Long-time numerical simulation of the wave equation in heterogeneous media

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The wave equation in heterogeneous media models propagation in environments with material properties that vary at a fine scale. It arises for instance in geophysics and acoustics. The equation writes

$$\partial_{tt} u^\varepsilon(t,x) = \nabla_x \cdot (a^\varepsilon(x) \nabla u^\varepsilon(t,x)) \quad \text{in } (0,T] \times \Omega, \tag{1}$$

where a^ε is the tensor that represents the medium and Ω is an open bounded subset of $\mathbb{R}^d$. The scalar parameter $\varepsilon > 0$ is the characteristic length of the variation of a^ε in space. In the multiscale regime, we assume that $\varepsilon \ll |\Omega| = 1$. In the periodic case, classical homogenization theory [4] provides an effective description for final times T independent of ε.
From a numerical standpoint, accurately approximating solutions to such PDEs with standard methods such as finite element methods (FEM) requires a mesh size $H \ll \varepsilon$ to resolve the oscillations of the coefficients. This results in large linear systems whose resolution becomes computationally prohibitive as ε becomes asymptotically small. Thus the need for alternative methods based on the homogenization paradigm.
For large times, typically of order ε^{-2}, even the homogenization based approach fails. The simulation is challenging and requires the addition of dispersive corrective terms [1, 2].

This talk is devoted to the construction of a numerical approach (inspired from multiscale finite element methods [6, 5]) for the simulation of the wave equation in heterogeneous media *over long times*.

The approach is a combination of the construction of local basis functions for the spectral problem associated with the dynamics, and a Richardson extrapolation to better capture the fine scale behavior [3]. The results show that the method allows for an economical simulation over the long times and also improves on the accuracy of existing methods for the simulation over shorter times. The performance carries over to non-periodic media.

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