

# A structure-preserving Hybrid Finite Volume scheme for an anisotropic cross-diffusion system

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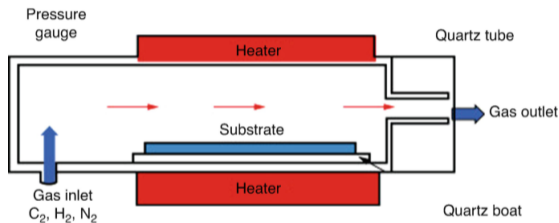


*CANUM 2026 – Modélisation et approximation numérique de systèmes multi-espèces*

- ① Introduction to the model
- ② Properties of the continuous model
- ③ Hybrid Finite Volume discretisation
- ④ Main results
- ⑤ Conclusion

# Motivation: Modelling anisotropic diffusion in solids

It describes the **diffusion of  $I + 1$  chemical species** in solids<sup>1</sup>, in a (nonempty) connected bounded open domain  $\Omega \subset \mathbb{R}^d$  over time:



**Example:** Diffusion in solar panel production.

The unknowns of the model are the species volume fractions  $u_0(x, t), \dots, u_I(x, t)$ .

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<sup>1</sup>Anisotropic version of cross-diffusion models studied in [Bakhta and Ehrlicher 2018](#); [Cancès and Gaudeul 2020](#)

# The continuous model for $I + 1$ species

We consider multi-species diffusion systems of the form:

$$\partial_t u_i - \nabla \cdot \left( \sum_{j \neq i} \Lambda_{ij}(x) (u_j \nabla u_i - u_i \nabla u_j) \right) = 0, \quad i \in \mathcal{I}_0.$$

with:

- $\Lambda_{ij} = \Lambda_{ji} \in L^\infty(\Omega; \mathbb{R}_s^{d \times d})$  and there exists  $0 < \lambda_b < \lambda_\# < \infty$  s.t.  $\forall \xi \in \mathbb{R}^d$

$$\lambda_b \|\xi\|^2 \leq \xi \cdot \Lambda_{ij}(x) \xi \leq \lambda_\# \|\xi\|^2.$$

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- no-flux boundary conditions and initial data, for all  $i \in \mathcal{I}_0$ ,

$$u_i(0, x) = u_i^{\text{in}}(x) \geq 0, \quad \sum_{i=0}^I u_i^{\text{in}}(x) = 1, \quad \text{a.e. in } \Omega, \quad \text{and} \quad \int_{\Omega} u_i^{\text{in}}(x) \, dx > 0.$$

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The volume-filling constraint is propagated:  $\sum_{i \in \mathcal{I}_0} u_i(t, x) = 1.$

# The continuous model for $I$ species

Therefore, the diffusion can be expressed in terms of the  $I$  volume fractions  $\mathbf{u} := (u_1, \dots, u_I)^T$  only:

$$\partial_t \mathbf{u} - \operatorname{div}(D(x, \mathbf{u}) \nabla \mathbf{u}) = 0,$$

where we set

$$u_0 = 1 - \sum_{i=1}^I u_i,$$

and

$$D_{ii}(x, \mathbf{u}) = \sum_{1 \leq j \neq i \leq I} (\Lambda_{ij}(x) - \Lambda_{i0}(x)) u_j + \Lambda_{i0}(x), \quad \text{and} \quad D_{ij}(x, \mathbf{u}) = -(\Lambda_{ij}(x) - \Lambda_{i0}(x)) u_i, \quad \text{for } j \neq i.$$

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In the following, we call

$$\mathcal{A} := \left\{ \mathbf{u} = (u_1, \dots, u_I) : u_i > 0 \text{ and } \sum_{i=1}^I u_i < 1 \right\}.$$

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# Properties of the continuous model

- ▷ **Mass Conservation:** For each species  $i \in \mathcal{I}_0$ ,

$$\int_{\Omega} u_i(t, x) dx = \int_{\Omega} u_i^{\text{in}}(x) dx > 0$$

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- ▷ **Formal Gradient Flow Structure:**

$$\partial_t \mathbf{u} - \operatorname{div}(M(x, \mathbf{u}) \nabla \mathbf{w}) = 0, \quad \text{with} \quad \begin{cases} M(x, \mathbf{u}) := D(x, \mathbf{u}) (D^2 h(\mathbf{u}))^{-1} \\ \mathbf{w} := h'(\mathbf{u}) = \left( \log \left( \frac{u_i}{u_0} \right) \right)_{1 \leq i \leq I} \end{cases}$$

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- ▷ **Entropy Decay:**

$$\frac{d}{dt} \mathcal{H}(\mathbf{u}(t)) = -\mathcal{R}(\mathbf{u}(t)) \leq 0, \quad \text{with} \quad \mathcal{R}(\mathbf{u}) := \int_{\Omega} \nabla \mathbf{w} : M(x, \mathbf{u}) \nabla \mathbf{w} \geq 0.$$

# Existence of global-in-time bounded weak solutions<sup>1</sup>

- 1) **Existence of the (discrete-in time and regularized) model:** In the entropy variables and reconstruct  $\mathbf{u}$  a posteriori via the inverse map:

$$\mathbf{w} = h'(\mathbf{u}) \quad \Rightarrow \quad \mathbf{u} = (h')^{-1}(\mathbf{w}) = \left( \frac{e^{w_i}}{1 + \sum_{j=1}^n e^{w_j}} \right)_{1 \leq i \leq I} \in \mathcal{A} \subseteq [0, 1]^I.$$

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- 2) **Compactness:** It follows from the fact that

$$\begin{aligned} \mathcal{R}(\mathbf{u}) &= \int_{\Omega} \nabla \mathbf{w} : M(x, \mathbf{u}) \nabla \mathbf{w} = \frac{1}{2} \sum_{i \neq j} \int_{\Omega} \Lambda_{ij} u_i u_j |\nabla \ln u_i - \nabla \ln u_j|^2 \\ &\geq \lambda_b \sum_{i=0}^I \int_{\Omega} \frac{|\nabla u_i|^2}{u_i} \geq \lambda_b \sum_{i=0}^I \int_{\Omega} |\nabla u_i|^2 \end{aligned}$$

which provides uniform  $L^2(0, T; H^1(\Omega)^I)$ -bounds, with  $T > 0$  some final time.

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- 1) Work on a **general mesh**

# Motivations and difficulties

- 1) Work on a **general mesh**
- 2) Preserve the **properties** of the continuous model
  - positivity and bounds of the volume fractions
  - decay of entropy,
  - preservation of the form of the steady states.
  - conservation of the total mass
  - conservation of the volumic constraint

# Motivations and difficulties

- 1) Work on a **general mesh**
- 2) Preserve the **properties** of the continuous model
  - positivity and bounds of the volume fractions
  - decay of entropy,
  - preservation of the form of the steady states.
  - conservation of the total mass
  - conservation of the volumic constraint
- 3) Have **unconditional convergence**

## Entropy-preserving FV

### Issues:

- Mostly isotropic diffusion
- TPFA-based methods
- Require orthogonal meshes

**Examples:** Cancès and Gaudeul 2020;  
Jüngel and Zurek 2020

## Finite differences / FEM

### Issues:

- Restricted mesh families
- Positivity/bounds preservation

**Examples:** Sun, Carrillo, and Shu 2019;  
Braukhoff, Perugia, and Stocker 2022;  
Aznaran et al. 2025; Gómez, A. Jüngel,  
and Perugia 2025

## Open problem

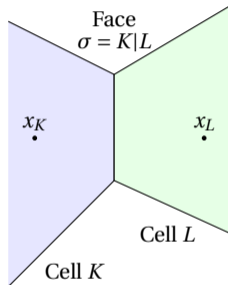
**Structure-preserving schemes for cross-diffusion on general meshes**  
Ehrlacher, Massimini, and Moatti 2026; Berrens and Eymard 2026

# Finite Volume & Cross-diffusion: how to approximate the normal fluxes?

1) We integrate each equation over  $K$ :

$$\int_K \partial_t u_i + \int_K \nabla \cdot F_i = 0, \quad i \in \mathcal{I} := \{1, \dots, I\},$$

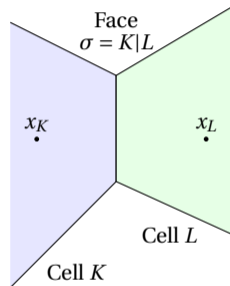
and solve the system in  $u_{iK} = \text{constant over } K$ .



2) We use the divergence theorem to rewrite the divergence term as a boundary integral:

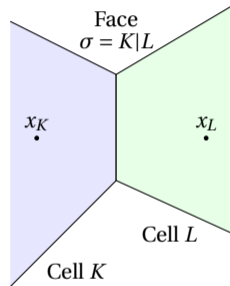
$$\int_K \partial_t u_i + \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} F_i \cdot \mathbf{n}_{K,\sigma} = 0, \quad i \in \mathcal{I} := \{1, \dots, I\},$$

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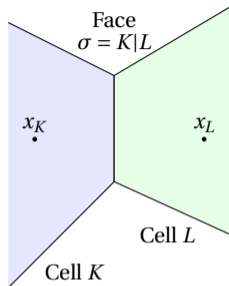
3) We use backward Euler (with  $t^n = n\Delta t$ ,  $n \geq 1$ ):

$$\frac{u_{iK}^n - u_{iK}^{n-1}}{\Delta t} |K| + \sum_{\sigma \in \mathcal{E}_K} F_{i,K\sigma}^n = 0, \quad i \in \mathcal{I} := \{1, \dots, I\}.$$



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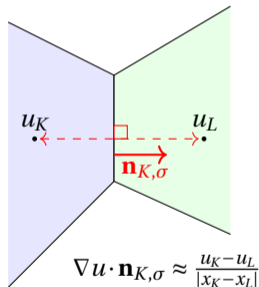
Difficulty: **Nonlinearity of the flux:** how to define

$$F_{i,K\sigma}^n \approx - \left( \sum_{j=1}^I D_{ij}(x, \mathbf{u}) \nabla u_j \cdot \mathbf{n} \right)_{\sigma} ?$$

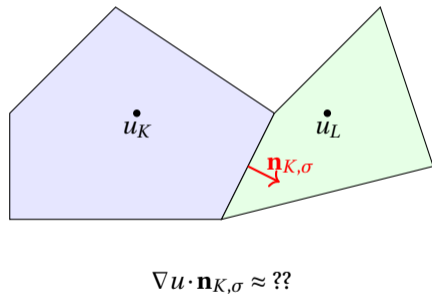
How to approximate  $u_i$  on  $\sigma$ ?

# Finite Volume & Anisotropy: how to approximate the normal gradient?

Orthogonal mesh



General mesh



# Why Hybrid Finite Volume?

On a **general mesh** (but for other simpler diffusion models):

- [Chainais-Hillairet et al. 2022](#) HFV for the heat equation
- [Moatti 2023](#) HFV for semiconductor models
- ...

They obtained **positivity/bounds** on the solutions!

# Mesh definition

General polygonal/polyhedral meshes

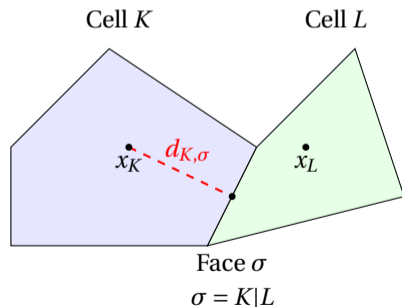
$\mathcal{M}$  : set of cells,  $\mathcal{E}$  : set of edges,  $\mathcal{P} = \{x_K\}_{K \in \mathcal{M}}$  : set of cell centres

such that for each  $K \in \mathcal{M}$ :

- $x_K \in K$  and  $K$  is star-shaped with respect to  $x_K$
- $d_{K,\sigma} := d(x_K, \mathcal{H}_\sigma) > 0$ , where  $\mathcal{H}_\sigma$  is the hyperplane containing  $\sigma$

$$\Rightarrow \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} |\sigma| = d|K|$$

where  $\mathcal{E}_K$  is the set of edges belonging to  $\partial K$ .



# Discrete unknowns

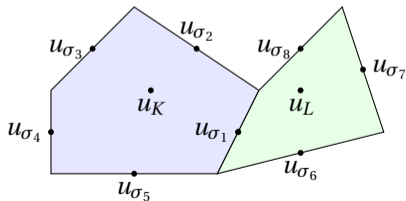
## Unknowns:

$$u_{i,K}, u_{i,\sigma}, \quad K \in \mathcal{M}, \sigma \in \mathcal{E}, i \in \mathcal{I}$$

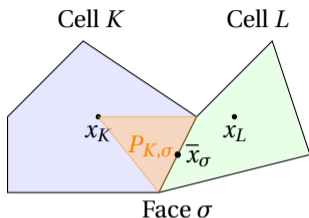
## Notation

$$\mathbf{u}_K = (u_{i,K})_{i \in \mathcal{I}}, \quad \mathbf{u}_\sigma = (u_{i,\sigma})_{i \in \mathcal{I}}$$

$$u_{i,\mathcal{D}} = ((u_{i,K})_{K \in \mathcal{M}}, (u_{i,\sigma})_{\sigma \in \mathcal{E}}), \quad \mathbf{u}_{\mathcal{D}} = (u_{i,\mathcal{D}})_{i \in \mathcal{I}}$$



# Discrete gradient



$P_{K,\sigma}$ : pyramid with base  $\sigma$  and apex  $x_K$

$\mathbf{n}_{K,\sigma}$ : outward unit normal to  $K$  at  $\sigma$

$v_{\mathcal{D}} = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}})$

On each cell  $K$ ,  $\nabla_K v_{\mathcal{D}}$  is piecewise constant and defined<sup>2</sup> on each  $P_{K,\sigma}$  by

$$\nabla_K v_{\mathcal{D}}|_{P_{K,\sigma}} := \nabla_{K,\sigma} v_{\mathcal{D}} = G_{K,\sigma} v_{\mathcal{D}}$$

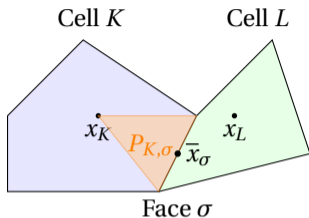
where:

$$G_{K,\sigma} v_{\mathcal{D}} := \frac{1}{|K|} \sum_{\sigma' \in \mathcal{E}_K} |\sigma'| (v_{\sigma'} - v_K) \mathbf{n}_{K,\sigma'} \quad (\text{consistent part})$$

---

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where:

$$G_{K,\sigma} v_{\mathcal{D}} := \frac{1}{|K|} \sum_{\sigma' \in \mathcal{E}_K} |\sigma'| (v_{\sigma'} - v_K) \mathbf{n}_{K,\sigma'} = \frac{1}{|K|} \sum_{\sigma' \in \mathcal{E}_K} |\sigma'| v_{\sigma'} \mathbf{n}_{K,\sigma'} \quad (\text{consistent part})$$

$$S_{K,\sigma} v_{\mathcal{D}} := \eta \frac{1}{d_{K,\sigma}} \left( v_{\sigma} - v_K - G_{K,\sigma} v_{\mathcal{D}} \cdot (\bar{x}_{\sigma} - x_K) \right) \mathbf{n}_{K,\sigma} \quad (\text{stabilisation})$$

<sup>2</sup>Eymard, Gallouët, and Herbin 2010

There exist constants  $c, C > 0$  such that, for all  $v_{\mathcal{D}}$ ,

$$c|v_{\mathcal{D}}|_{1,K}^2 \leq \|\nabla_K v_{\mathcal{D}}\|_{L^2(K)}^2 \leq C|v_{\mathcal{D}}|_{1,K}^2.$$

with

$$|v_{\mathcal{D}}|_{1,K}^2 = \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (v_K - v_{\sigma})^2,$$

and  $c, C$  depending only on  $\Omega$  and  $\theta_{\mathcal{D}}$ .

# Defining numerical fluxes via entropy variables

- ▷ In Hybrid Finite Volume, even if  $I = 1$  and  $D_{ij}(x, u) \equiv \Lambda(x)$  (**linear**), and we approximate

$$\int_{\sigma} F_i \cdot \mathbf{n}_{K,\sigma} = - \int_{\sigma} \Lambda(x) \nabla u \cdot \mathbf{n}_{K,\sigma} \quad \hookrightarrow \quad F_{K\sigma}(\underline{u}_{\mathcal{D}}) = \sum_{\sigma' \in \mathcal{E}_K} \mathbb{A}_K^{\Lambda, \sigma\sigma'} (u_K - u_{\sigma'}),$$

we do **not preserve** bounds on the solutions!!

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- ▷ In [Chainais-Hillairet et al. 2022](#), nonlinear reformulation in entropy variables

$$\nabla u = u \nabla \log u$$

is used to recover lost structural properties:

$$- \int_{\sigma} u \Lambda(x) \nabla \log u \cdot \mathbf{n}_{K,\sigma} \quad \hookrightarrow \quad r_K(\underline{u}_{\mathcal{D}}) F_{K\sigma}(\log(\underline{u}_{\mathcal{D}})),$$

where

$$r_K(\underline{u}_{\mathcal{D}}) := \frac{1}{|\mathcal{E}_K|} \sum_{\sigma \in \mathcal{E}_K} \frac{u_K + u_{\sigma}}{2}$$

is the cell reconstruction.

# Discretising the gradient flow formulation

- ▷ In our case, fluxes are already nonlinear. The key design choice is to **discretise the gradient flow formulation!**
- ▷ We hence discretise the gradient-flow formulation in the **entropy variables  $\mathbf{w}$**

$$\partial_t \mathbf{u}(\mathbf{w}) - \operatorname{div}(\mathbf{M}(\mathbf{x}, \mathbf{u}) \nabla \mathbf{w}) = 0$$

where

$$M_{ii}(\mathbf{x}, \mathbf{u}) = \Lambda_{i0}(\mathbf{x}) u_0 u_i + \sum_{j \neq i} \Lambda_{ij}(\mathbf{x}) u_i u_j, \quad \text{and} \quad M_{ij}(\mathbf{x}, \mathbf{u}) = -\Lambda_{ij}(\mathbf{x}) u_i u_j, \quad \text{for } j \neq i,$$

- ▷ The volume fractions are reconstructed via

$$\mathbf{u} = \left( \frac{e^{w_i}}{1 + \sum_{j=1}^I e^{w_j}} \right)_{1 \leq i \leq I}$$

# Discrete bilinear form

Given a cell  $K \in \mathcal{M}$ , we define an approximation of

$$(\mathbf{u}; \mathbf{w}, \cdot) \mapsto \int_K \nabla : \mathbf{M}(\cdot, \mathbf{u}) \nabla \mathbf{w} = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \int_K M_{ij}(\cdot, \mathbf{u}) \nabla w_j \cdot \nabla v_i,$$

under the form

$$T_K(\underline{\mathbf{u}}_K; \underline{\mathbf{w}}_K, \underline{\mathbf{v}}_K) := \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \int_K \widetilde{M}_{ij,K}(\cdot, \underline{\mathbf{u}}_K) \nabla_K \underline{w}_{jK} \cdot \nabla_K \underline{v}_{iK},$$

for all  $(\underline{\mathbf{u}}_K, \underline{\mathbf{w}}_K, \underline{\mathbf{v}}_K) \in (\underline{V}_K^{\mathcal{I}})^3$ , where

$$(1) \quad \begin{aligned} \widetilde{M}_{ii,K}(x, \underline{\mathbf{u}}_K) &:= \Lambda_{i0}(x) r_K(\underline{\mathbf{u}}_{0K}) r_K(\underline{\mathbf{u}}_{iK}) + \sum \Lambda_{ij}(x) r_K(\underline{\mathbf{u}}_{iK}) r_K(\underline{\mathbf{u}}_{jK}), \\ \widetilde{M}_{ij,K}(x, \underline{\mathbf{u}}_K) &:= -\Lambda_{ij}(x) r_K(\underline{\mathbf{u}}_{iK}) r_K(\underline{\mathbf{u}}_{jK}), \quad \text{for } j \neq i. \end{aligned}$$

with  $\underline{\mathbf{u}}_{0K} = \underline{\mathbf{1}}_K - \sum_{i \in \mathcal{I}} \underline{\mathbf{u}}_{iK}$

For any  $\underline{v}_K \in \underline{V}_K$ ,

$$r_K(\underline{v}_K) := \frac{1}{|\mathcal{E}_K|} \sum_{\sigma \in \mathcal{E}_K} m(v_K, v_\sigma),$$

where  $m: (\mathbb{R}_+^*)^2 \rightarrow \mathbb{R}_+^*$  is the *arithmetic mean*:

$$m(x, y) := \frac{x+y}{2}, \quad \text{for all } (x, y) \in (\mathbb{R}_+^*)^2.$$

## Definition of the scheme

Find  $\underline{w}_{\mathcal{D}}^n \in \underline{V}_{\mathcal{D}}^{\mathcal{I}}$  for  $1 \leq n \leq N$  such that:

$$\left\{ \begin{array}{l} \frac{1}{\Delta t} (\mathbf{u}_{\mathcal{M}}^n - \mathbf{u}_{\mathcal{M}}^{n-1}, \mathbf{v}_{\mathcal{M}})_{L^2(\Omega)^{\mathcal{I}}} + T_{\mathcal{D}}(\underline{u}_{\mathcal{D}}^n; \underline{w}_{\mathcal{D}}^n, \underline{v}_{\mathcal{D}}) = 0, \quad \forall \underline{v}_{\mathcal{D}} \in \underline{V}_{\mathcal{D}}^{\mathcal{I}}, \\ \underline{u}_{i\mathcal{D}}^n = \frac{e^{\underline{w}_{i\mathcal{D}}^n}}{1 + \sum_{j \in \mathcal{I}} e^{\underline{w}_{j\mathcal{D}}^n}}, \quad \forall i \in \mathcal{I}, \\ u_{iK}^0 = \frac{1}{|K|} \int_K u_i^{\text{in}}, \quad \forall K \in \mathcal{M}, i \in \mathcal{I}_0. \end{array} \right.$$

- ① Introduction to the model
- ② Properties of the continuous model
- ③ Hybrid Finite Volume discretisation
- ④ Main results**
- ⑤ Conclusion

# Properties of our HFV scheme

- ▷ **Discrete Mass Conservation:** For each species  $i$ ,

$$\int_{\Omega} u_{i\mathcal{M}}^n = \int_{\Omega} u_{i\mathcal{M}}^0 = \int_{\Omega} u_i^{\text{in}}$$

- ▷ **Discrete Entropy Functional:** :

$$\mathcal{H}(\underline{\mathbf{u}}_{\mathcal{D}}) := \mathcal{H}(\mathbf{u}_{\mathcal{M}}) = \int_{\Omega} h(\mathbf{u}_{\mathcal{M}}) = \sum_{i \in \mathcal{I}_0} \int_{\Omega} u_{i\mathcal{M}} \log(u_{i\mathcal{M}}),$$

- ▷ **Discrete Entropy Decay:**

$$\frac{\mathcal{H}(\mathbf{u}_{\mathcal{M}}^n) - \mathcal{H}(\mathbf{u}_{\mathcal{M}}^{n-1})}{\Delta t} + \mathcal{R}_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}^n) \leq 0, \quad \forall 1 \leq n \leq N,$$

with  $\mathcal{R}_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}(\underline{\mathbf{w}}_{\mathcal{D}})) := T_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}; \underline{\mathbf{w}}_{\mathcal{D}}, \underline{\mathbf{w}}_{\mathcal{D}})$ ,

## Theorem (Existence)

For any mesh  $\mathcal{D} = (\mathcal{M}, \mathcal{E})$  and time step  $\Delta t > 0$ , the scheme admits a solution such that:

- $0 < u_{iK}^n < 1, \quad 0 < u_{i\sigma}^n < 1,$
- $\sum_{i=0}^I u_{iK}^n = 1, \quad \sum_{i=0}^I u_{i\sigma}^n = 1,$

for all  $K \in \mathcal{M}$  and all  $\sigma \in \mathcal{E}$ .

---

Indeed:

$$\underline{u}_{i\mathcal{D}} = \frac{e^{w_{i\mathcal{D}}}}{1 + \sum_{j=1}^I e^{w_{j\mathcal{D}}}} \in (0, 1), \quad 1 \leq i \leq I,$$

and

$$\sum_{i=0}^I \underline{u}_{i\mathcal{D}} = 1 \quad \text{volumic constraint}$$

# Compactness via dissipation

Let  $\underline{w}_{\mathcal{D}} \in V_{\mathcal{D}}^{\mathcal{J}}$ , and  $\underline{u}_{\mathcal{D}} \in V_{\mathcal{D}}^{\mathcal{J}}$  the corresponding concentrations. The following inequality holds:

$$\begin{aligned}\mathcal{R}_{\mathcal{D}}(\underline{u}_{\mathcal{D}}) &\geq C_{\log} \sum_{i \in \mathcal{I}_0} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} m^{\log}(u_{iK}, u_{i\sigma}) (\log(u_{iK}/u_{i\sigma}))^2. \\ &\geq C_{\log} \sum_{i \in \mathcal{I}_0} |\underline{u}_{i\mathcal{D}}|_{1,\mathcal{D}}^2,\end{aligned}$$

- ▷  $m(x, y) \geq m^{\log}(x, y)$  is employed to reproduce a "discrete chain rule"

$$m^{\log}(u_{iK}, u_{i\sigma}) \log(u_{iK}/u_{i\sigma}) = u_{iK} - u_{i\sigma}$$

- ▷ the arithmetic mean is essential to deduce a volume-filling constraint

$$\sum_{i \in \mathcal{I}_0} m(u_{iK}, u_{i\sigma}) = 1$$

# Consistency of the "flux" term

1. The dissipation gives uniform bounds on the discrete gradient of the volume fractions

$$\infty > \mathcal{H}(\mathbf{u}^{\text{in}}) + C_{\mathcal{H}} \geq \sum_{n=1}^N \Delta t \mathcal{R}_{\mathcal{D}}(\underline{\mathbf{u}}_{\mathcal{D}}^n) \geq \sum_{i \in \mathcal{I}_0} \sum_{n=1}^N \Delta t \|\underline{u}_{i\mathcal{D}}^n\|_{1,\mathcal{D}}^2, \quad \forall N \in \mathbb{N}^*.$$

Hence the flux defined via an approximation of the tensor  $D$  converges to the continuous one

$$\sum_{i=1}^I \sum_{j=1}^I \int_K \tilde{D}_{ij}(\cdot, \underline{\mathbf{u}}_{\mathcal{D}}) \nabla_K \underline{u}_{jK} \cdot \nabla_K \underline{v}_{iK} \quad \rightarrow \quad \sum_{i=1}^I \sum_{j=1}^I \int_K D_{ij}(\cdot, u) \nabla u_j \cdot \nabla v_i.$$

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2. In general

$$\sum_{i=1}^I \sum_{j=1}^I \int_K \tilde{D}_{ij}(\cdot, \underline{\mathbf{u}}_{\mathcal{D}}) \nabla_K \underline{u}_{jK} \cdot \nabla_K \underline{v}_{iK} \quad \neq \quad \sum_{i=1}^I \sum_{j=1}^I \int_K \tilde{M}_{ij}(\cdot, \underline{\mathbf{u}}_{\mathcal{D}}) \nabla_K \underline{w}_{jK} \cdot \nabla_K \underline{v}_{iK}.$$

We need to show they are consistent to conclude!

## Theorem (Convergence)

Let  $(\mathcal{D}_\ell, \Delta t_\ell)$  be a refining sequence, i.e. such that  $h_{\mathcal{D}_\ell, \Delta t_\ell} \rightarrow 0$  and  $\theta_{\mathcal{D}_\ell} \leq \theta < \infty$ . Then, up to a subsequence,

$$\begin{aligned}u_{i,\ell} &\rightarrow u_i \quad \text{strongly in } L^2((0, T); L^2(\Omega)), \\r_\ell(u_{i,\ell}) &\rightarrow u_i \quad \text{strongly in } L^2((0, T); L^2(\Omega)), \\\nabla_\ell u_{i,\ell} &\rightharpoonup \nabla u_i \quad \text{weakly in } L^2((0, T); L^2(\Omega)),\end{aligned}$$

where  $u$  is a weak solution of the continuous system.

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- ▷ We constructed the first HFV scheme for anisotropic cross-diffusion systems on general meshes, using entropy variables.
- ▷ The scheme converges unconditionally and it is fully consistent with the structure of the problem:
  - preservation of non-negativity,
  - volume constraints,
  - mass conservation,
  - entropy dissipation.

- ▷ Numerical simulations (ongoing work).
- ▷ Extension of the method to other cross-diffusion systems: SKT (with [Ehrlacher](#), [Harnist](#), [Moatti](#), [Zurek](#)), Maxwell–Stefan, Quastel<sup>3</sup>, ...).

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## Thank you for your attention!

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<sup>3</sup>[Quastel 1992](#)