximate solutions

Artificial pressure

$$p_\delta(\rho,\chi) = (1-\chi)a_f\rho^{\gamma_f} + \chi a_b\rho^{\gamma_b} + \delta\rho^\beta, \qquad \beta \geq \max\{8, 2\gamma_f, 2\gamma_b\}.$$

Regularized continuity equation

$$\partial_t\rho + \nabla\cdot(\rho u) = \varepsilon\Delta\rho.$$

Penalized momentum equation

$$\varphi \in \mathcal{D}((0,T)\times\Omega), \quad n\int_0^T\int_\Omega \chi(u-\Pi u)(\varphi-\Pi\varphi)\,\mathrm{d}x\,\mathrm{d}t, \quad \partial_t\chi + \Pi u\cdot\nabla\chi = 0.$$

Faedo–Galerkin approximation

$$u_N(t,x) = \sum_{i=1}^N \alpha_i(t)\psi_i(x), \quad \alpha_i \in C([0,T]), \quad \{\psi_i\}_{i\geq 1} \subset L^2(\Omega) \text{ orthonormal}.$$

Limit passage

$$N\to\infty, \qquad n\to\infty, \qquad \varepsilon\to 0, \qquad \delta\to 0.$$

Existence of the color function χ

Transport equation satisfied by χ

$$\partial_t \chi + \Pi u \cdot \nabla \chi \text{ in } \mathcal{D}'([0,T) \times \mathbb{R}^3), \; \chi|_{t=0} = \mathbb{1}_{\mathcal{B}_0} \text{ in } \Omega, \; \mathcal{B}_0 = \mathcal{B}(x_0, R_0), \; R_0 > 0$$

Definition of the projection operator

$$\Pi u(t,z) = \frac{3}{4\pi R(t)^3} \int_\Omega \chi u \mathrm{d}y + \frac{5}{4\pi R(t)^5} \left(\int_\Omega \chi (y - x(t)) \cdot u \mathrm{d}y \right) (z - x(t)),$$

$$R = \left(\frac{3}{4\pi} \int_{\mathbb{R}^3} \chi \mathrm{d}y \right)^{\frac{1}{3}}, \quad x = \frac{3}{4\pi R^3} \left(\int_{\mathbb{R}^3} \chi y \mathrm{d}y \right).$$

Proposition

Let $u \in \mathcal{C}([0,T], \mathcal{D}(\overline{\Omega}))$, if T satisfies

$$T \|u\|_{L^\infty(0,T;L^2(\Omega))} \leq g(R_0)$$

then there exists a unique solution and

$$\chi = \mathbb{1}_{\mathcal{B}}, \; \mathcal{B}(t) = \mathcal{B}(x(t), R(t)), \; R(t) \geq R_0/2 \quad \forall t \in [0,T].$$

Limit for $n \to \infty$

Energy inequality

$$\tilde{E}_\delta(\rho_n, \rho_n u_n) + \int_0^T \int_\Omega \left(2\mu_n \left| \mathbb{D}(u_n) - \frac{1}{3}(\nabla \cdot u_n)\mathbb{I}_3 \right|^2 + \nu_n |\nabla \cdot u_n|^2 \right) \mathrm{d}y\mathrm{d}\tau$$
$$+ \delta\varepsilon \int_0^T \int_\Omega (\rho_n)^{\beta-2} |\nabla \rho_n|^2 \mathrm{d}y\mathrm{d}\tau + n \int_0^T \int_\Omega \chi_n |u_n - \Pi_n u_n|^2 \mathrm{d}y\mathrm{d}\tau$$
$$\leq a_f \int_0^T \int_\Omega (1-\chi_n)\rho_n^{\gamma_f} \nabla \cdot u_n \mathrm{d}y\mathrm{d}\tau + a_b \int_0^t \int_\Omega \chi_n \rho_n^{\gamma_b} \nabla \cdot u_n \mathrm{d}y\mathrm{d}\tau -$$
$$\int_0^T \int_\Omega \rho_n g \cdot u_n \mathrm{d}y\mathrm{d}\tau + \int_0^T \int_\Omega \chi_n \frac{\kappa_b}{R_n} \nabla \cdot u_n \mathrm{d}y\mathrm{d}\tau + \tilde{E}_\delta(\rho_0, q_0).$$

Convergences

Step 1: The energy estimates yield

- $u_n \to u$ weakly in $L^2(0, T; H^1(\Omega))$,
- $\chi_n (u_n - \Pi_n u_n) \to 0$ strongly in $L^2((0, T) \times \Omega)$.

Step 2: Using the DiPerna–Lions theory for transport equations, we obtain

$$\chi_n \to \mathbb{1}_{\mathcal{B}} \text{ weakly-* in } L^\infty((0, T) \times \mathbb{R}^3), \text{ strongly in } \mathcal{C}([0, T]; L^p(\mathbb{R}^3)).$$

Step 3: Combining these results, we deduce

$$\mathbb{1}_{\mathcal{B}}(u - \Pi u) = 0 \text{ and } u = \Pi u \text{ for a.e. } (t, x) \in (0, T) \times \mathcal{B}(t).$$

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$$\leq a_f \int_0^T \int_\Omega (1-\chi_n)\rho_n^{\gamma_f} \nabla \cdot u_n \mathrm{d}y\mathrm{d}\tau + a_b \int_0^t \int_\Omega \chi_n \rho_n^{\gamma_b} \nabla \cdot u_n \mathrm{d}y\mathrm{d}\tau -$$
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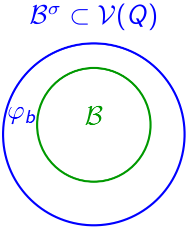
$$\mathbb{1}_{\mathcal{B}}(u - \Pi u) = 0 \text{ and } u = \Pi u \text{ for a.e. } (t,x) \in (0,T) \times \mathcal{B}(t).$$

Convergence of the non-linear terms in the momentum equation

Convective term: $\displaystyle\int_0^T \int_\Omega (\rho_n u_n \otimes u_n) : \mathbb{D}(\varphi)\mathrm{d}y\mathrm{d}\tau$ for $\varphi \in \mathcal{T}(Q)$

$$\mathcal{T}(Q) = \left\{ \begin{matrix} \varphi \in \mathcal{D}((0,T) \times \Omega) \ : \ \varphi = \varphi_b \text{ in } \mathcal{V}(\overline{Q}) \\ \varphi_b = V_b + \omega_b \times (x - x_b) + \dfrac{\Lambda_b}{3}(x - x_b) \end{matrix} \right\}$$

Let $\varphi \in \mathcal{T}(Q)$ and $\sigma \in \mathbb{R}_+^*$ such that



Convergence of the convective term

Decomposition of the convective term

$$\int_0^T \int_\Omega (\rho_n u_n \otimes u_n) : \mathbb{D}(\varphi) \mathrm{d}y\mathrm{d}\tau - \int_0^T \int_\Omega (\rho u \otimes u) \mathbb{D}(\varphi) \mathrm{d}y\mathrm{d}\tau = M_1 + M_2 + M_3 + M_4 + M_5$$

Algebraic decomposition

$$\mathbb{D}(\varphi) = (\mathbb{1}_\Omega - \mathbb{1}_{\mathcal{B}^\sigma}) \mathbb{D}(\varphi) + \mathbb{1}_{\mathcal{B}^\sigma} \mathbb{D}(\varphi_b),$$

$$\mathbb{1}_{\mathcal{B}^\sigma} = (\mathbb{1}_{\mathcal{B}^\sigma} - \mathbb{1}_{\mathcal{B}}) + (\mathbb{1}_{\mathcal{B}} - \chi_n) + \chi_n,$$

$$u_n = (u_n - \Pi_n u_n) + \Pi_n u_n.$$

Remainder terms

$$M_2(\sigma, n) = \int_0^T \int_\Omega (\mathbb{1}_{\mathcal{B}^\sigma} - \mathbb{1}_{\mathcal{B}}) \rho_n u_n \otimes u_n : \mathbb{D}(\varphi)\, dy\, d\tau - \int_0^T \int_\Omega (\mathbb{1}_{\mathcal{B}^\sigma} - \mathbb{1}_{\mathcal{B}}) \rho u \otimes u : \mathbb{D}(\varphi)\, dy\, d\tau,$$

$$M_3(\sigma, n) = \int_0^T \int_\Omega (\mathbb{1}_{\mathcal{B}} - \chi_n) \rho_n u_n \otimes u_n : \mathbb{D}(\varphi)\, dy\, d\tau,$$

$$M_4(\sigma, n) = \int_0^T \int_\Omega \chi_n \rho_n u_n \otimes (u_n - \Pi_n u_n) : \mathbb{D}(\varphi)\, dy\, d\tau.$$

Key convergences

$$\mathbb{1}_{\mathcal{B}^\sigma} - \mathbb{1}_{\mathcal{B}} \xrightarrow{\sigma \to 0} 0 \text{ in } \mathcal{C}([0, T]; L^p(\Omega)),$$

$$\mathbb{1}_{\mathcal{B}} - \chi_n \xrightarrow{n \to \infty} 0 \text{ in } \mathcal{C}([0, T]; L^p(\Omega)),$$

$$\chi_n (u_n - \Pi_n u_n) \xrightarrow{n \to \infty} 0 \text{ in } L^2((0, T) \times \Omega)$$

Convergence of the convective term

Convergence of the term M_5:

$$M_5 = \int_0^T \mathbb{D}(\varphi_b) \int_\Omega \chi_n \rho_n u_n \cdot \Pi_n u_n \mathrm{d}y \mathrm{d}\tau - \int_0^T \mathbb{D}(\varphi_b) \int_\Omega \mathbb{1}_{\mathcal{B}} \rho u \cdot \Pi u \mathrm{d}y \mathrm{d}\tau$$

We set

$$\Pi_n u_n = V_n + \omega_n \times (x - x_n) + \frac{\Lambda_n}{3}(x - x_n), \quad \Pi u = V + \omega \times (x - x_b) + \frac{\Lambda}{3}(x - x_b)$$

with

$$V_n \to V, \ \omega_n \to \omega, \ \Lambda_n \to \Lambda \text{ weakly in } L^2(0, T), \ x_n \to x_b \text{ in } \mathcal{C}([0, T]).$$

We introduce

$\tilde{V}_n = (\rho_n u_n, \chi_n)_\Omega, \ \tilde{\omega}_n = ((x - x_n) \times \rho_n u_n, \chi_n)_\Omega, \ \tilde{\Lambda}_n(t) = ((x - x_n) \cdot \rho_n u_n, \chi_n)_\Omega,$
$\tilde{V}_b = (\rho_b u_b, \mathbb{1}_{\mathcal{B}})_\Omega, \ \tilde{\omega} = ((x - x_b) \times \rho u, \mathbb{1}_{\mathcal{B}})_\Omega, \ \tilde{\Lambda}(t) = ((x - x_b) \cdot \rho u, \mathbb{1}_{\mathcal{B}})_\Omega.$

such that

$$M_5(\sigma, n) = \int_0^T \mathrm{div}\,(\varphi_b)(V_n \cdot \tilde{V}_n + \omega_n \cdot \tilde{\omega}_n + \Lambda_n \tilde{\Lambda}_n - V \cdot \tilde{V} - \omega \cdot \tilde{\omega} - \Lambda \tilde{\Lambda}) \mathrm{d}\tau.$$

Convergence of the convective term

We focus on

$$\tilde{V}_n = (\rho_n u_n, \chi_n)_\Omega.$$

The momentum equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (\rho_n u_n)\cdot\varphi\right) = \int_\Omega (\rho_n u_n \otimes u_n) : \mathbb{D}(\varphi) + p_\delta(\rho_n, \chi_n)\nabla\cdot\varphi \mathrm{d}y\mathrm{d}\tau$$
$$- \int_\Omega 2\mu_n \left(\mathbb{D}(u_n) - \frac{1}{3}(\nabla\cdot u_n)\mathbb{I}_3\right) : \left(\mathbb{D}(\varphi) - \frac{1}{3}(\nabla\cdot\varphi)\mathbb{I}_3\right) - \int_\Omega \nu_n \nabla\cdot u_n \nabla\cdot\varphi \mathrm{d}y$$
$$- \varepsilon \int_\Omega \nabla\rho_n \cdot \nabla u_n \cdot \varphi \mathrm{d}y - n\int_\Omega \chi_n(u_n - \Pi_n u_n)(\varphi - \Pi_n\varphi)\mathrm{d}y\mathrm{d}\tau - \rho_n g\cdot\varphi + \int_\Omega \frac{\kappa_b}{R_n}\nabla\cdot\varphi \mathrm{d}y$$

$\chi_n e_i$ is not regular enough.

We build a regularization of χ_n

- $\chi_n^\nu \in \mathcal{C}([0,T], \mathcal{D}(\Omega))$,
- $\chi_n^\nu \xrightarrow{\nu\to 0} \chi_n$ in $\mathcal{C}([0,T]; L^p(\Omega))$ uniformly in n
- $\chi_n(\Pi_n(\chi_n^\nu e_i) - \chi_n^\nu e_i) = 0$

$\tilde{V}_n^\nu$ is uniformly continuous on $[0,T]$ uniformly in n.

Arzela-Ascoli theorem implies

$$\tilde{V}^\nu_{\phi_\nu(n)} \to \tilde{V}^\nu \text{ in } \mathcal{C}([0,T]).$$

Convergence of the convective term

An argument of diagonal extraction then implies that, up to a subsequence,

$$\tilde{V}_n \to \tilde{V},\ \tilde{\omega}_n \to \tilde{\omega},\ \tilde{\Lambda}_n \to \tilde{\Lambda} \text{ in } L^2(0,T).$$

We conclude

$$\int_0^T \mathbb{D}(\varphi_b) \int_\Omega \chi_n \rho_n u_n \cdot \Pi_n u_n \mathrm{d}y \mathrm{d}\tau \xrightarrow{n\to\infty} \int_0^T \mathbb{D}(\varphi_b) \int_\Omega \mathbb{1}_{\mathcal{B}} \rho u \cdot \Pi u \mathrm{d}y \mathrm{d}\tau.$$

Let $\theta > 0$, there exists $(\sigma_0, n_0) \in (0,\infty) \times \mathbb{N}$ such that for all $0 < \sigma \leq \sigma_0$ and $n \geq n_0$,

$$|M_2(\sigma,n)| \leq \frac{\theta}{6},\ |M_3(\sigma,\theta)| \leq \frac{\theta}{6},\ |M_4(\sigma,\theta)| \leq \frac{\theta}{6}, |M_5(\sigma,n)| \leq \frac{\theta}{6}$$

Convergence of the convective term

Convergence of M_1:

Using standard arguments, we have

$$\rho_n u_n \otimes u_n \xrightarrow{n\to\infty} \rho u \otimes u \text{ in } L^1(Q_f^{\sigma_0})$$

where

$$Q_f^{\sigma_0} = (0,T)\setminus\overline{Q^{\sigma_0}}, \quad \overline{Q^{\sigma_0}} = \{(t,x) \in (0,T)\times\Omega \mid x \in \overline{\mathcal{B}^{\sigma_0}(t)}\}.$$

so that we can choose $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$\left|\int_0^T \int_\Omega \chi_n \rho_n u_n \otimes \Pi_n u_n : \mathbb{D}(\varphi)\mathrm{d}y\mathrm{d}\tau - \int_0^T \int_\Omega \mathbb{1}_{\mathcal{B}} \rho u \otimes \Pi u : \mathbb{D}(\varphi)\mathrm{d}y\mathrm{d}\tau\right| \leq \frac{\theta}{6}$$

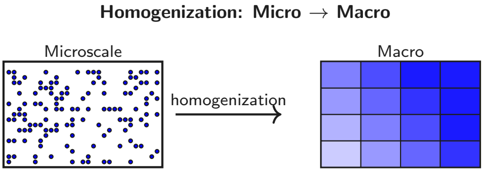
We conclude

$$\int_0^T \int_\Omega \rho_n u_n \otimes u_n : \mathbb{D}(\varphi)\mathrm{d}y\mathrm{d}\tau \xrightarrow{n\to\infty} \int_0^T \int_\Omega \rho u \otimes u : \mathbb{D}(\varphi)\mathrm{d}y\mathrm{d}\tau.$$

Current Research

Homogenization

- Homogenized (averaged) models
- Asymptotic limit as the number of bubbles becomes large



Numerical Methods

- Finite volume method
- One-dimensional staggered grid
- Tailored to fluid–bubble flows