A nonsmooth extension of the Brezzi-Rappaz-Raviart approximation theorem

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What is the BRR theorem about

- Very often, solving a nonlinear PDE may be rephrased as finding an element $\bar{x} \in X$ such that $F(\bar{x}) = 0$ where $F \colon X \to Y$ and X, Y are function spaces.
- Similarly, solving a (conformal) numerical scheme for the PDE amounts to finding $\bar{x}_h \in X$ such that $F_h(\bar{x}_h) = 0$, where $F_h \colon X \to Y$ encodes the scheme and $h > 0$ describes the precision of the discretization.
- The Brezzi-Rappaz-Raviart theorem (BRR) roughly states that if
  - there exists $\bar{x} \in X$ such that $F(\bar{x}) = 0$;
  - the numerical scheme is consistent;
  - the linearization of the PDE enjoys some stability properties;

  then there exist solutions $\bar{x}_h \in X$ to $F_h(\bar{x}_h) = 0$ for all $h > 0$ small enough and the error $\|\bar{x} - \bar{x}_h\|_X$ is proportional to the consistency error.

The Brezzi-Rappaz-Raviart approximation theorem

Theorem (Brezzi-Rappaz-Raviart 1980)

Let X *and* Y *be Banach spaces, let* $F\colon X \to Y$ *be continuous and let* $\bar{x} \in X$ *be such that* $F(\bar{x}) = 0$*. For every* $h > 0$*, let* $F_h\colon X \to Y$ *be continuous and assume that*

- $F_h(\bar{x}) \xrightarrow{h\to 0} 0$*;*
- F *and* F_h *are Fréchet differentiable at* $\bar{x}$*, with*

  $$\lim_{h\to 0} \|dF[\bar{x}] - dF_h[\bar{x}]\|_{\mathcal{L}(X,Y)} = 0,$$

  and there exists $c\colon \mathbb{R}_+ \to \mathbb{R}_+$*, nondecreasing with* $c(0^+) = 0$*, such that, for all* $x,\ x' \in B_X(\bar{x}, r)$ *and* $r > 0$*, we have*

  $$\begin{aligned} \left\|F(x) - F(x') - dF[\bar{x}](x - x')\right\|_Y &\le c(r)\left\|x - x'\right\|_X,\\ \left\|F_h(x) - F_h(x') - dF_h[\bar{x}](x - x')\right\|_Y &\le c(r)\left\|x - x'\right\|_X; \end{aligned}$$

- $dF[\bar{x}]$ *is an isomorphism.*

Then there exists a neighborhood $\mathcal{O}$ *of* $\bar{x}$ *such that, for* h *small enough, there exists a unique* $\bar{x}_h \in \mathcal{O}$ *such that* $F_h(\bar{x}_h) = 0$ *and we have the error estimate*

$$\|\bar{x} - \bar{x}_h\|_X \le 2\left\|dF[\bar{x}]^{-1}\right\|_{\mathcal{L}(Y,X)} \|F_h(\bar{x})\|_Y$$

Sketch of proof.

- Since $dF[\bar{x}]$ is an isomorphism and $dF_h[\bar{x}] \xrightarrow{h\to 0} dF[\bar{x}]$, $dF_h[\bar{x}]$ is also an isomorphism for h small enough.
- From the inverse function theorem there exist neighborhoods U and V of $\bar{x}$ and $F_h(\bar{x})$ such that F_h is bijective from U to V and $\mathrm{Lip}_V\left(F_h^{-1}\right) \leq 2\left\|dF[\bar{x}]^{-1}\right\|_{\mathcal{L}(Y,X)}$.
- Using consistency we have $0 \in V$ for h small enough and then

$$\|\bar{x} - \bar{x}_h\|_X = \left\|F_h^{-1}(F_h(\bar{x})) - F_h^{-1}(0)\right\|_X \leq 2\left\|dF[\bar{x}]^{-1}\right\|_{\mathcal{L}(Y,X)} \|F_h(\bar{x})\|_Y$$

□

- Main ingredients:
  - local invertibility of F_h on a neighborhood of $F_h(\bar{x})$, uniformly in h;
  - uniform bound on $\mathrm{Lip}\left(F_h^{-1}\right)$.

Example: Viscous Hamilton-Jacobi equations

- Consider the equation

$$\begin{cases} -\Delta u(x) + H(Du(x)) + \lambda u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{HJ}$$

  where Ω is a bounded convex domain, $H\colon \mathbb{R}^d \to \mathbb{R}$ is C^1 and Lipschitz continuous, $\lambda > 0$ and $f \in L^2(\Omega)$.

- Solving (HJ) (in the weak sense) is equivalent to finding $\bar{u} \in H_0^1(\Omega)$ such that $F(\bar{u}) = 0$, where $F = I + T \circ G$ with

$$G\colon H_0^1(\Omega) \ni u \mapsto H(Du) - f \in L^2(\Omega)$$

  and $T\colon L^2(\Omega) \ni g \mapsto T(g) \in H_0^1(\Omega)$ is the solution operator associated to

$$\begin{cases} -\Delta v(x) + \lambda v(x) = g(x) & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

- The (strict) differential at $\bar{u}$ is given by $dF[\bar{u}] = I + T \circ dG[\bar{u}]$, where
$$dG[\bar{u}](v) = H_p(D\bar{u}) \cdot Dv.$$
- The invertibility of $dF[\bar{u}]$ is equivalent to the well-posedness of the linearized problem
$$\begin{cases} -\Delta v(x) + H_p(D\bar{u}(x)) \cdot Dv(x) + \lambda v(x) = g(x) & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega, \end{cases}$$
for all $g \in L^2(\Omega)$.
- We look for $\mathbb{P}^1$-Lagrange finite element approximations to (HJ), i.e., solutions $\bar{u}_h \in H^1_0(\Omega)$ to $F_h(u_h) = 0$ where $F_h = I + T_h \circ G$ and T_h is the $\mathbb{P}^1$-Lagrange finite element approximation of T.
- Consistency:
$$\|F_h(\bar{u})\|_{H^1} = \|F_h(\bar{u}) - F(\bar{u})\|_{H^1} \le \|T - T_h\|_{\mathcal{L}(L^2,H^1)} \|G(\bar{u})\|_{L^2} = O(h).$$
- Therefore, the BRR theorem applies and there exist solutions $\bar{u}_h \in X$ to $F_h(\bar{u}_h) = 0$ and
$$\|\bar{u}_h - \bar{u}\|_{H^1} = O(h).$$

Metric regularity

- If H is a general Lipschitz continuous Hamiltonian, we do not expect the differentiability of the mapping F and we cannot apply the standard BRR theorem.
- We therefore need to generalize the BRR theorem. We rely on the notion of metrically regular mappings (developed in the context of variational analysis).

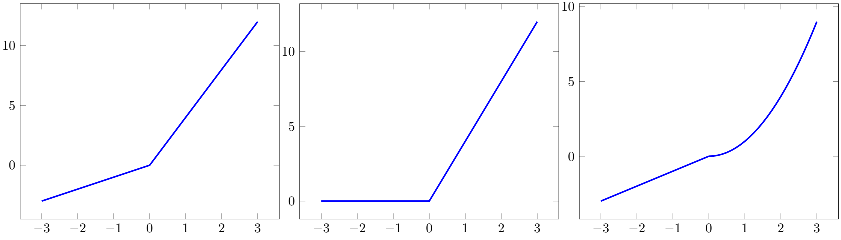
Definition

A continuous mapping $F\colon X \to Y$ is strongly metrically regular at $\bar{x} \in X$ if there exist neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $\bar{x}$ and $F(\bar{x})$, respectively, and $K > 0$ such that F is bijective from $\mathcal{U}$ to $\mathcal{V}$ and

$$\left\|F^{-1}(y) - F^{-1}(y')\right\|_X \leq K \left\|y - y'\right\|_Y \quad \text{for all } y,\, y' \in \mathcal{V}. \tag{$\star$}$$

- In order to generalize the BRR theorem, we need to ensure that F_h is uniformly strongly metrically regular for h small enough.

Examples



(a) $\max\{x, 4x\}$ (b) $\max\{0, 4x\}$ (c) $x\mathbb{1}_{\mathbb{R}_-}(x) + x^2\mathbb{1}_{\mathbb{R}_+}(x)$

- The function $f : x \mapsto \max\{x, 4x\}$ is strongly metrically regular near 0 with $\mathrm{Lip}(f^{-1}) = \frac{1}{4}$.
- The function $f : x \mapsto \max\{0, 4x\}$ is not strongly metrically regular near 0 because it is not locally surjective near 0.
- The function $f : x \mapsto x\mathbb{1}_{\mathbb{R}_-}(x) + x^2\mathbb{1}_{\mathbb{R}_+}(x)$ is not strongly metrically regular near 0 because $f^{-1}(y) = y\mathbb{1}_{\mathbb{R}_-}(y) + \sqrt{|y|}\mathbb{1}_{\mathbb{R}_+}(y)$ is not Lipschitz continuous at 0.

Milyutin's perturbation theorem

Theorem (Milyutin)

Let

- $F\colon X \to Y$ *be continuous and strongly metrically regular near* $\bar{x}$*, i.e.,* $R > 0$ *such that* F *is bijective from* $B_X(\bar{x}, R)$ *to* $B_Y(F(\bar{x}), R)$ *and* $K := \operatorname{Lip}_{B(F(\bar{x}),R)} < +\infty$*;*
- $g\colon X \to Y$ *be Lipschitz continuous on* $B(\bar{x}, R)$ *with* $\mu := \operatorname{Lip}_{B_X(\bar{x},R)}(g) < \min\{K^{-1}, R\}$*.*

Then $F + g$ *is bijective from* $B_X(\bar{x}, R - \mu)$ *to* $B_Y((F + g)(\bar{x}), R/2)$ *and*

$$\operatorname{Lip}_{B_Y((F+g)(\bar{x}),R/2)}(F + g) \leq \frac{K}{1 - K\mu}.$$

In other words, $F + g$ *is also strongly metrically regular near* $\bar{x}$

Writing $F_h = F + (F_h - F)$, if we have a small uniform bound on $\operatorname{Lip}(F_h - F)$, we obtain the uniform invertibility of F_h with a uniform bound on $\operatorname{Lip}\left(F_h^{-1}\right)$.

Generalized BRR theorem

Theorem (Generalized BRR, B.-Ley-Silva 2025)

Let $F,\ F_h \colon X \to Y$ *be continuous, and* $\bar{x} \in X$*. Assume that*

- $\|F_h(\bar{x})\|_Y \xrightarrow{h\to 0} 0$*;*
- F *is strongly metrically regular at* $\bar{x} \in X$*;*
- *there exist* $R > 0$ *and* $L \colon \mathbb{R}_+^* \to \mathbb{R}_+$*, increasing and such that* $\lim_{h\to 0} L(h) = 0$*, such that the mapping* $\Phi_h := F_h - F$ *satisfies*

$$\|\Phi_h(x) - \Phi_h(y)\|_Y \leq L(h)\,\|x - y\|_X \quad \text{for all } x, y \in B_X(\bar{x}, R).$$

Then there exists $h_0 > 0$ *and a neighborhood* $\mathcal{O}$ *of* $\bar{x}$ *such that, for all* $0 < h \leq h_0$*, there exists a unique* $\bar{x}_h \in \mathcal{O}$ *such that* $F_h(x_h) = 0$ *and we have the estimate*

$$\|\bar{x} - \bar{x}_h\|_X \leq C\,\|F_h(\bar{x})\|_Y\,.$$

Clarke's inverse function theorem

- For a locally Lipschitzian function $F\colon \mathbb{R}^n \to \mathbb{R}^n$, Rademarcher's theorem states that there exists a negligible set $\mathcal{N} \subset \mathbb{R}^n$ such that F is differentiable on $D := \mathbb{R}^n \setminus \mathcal{N}$.
- Clarke's generalized Jacobian at $\bar{x} \in \mathbb{R}^n$ is then defined by

$$\partial^C F(\bar{x}) := \operatorname{co}\left\{ A \in \mathbb{R}^{n\times n} : A = \lim_{k\to\infty} JF(x_k),\, x_k \xrightarrow{k\to\infty} \bar{x},\, x_k \in D \right\}.$$

Theorem (Clarke's inverse function theorem, 1976)

Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitzian and $\bar{x} \in \mathbb{R}^n$. Assume that each $A \in \partial^C F(\bar{x})$ is invertible. Then F is strongly metrically regular near $\bar{x}$.

Infinite dimensional version

Theorem (Cibulka-Fabian-Ioffe, 2015)

Let X and Y be reflexive Banach spaces and let $\bar{x} \in X$. Let further $F\colon X \to Y$ be continuous and let $\mathcal{A}$ be a bounded convex subset of $\mathcal{L}(X, Y)$ such that

- *for every $\epsilon > 0$ there is $\delta > 0$ such that, for any $x, x' \in B_X(\bar{x}, \delta)$, there exists $A \in \mathcal{A}$ such that*

$$\|F(x') - F(x) - A(x' - x)\|_Y < \epsilon \|x' - x\|_X,$$

- *each $A \in \mathcal{A}$ is an isomorphism and there is $\kappa > 0$ such that $\left\|A^{-1}\right\|_{\mathcal{L}(Y,X)} \leq \kappa^{-1}$.*

Then F is strongly metrically regular near $\bar{x}$.

In the case of the Hamilton-Jacobi equation we choose

$$\mathcal{A}[\bar{u}] = \left\{ I + T \circ B : \begin{array}{l} \text{where } B(v) = b(x) \cdot Dv(x) \\ \text{for all } b\colon \Omega \to \mathbb{R}^d \text{ measurable selection in } \partial^C H(D\bar{u}(\cdot)) \end{array} \right\}.$$

Back to the Viscous Hamilton-Jacobi equation

- We come back to the equation

$$\begin{cases} -\Delta u(x) + H(Du(x)) + \lambda u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{HJ}$$

  and we only assume that H is merely Lipschitz continuous.

- The metric regularity condition amounts to verifying that the linear equation

$$\begin{cases} -\Delta v(x) + b(x) \cdot Dv(x) + \lambda v(x) = g(x) & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

  is well-posed for every measurable selection b in $\partial^C H(D\bar{u}(\cdot))$ and $g \in L^2(\Omega)$ (which is true!).

- Therefore, the generalized BRR theorem applies and there exist solutions $\bar{u}_h \in X$ to $F_h(\bar{u}_h) = 0$ and

$$\|\bar{u}_h - \bar{u}\|_{H^1} = O(h).$$

Other applications

- The nonsmooth BRR theorem can also be used in the context of second order mean field games such as

$$\begin{cases} -\Delta u + H(x, Du) + \lambda u = f(x, m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \operatorname{div}\left(m H_p(x, Du)\right) + \lambda m = \lambda m_0 & \text{in } \mathbb{T}^d, \end{cases} \tag{2}$$

  where H is an Hamiltonian of class $C^{1,1}$.

- We obtained the following quasi-optimal error estimate for Lagrange finite element approximations (u_h, m_h) to (2):

$$\|u - u_h\|_{H^1} + \|m - m_h\|_{H^1} \leq K \left(\inf_{(v_h, \rho_h) \in V_h \times V_h} \|u - v_h\|_{H^1} + \|m - \rho_h\|_{H^1} \right).$$

- It is also possible to consider time-dependent analogues to (2) and obtain error estimates for semi-discrete in space approximations.

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Thank you for your attention !