…

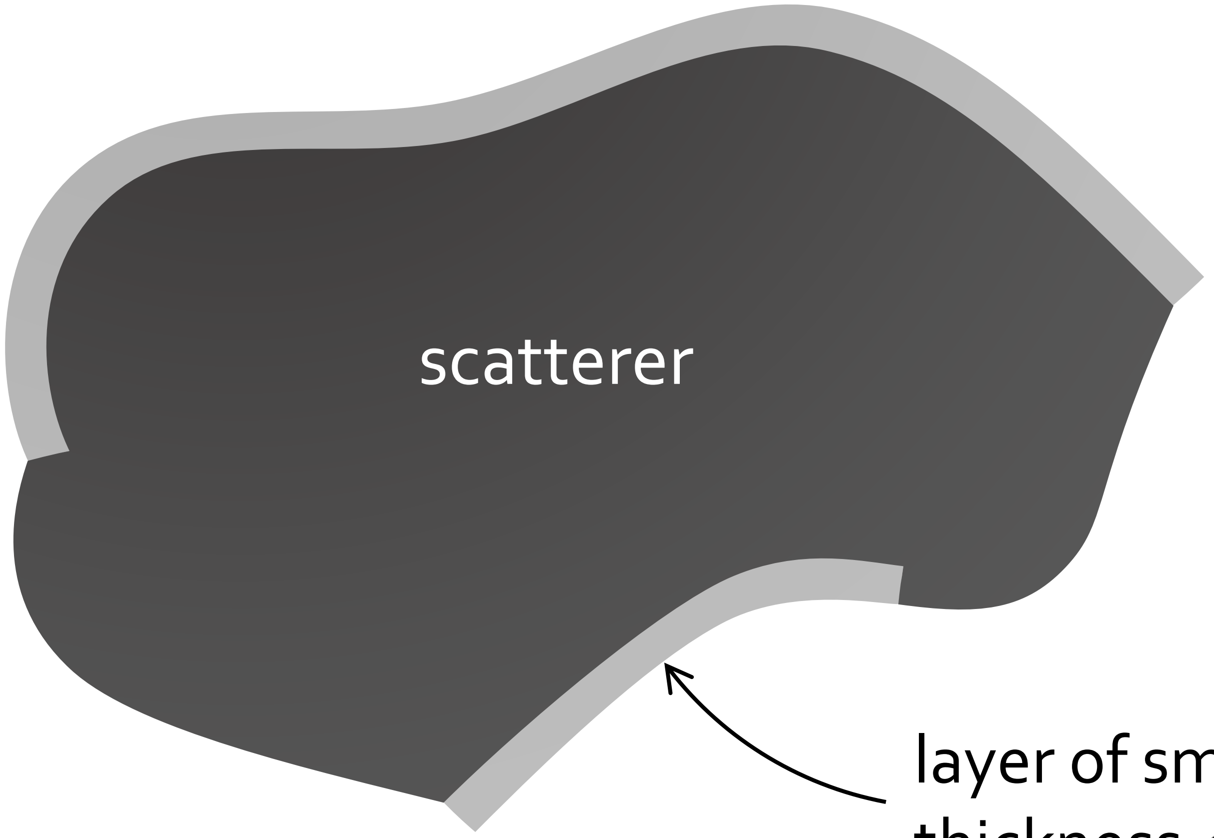
Modèles effectifs pour l'équation de Helmholtz dans un coin avec couche mince

Cédric Baudet, Sonia Fliss et Patrick Joly

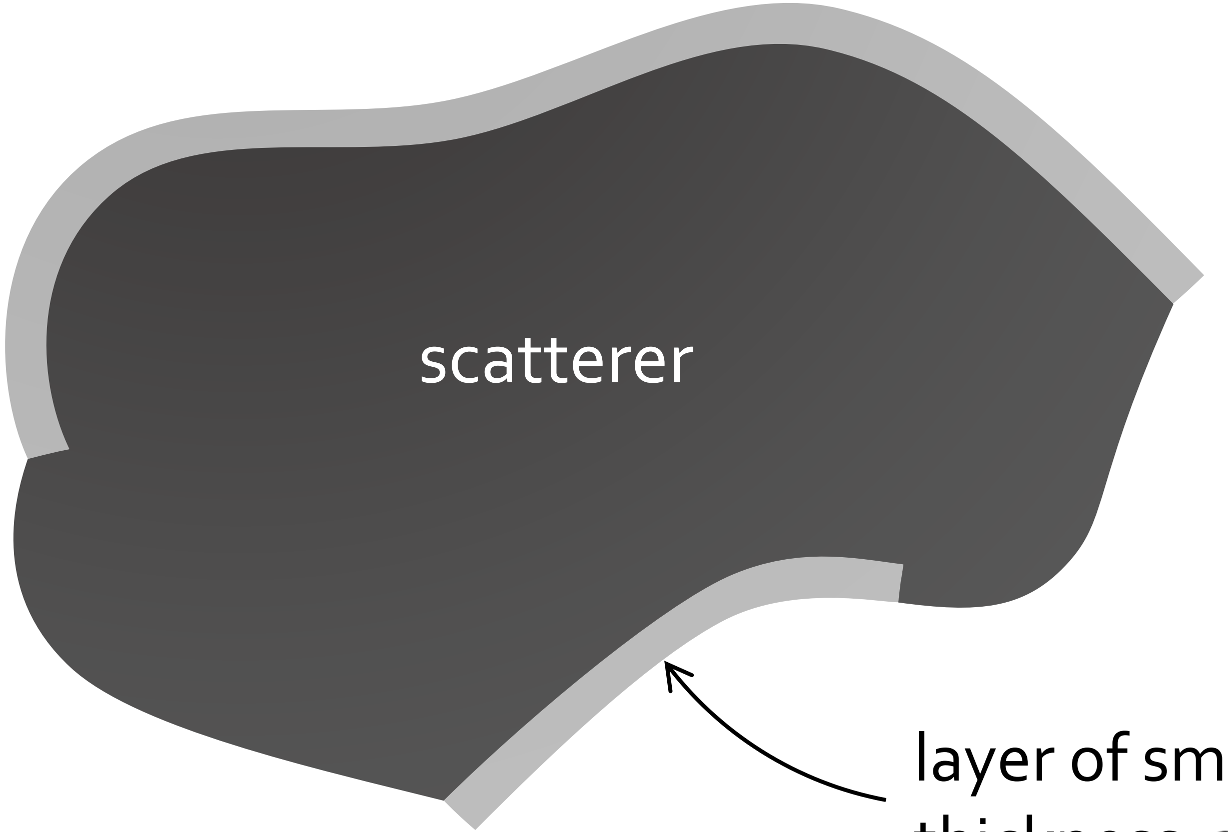
CANUM 2026

acoustics:
  honeycomb structures for noise reduction…

electromagnetism:
  thin dielectric layers, ferromagnetic films…



layer of small thickness $\varepsilon \ll \lambda$



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- honeycomb structures for noise reduction…

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⚠ excessive computational cost

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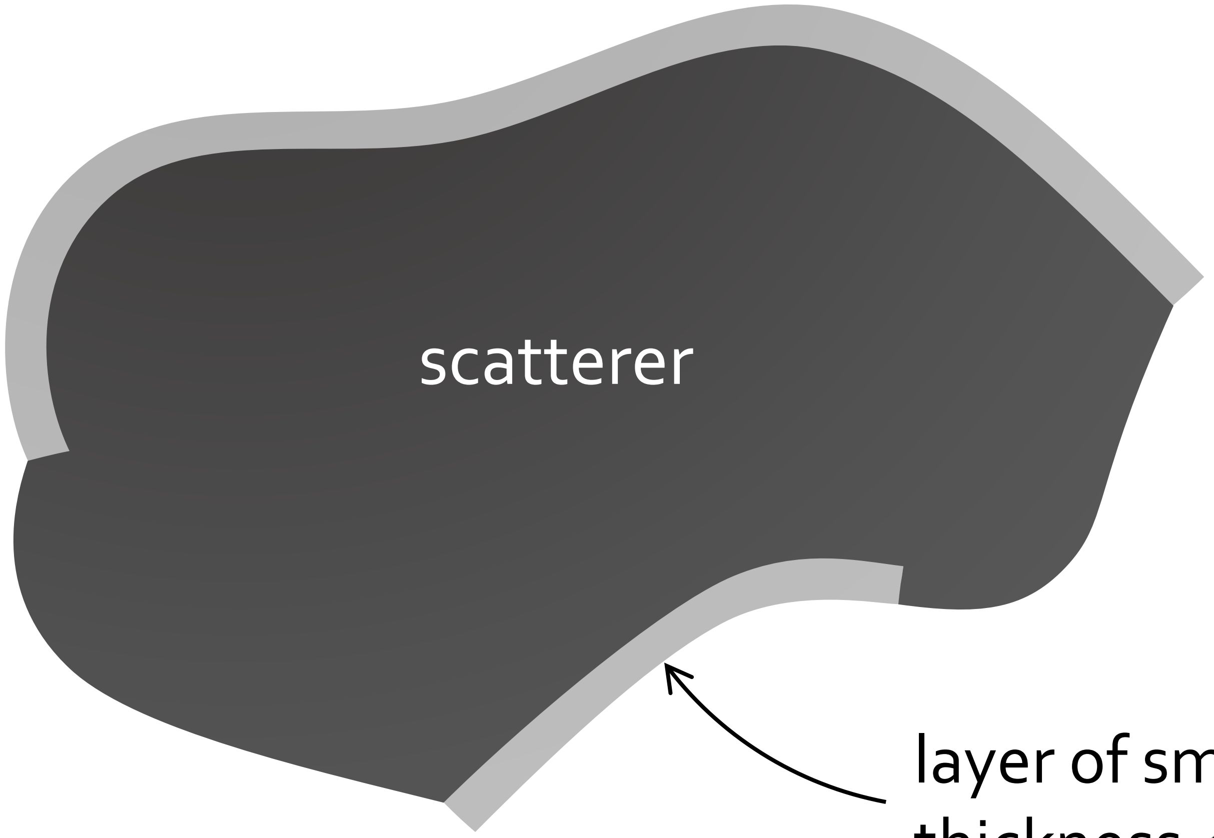

layer of small thickness $\varepsilon \ll \lambda$

excessive computational cost

→ asymptotic expansions & effective models

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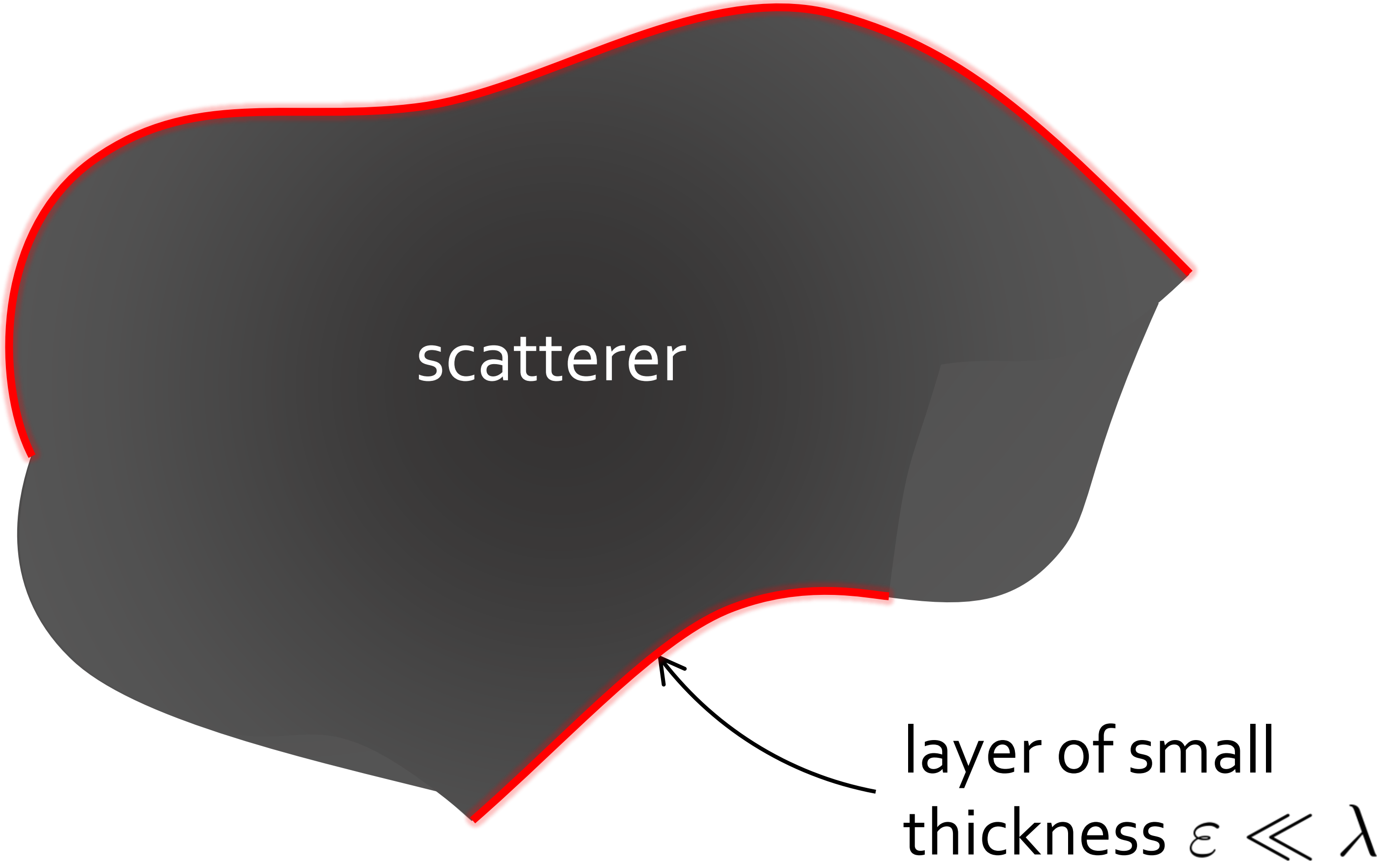

excessive computational cost

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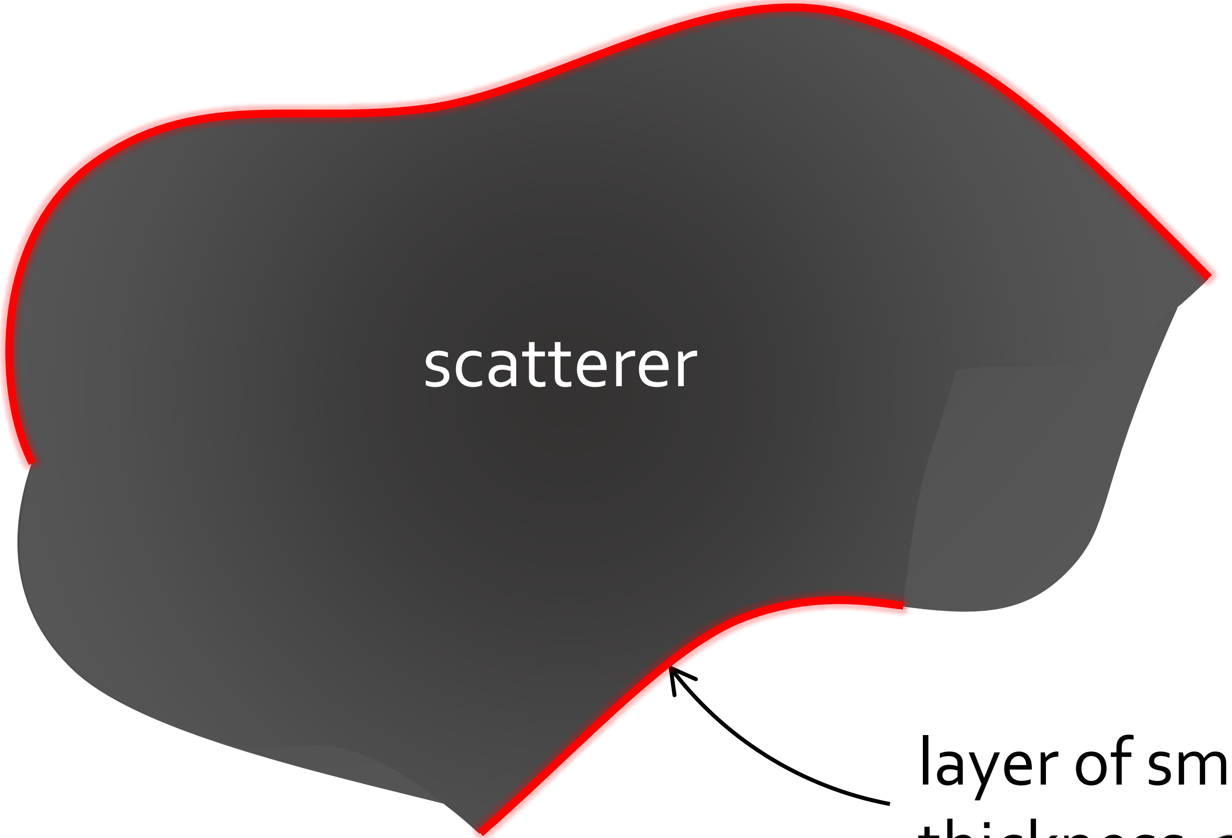

excessive computational cost

$\rightarrow \underbrace{\text{asymptotic expansions}}_{\text{Part 1}}\ \&\ \underbrace{\text{effective models}}_{\text{Part 2}}$

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  honeycomb structures for noise reduction…

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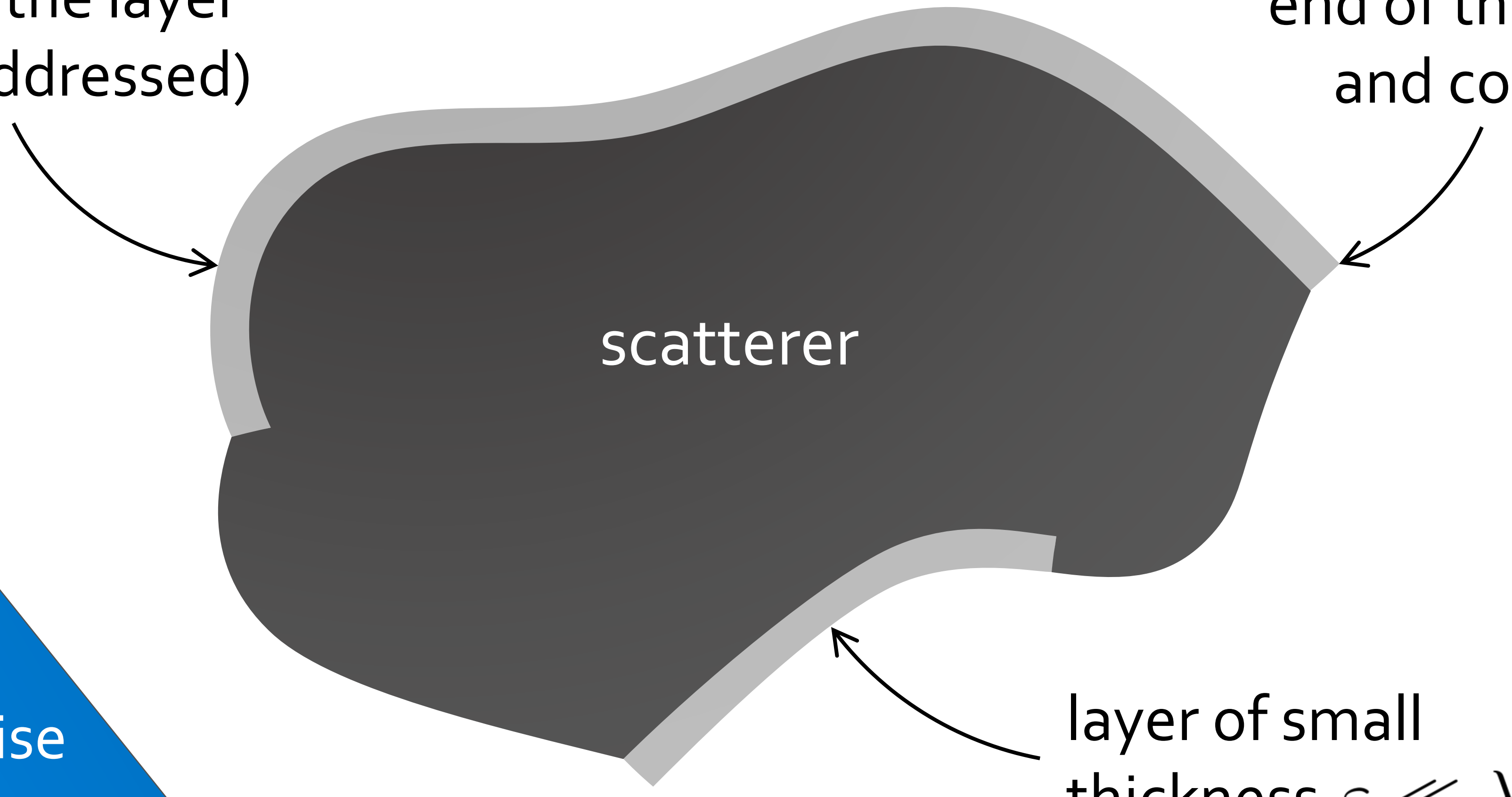


middle of the layer
(already addressed)

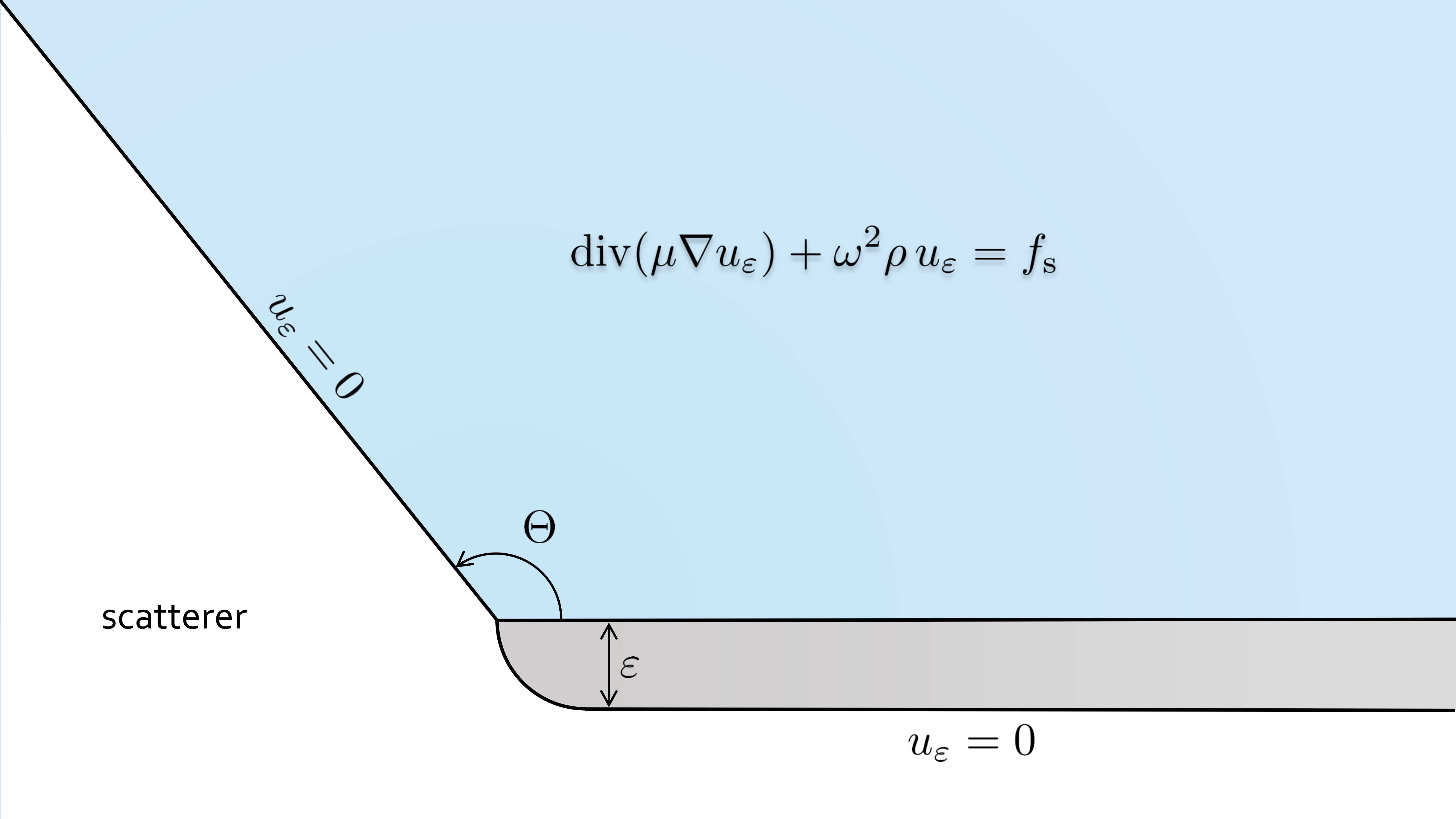
end of the layer
and corner

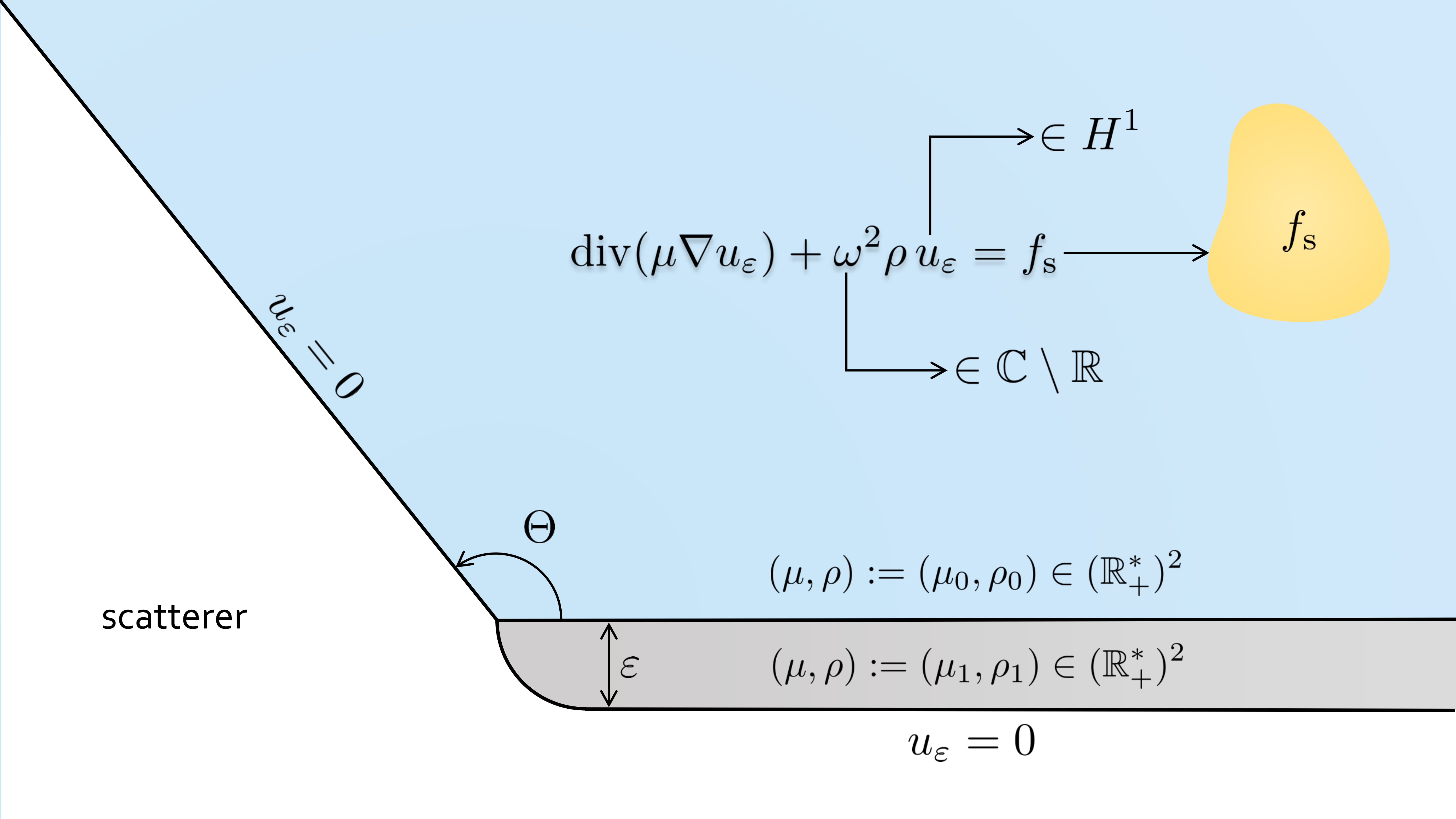
scatterer

layer of small
thickness $\varepsilon \ll \lambda$

excessive computational cost

→ $\underbrace{\text{asymptotic expansions}}_{\text{Part 1}}$ & $\underbrace{\text{effective models}}_{\text{Part 2}}$


$\mathrm{div}(\mu \nabla u_\varepsilon) + \omega^2 \rho\, u_\varepsilon = f_{\mathrm{s}}$
$u_\varepsilon = 0$
Θ
scatterer
ε
$u_\varepsilon = 0$



$$\mathrm{div}(\mu \nabla u_\varepsilon) + \omega^2 \rho\, u_\varepsilon = f_\mathrm{s}$$

$\in H^1$

$\in \mathbb{C} \setminus \mathbb{R}$

$(\mu, \rho) := (\mu_0, \rho_0) \in (\mathbb{R}_+^*)^2$

$(\mu, \rho) := (\mu_1, \rho_1) \in (\mathbb{R}_+^*)^2$

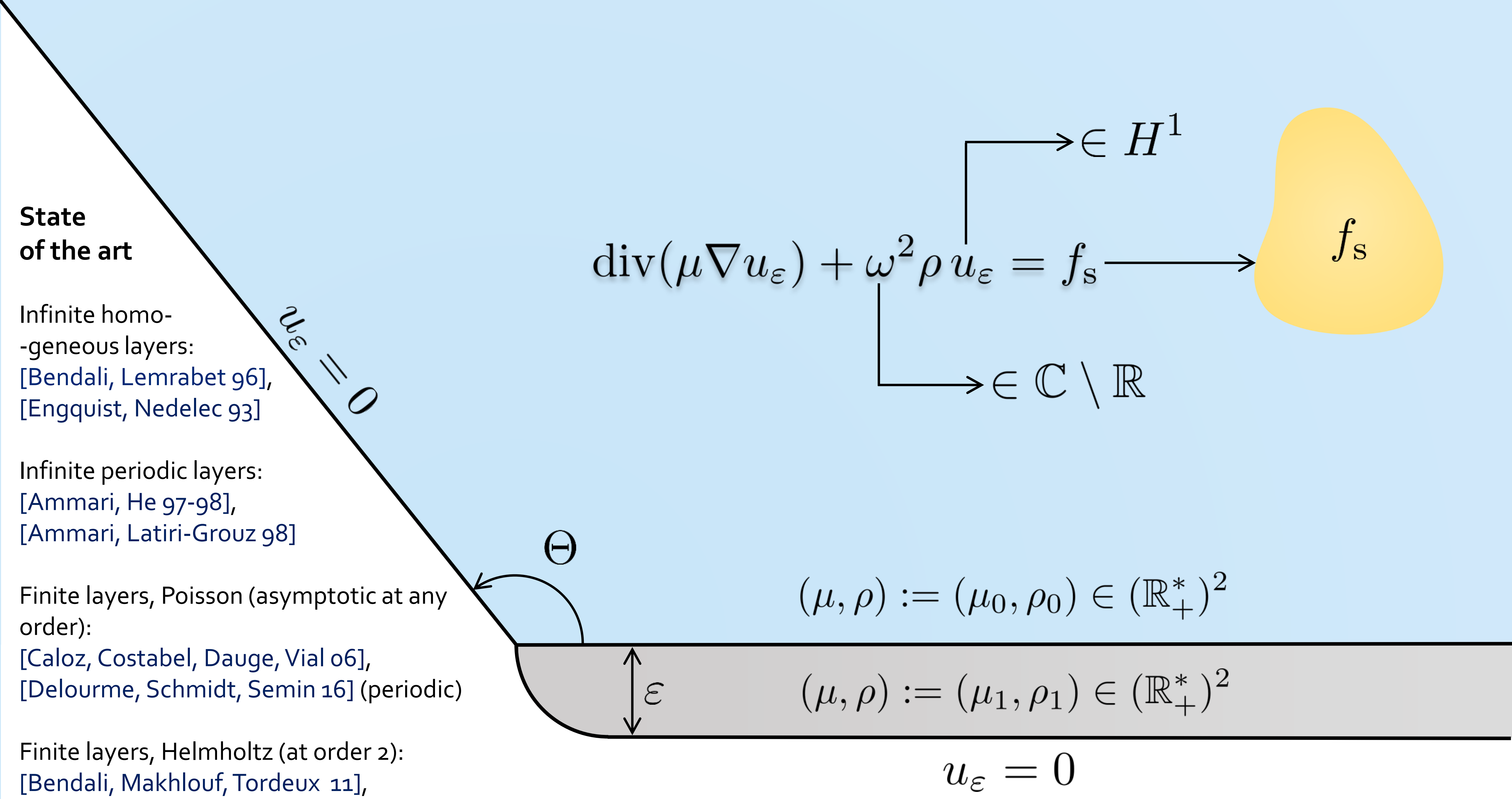
State of the art

Infinite homogeneous layers:
[Bendali, Lemrabet 96],
[Engquist, Nedelec 93]

Infinite periodic layers:
[Ammari, He 97-98],
[Ammari, Latiri-Grouz 98]

Finite layers, Poisson (asymptotic at any order):
[Caloz, Costabel, Dauge, Vial 06],
[Delourme, Schmidt, Semin 16] (periodic)

Finite layers, Helmholtz (at order 2):
[Bendali, Makhlouf, Tordeux 11],
[Delourme, Schmidt, Semin 18] (periodic).



$$\operatorname{div}(\mu \nabla u_\varepsilon) + \omega^2 \rho\, u_\varepsilon = f_\mathrm{s}$$

$$(\mu, \rho) := (\mu_0, \rho_0) \in (\mathbb{R}_+^*)^2$$

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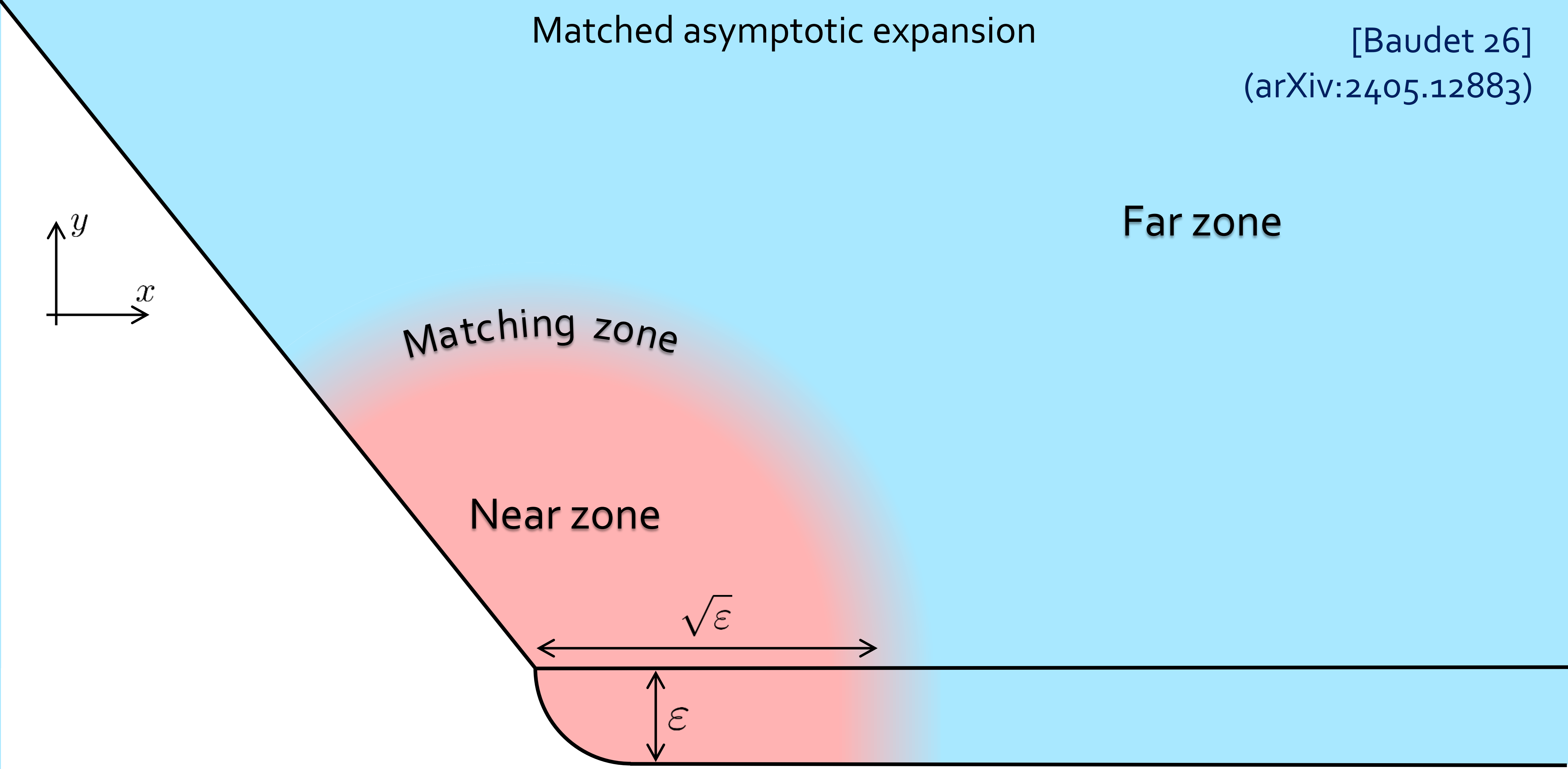
Matched asymptotic expansion

[Baudet 26]
(arXiv:2405.12883)



Matched asymptotic expansion

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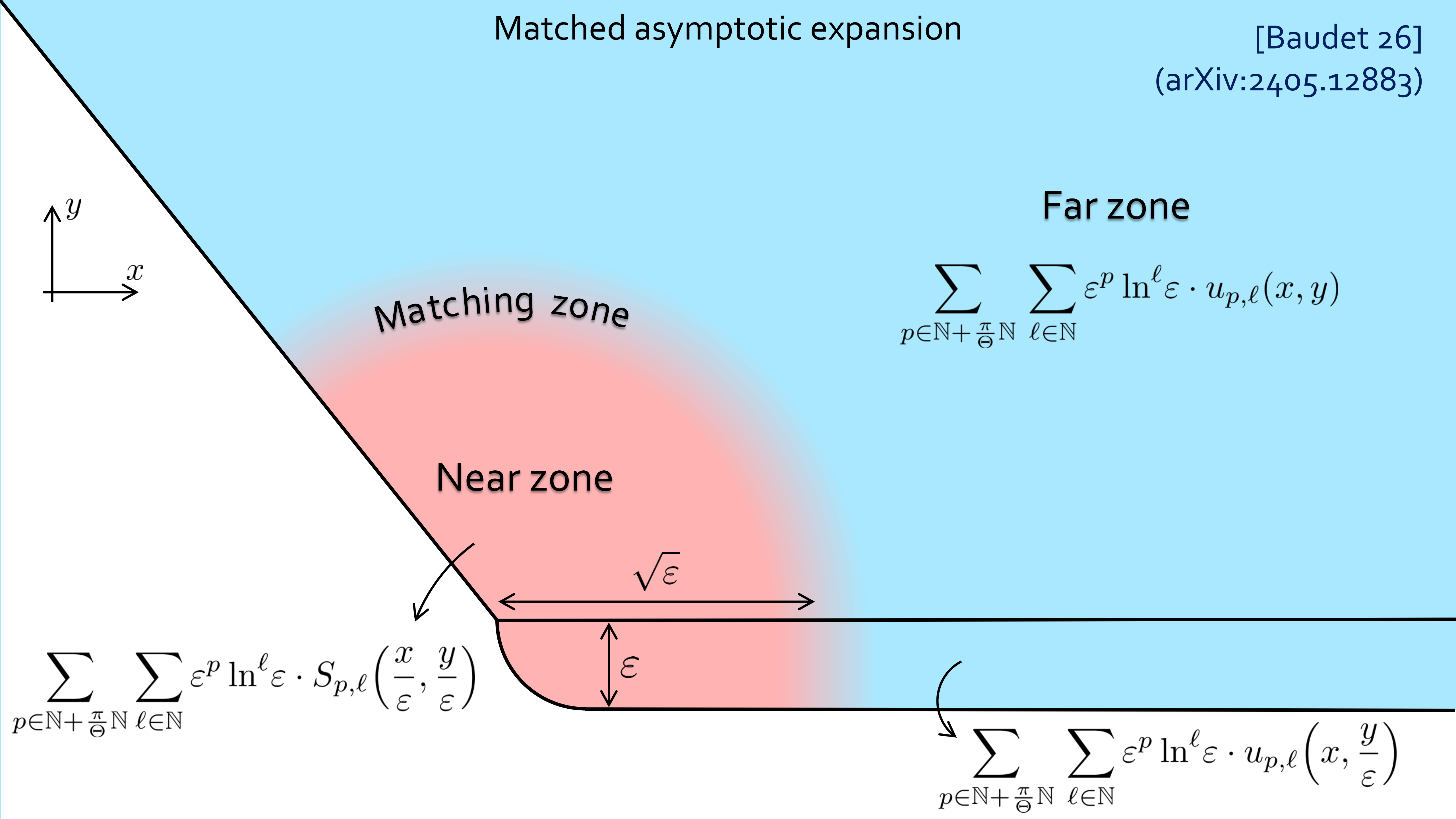

Far zone

$$\sum_{p\in\mathbb{N}+\frac{\pi}{\Theta}\mathbb{N}}\ \sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot u_{p,\ell}(x,y)$$

Matching zone

Near zone

$$\sum_{p\in\mathbb{N}+\frac{\pi}{\Theta}\mathbb{N}}\ \sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot S_{p,\ell}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)$$

$$\sum_{p\in\mathbb{N}+\frac{\pi}{\Theta}\mathbb{N}}\ \sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot u_{p,\ell}\left(x,\frac{y}{\varepsilon}\right)$$

Matched asymptotic expansion

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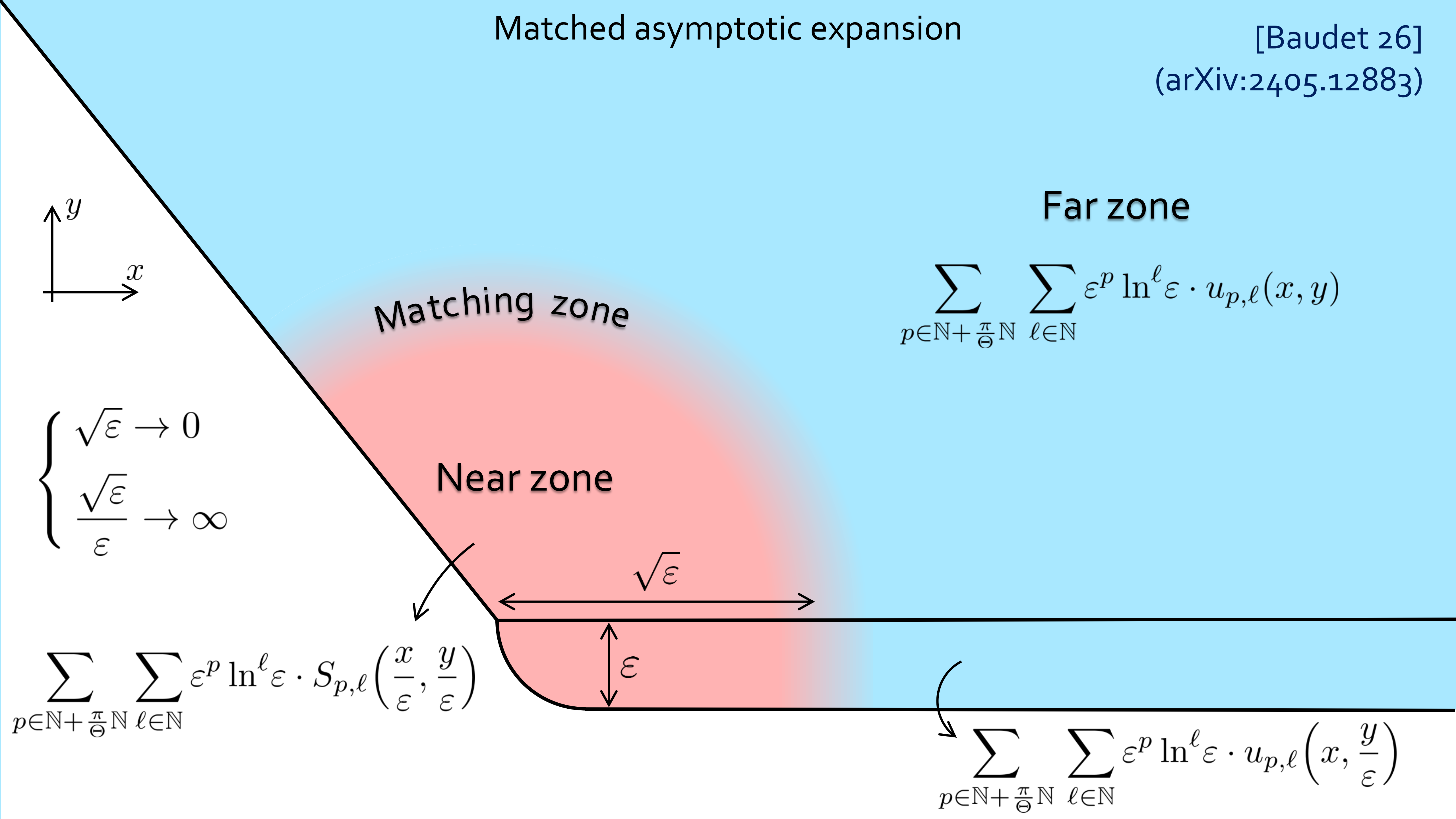


Matching zone

Far zone

$$\sum_{p\in\mathbb{N}+\frac{\pi}{\Theta}\mathbb{N}}\ \sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\cdot u_{p,\ell}(x,y)$$

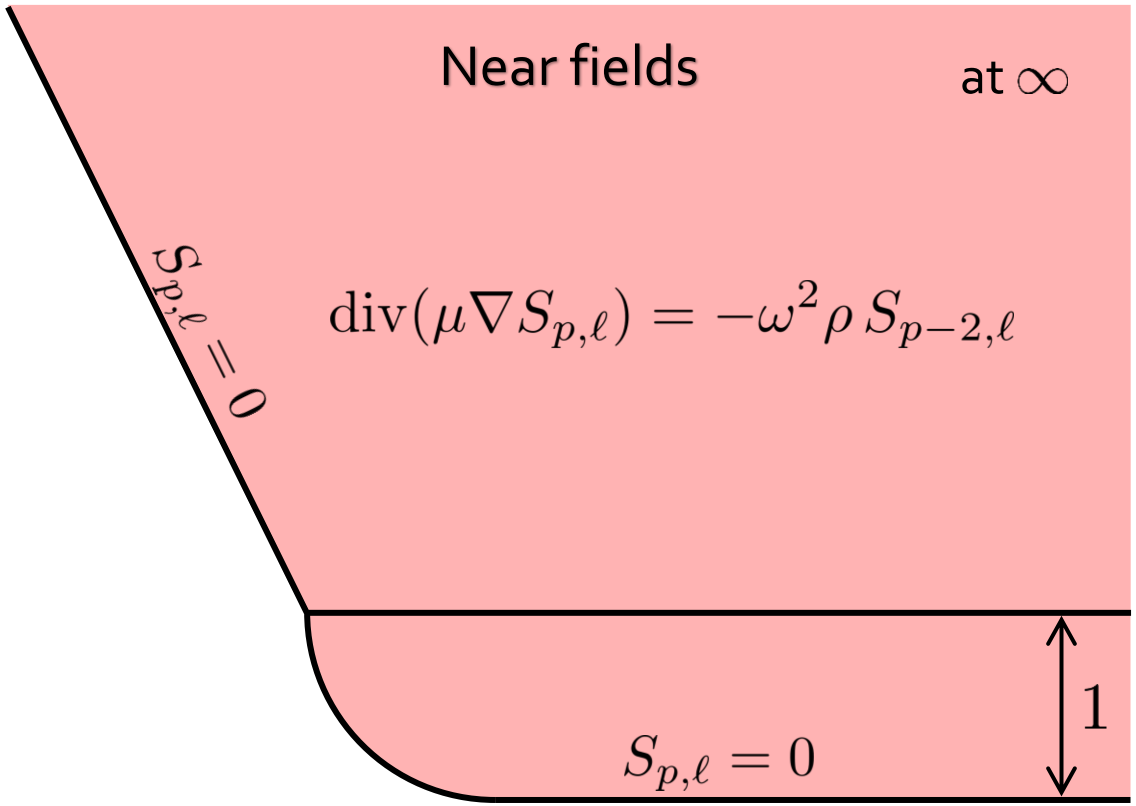
Near zone

$$\begin{cases}\sqrt{\varepsilon}\to 0\\ \dfrac{\sqrt{\varepsilon}}{\varepsilon}\to\infty\end{cases}$$

$$\sum_{p\in\mathbb{N}+\frac{\pi}{\Theta}\mathbb{N}}\ \sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\cdot S_{p,\ell}\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)$$

$$\sum_{p\in\mathbb{N}+\frac{\pi}{\Theta}\mathbb{N}}\ \sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\cdot u_{p,\ell}\Big(x,\frac{y}{\varepsilon}\Big)$$

Algorithm: $u_{p,\ell}$ and $S_{p,\ell}$ are built inductively w.r.t p



Near fields

at ∞

$$\mathrm{div}(\mu\nabla S_{p,\ell}) = -\omega^2\rho\, S_{p-2,\ell}$$

$S_{p,\ell} = 0$

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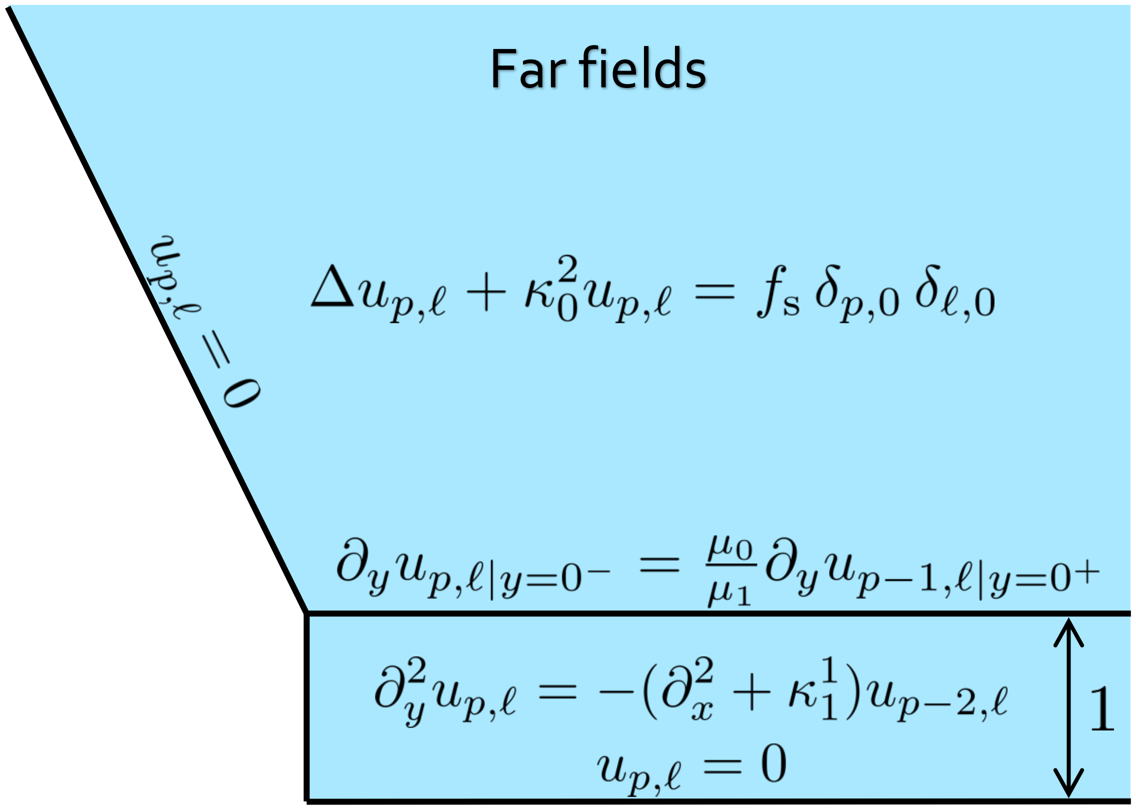


Far fields

$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0}$$

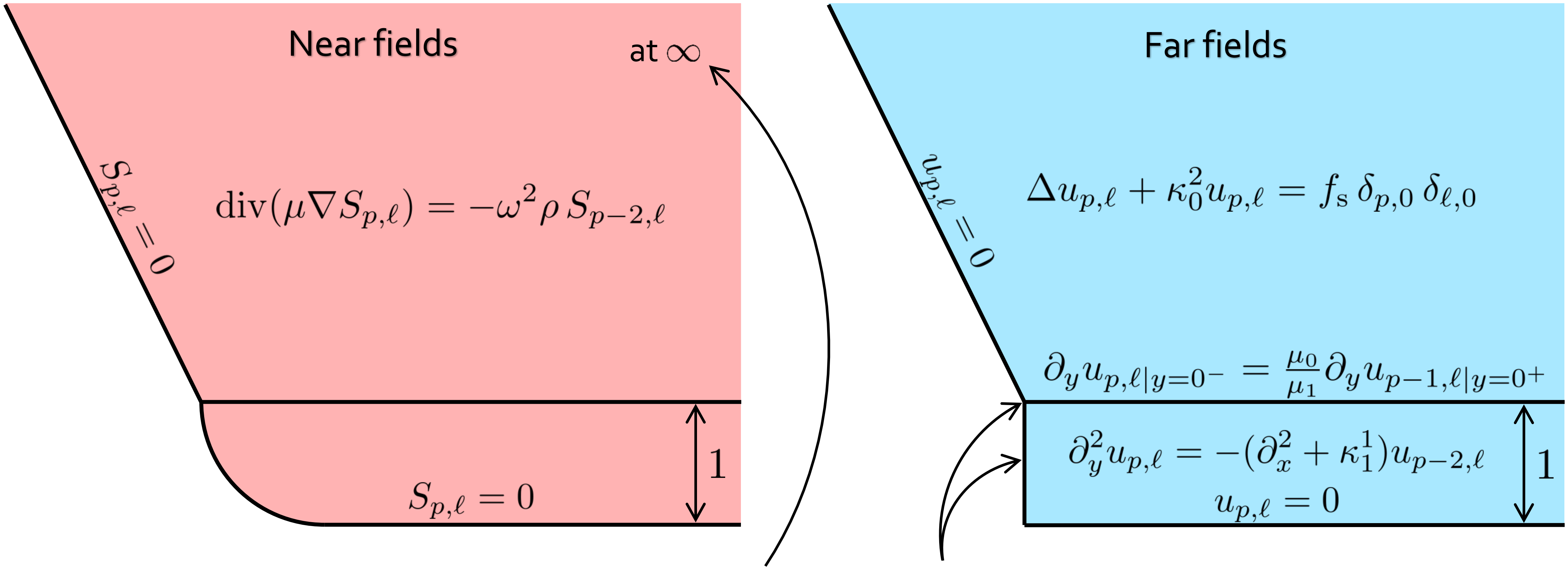
$u_{p,\ell} = 0$

$$\partial_y u_{p,\ell|y=0^-} = \frac{\mu_0}{\mu_1}\partial_y u_{p-1,\ell|y=0^+}$$

$$\partial_y^2 u_{p,\ell} = -(\partial_x^2 + \kappa_1^1)u_{p-2,\ell}$$

$$u_{p,\ell} = 0$$

Algorithm: $u_{p,\ell}$ and $S_{p,\ell}$ are built inductively w.r.t p



Near fields

at ∞

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Far fields

$u_{p,\ell} = 0$

$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\, \delta_{p,0}\, \delta_{\ell,0}$$

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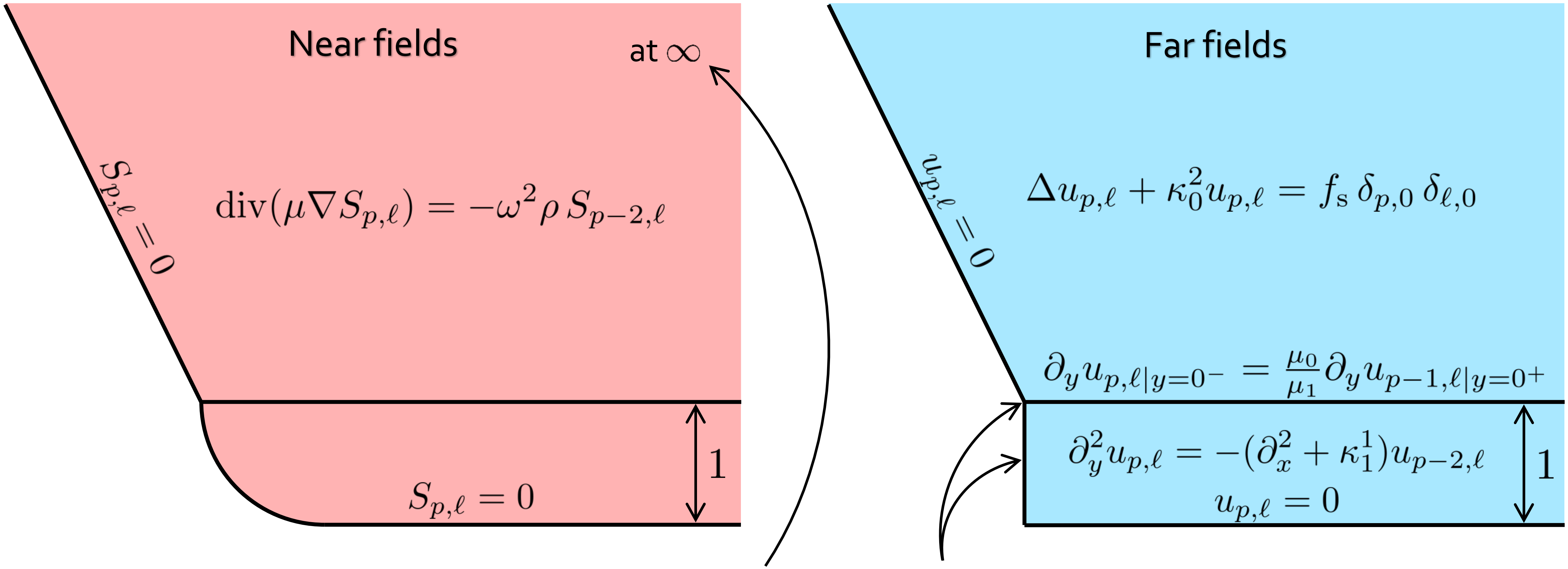
Diverging behaviors due to the matching condition

Algorithm: $u_{p,\ell}$ and $S_{p,\ell}$ are built inductively w.r.t p



Near fields

at ∞

$$\mathrm{div}(\mu\nabla S_{p,\ell}) = -\omega^2\rho\, S_{p-2,\ell}$$

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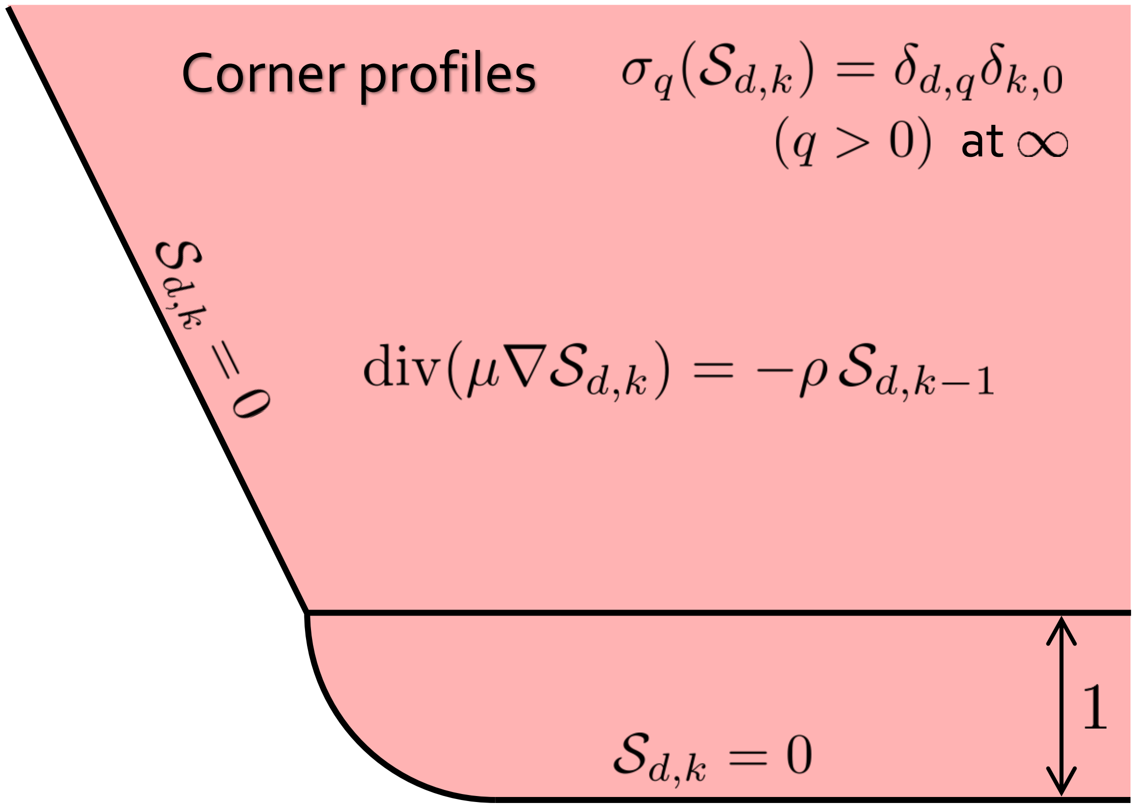
Diverging behaviors due to the matching condition

$$S_{p,\ell} \underset{r\to\infty}{\approx} \sum_{d\in\frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(S_{p,\ell})\, r^d \sin(d\theta) + \text{explicit} \quad \text{and} \quad u_{p,\ell} \underset{r\to 0}{\approx} \sum_{d\in\frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(u_{p,\ell})\, r^d \sin(d\theta) + \text{expl.}$$

Algorithm: $u_{p,\ell}$ and $S_{p,\ell}$ are built inductively w.r.t p

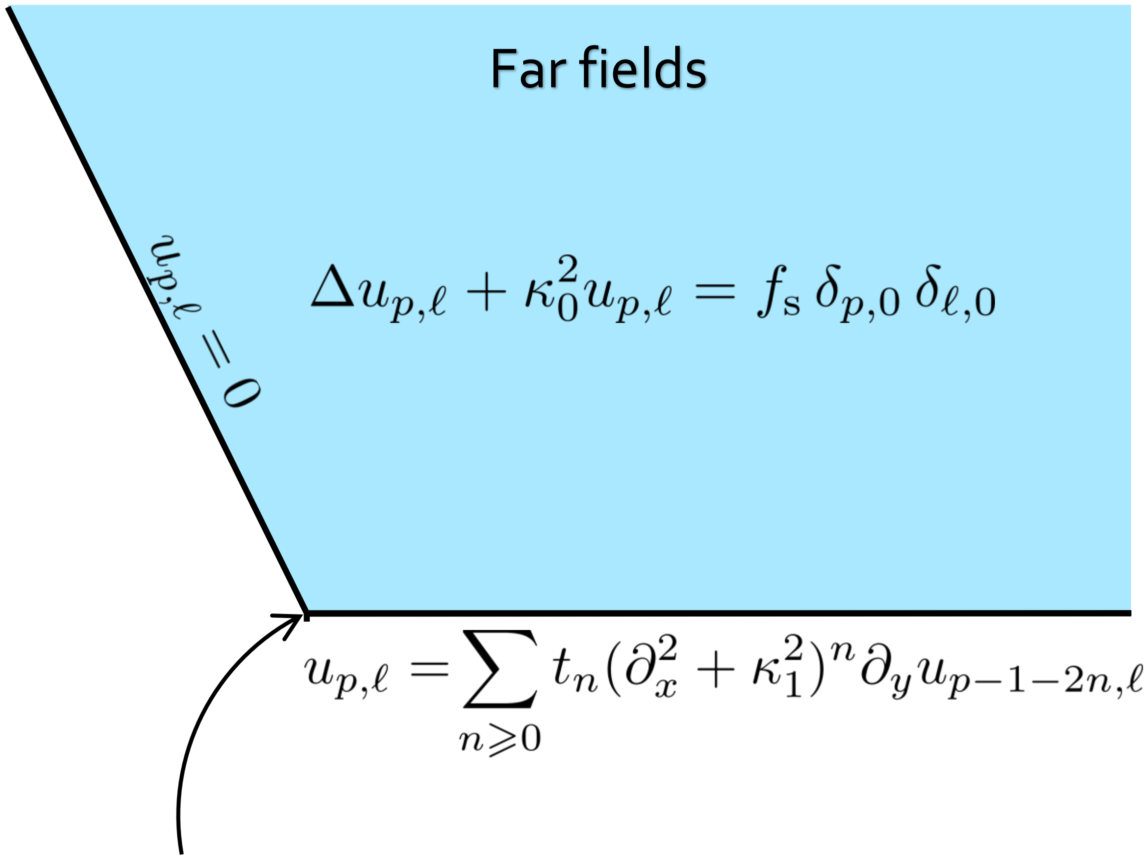


Corner profiles

$$\sigma_q(\mathcal{S}_{d,k}) = \delta_{d,q}\delta_{k,0}$$

$(q > 0)$ at ∞

$$\operatorname{div}(\mu\nabla\mathcal{S}_{d,k}) = -\rho\,\mathcal{S}_{d,k-1}$$



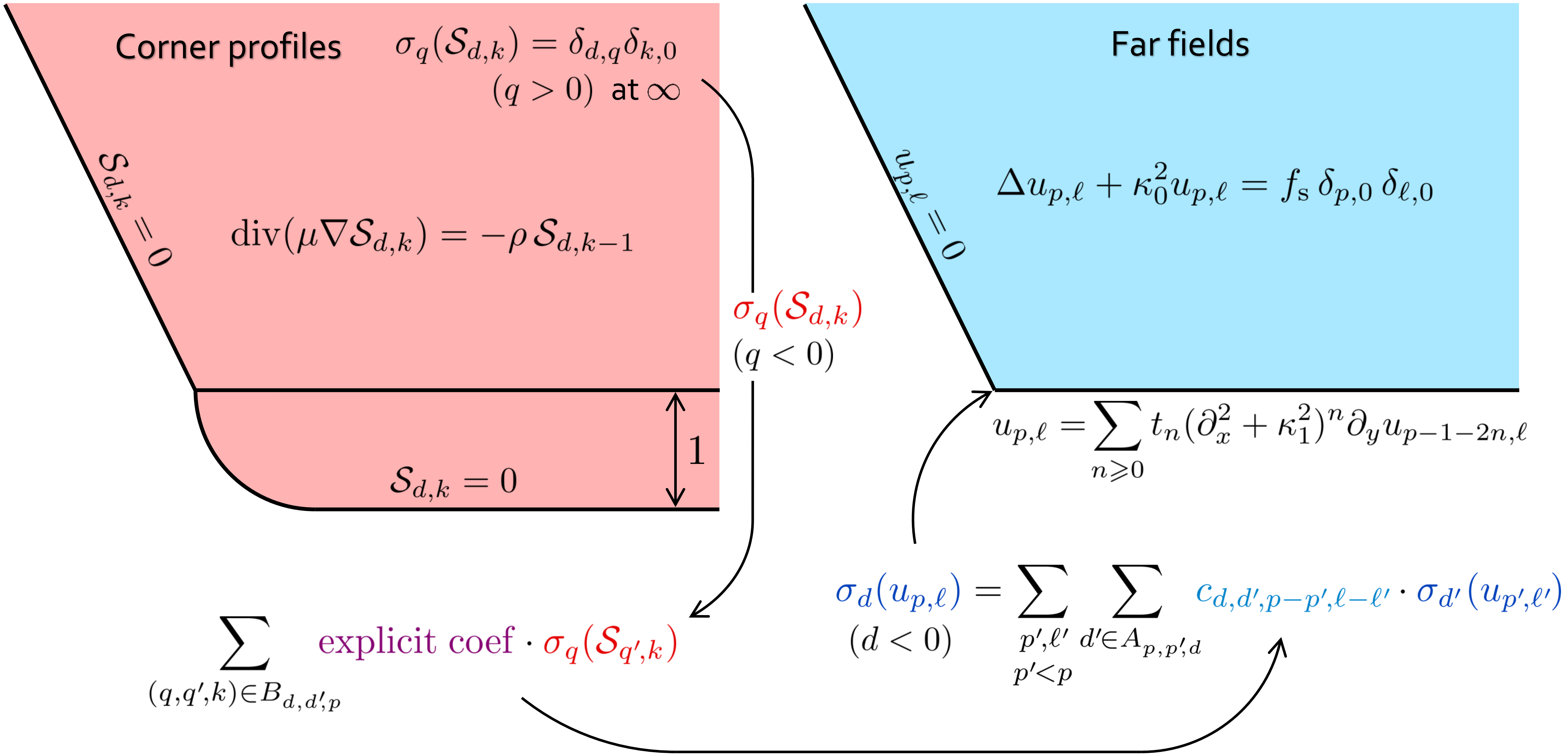
Far fields

$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0}$$

$$u_{p,\ell} = \sum_{n\geqslant 0} t_n(\partial_x^2 + \kappa_1^2)^n\partial_y u_{p-1-2n,\ell}$$

$$\underset{(d<0)}{\sigma_d(u_{p,\ell})} = \sum_{\substack{p',\ell' \\ p'<p}}\ \sum_{d'\in A_{p,p',d}} c^{u\leftarrow u}_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(u_{p',\ell'})$$

Algorithm



Step 1

Corner profiles

$\sigma_q(\mathcal{S}_{d,k}) = \delta_{d,q}\delta_{k,0}$
$(q>0)$ at ∞

$$\operatorname{div}(\mu\nabla\mathcal{S}_{d,k}) = -\rho\,\mathcal{S}_{d,k-1}$$

$\mathcal{S}_{d,k}=0$

$\mathcal{S}_{d,k}=0$

1

$\sigma_q(\mathcal{S}_{d,k})$
$(q<0)$

$$\sum_{(q,q',k)\in B_{d,d',p}} \text{explicit coef}\cdot\sigma_q(\mathcal{S}_{q',k})$$

Step 2

Far fields

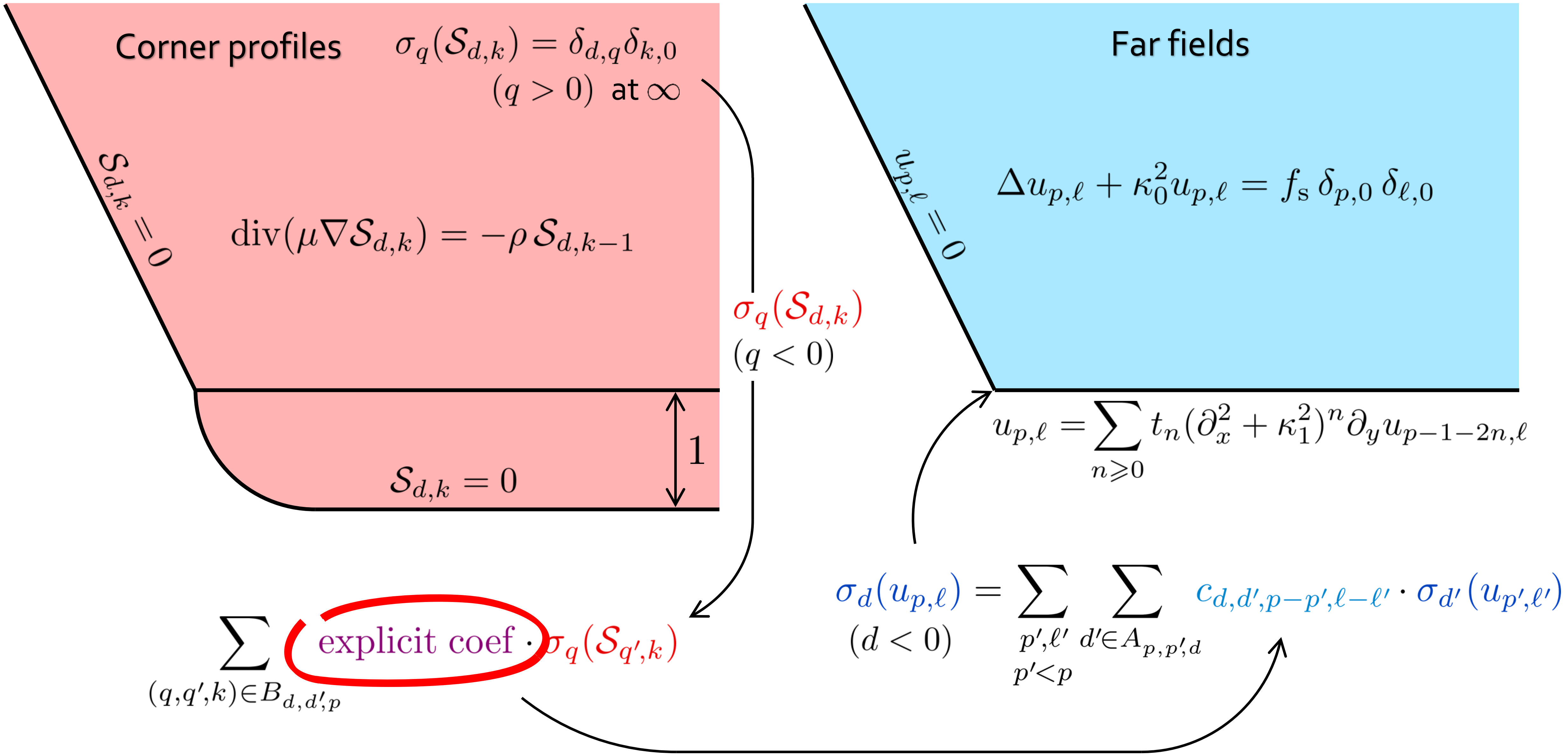
$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0}$$

$u_{p,\ell}=0$

$$u_{p,\ell} = \sum_{n\geqslant 0} t_n(\partial_x^2+\kappa_1^2)^n\partial_y u_{p-1-2n,\ell}$$

$$\underset{(d<0)}{\sigma_d(u_{p,\ell})} = \sum_{\substack{p',\ell'\\ p'<p}}\ \sum_{d'\in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(u_{p',\ell'})$$

Algorithm



Step 1

Corner profiles $\sigma_q(\mathcal{S}_{d,k}) = \delta_{d,q}\delta_{k,0}$

$(q > 0)$ at ∞

$\mathcal{S}_{d,k} = 0$

$$\mathrm{div}(\mu\nabla\mathcal{S}_{d,k}) = -\rho\,\mathcal{S}_{d,k-1}$$

$\mathcal{S}_{d,k} = 0$

$\sigma_q(\mathcal{S}_{d,k})$

$(q < 0)$

$$\sum_{(q,q',k)\in B_{d,d',p}} \boxed{\text{explicit coef}} \cdot \sigma_q(\mathcal{S}_{q',k})$$

Step 2

Far fields

$u_{p,\ell} = 0$

$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0}$$

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$$\sigma_d(u_{p,\ell}) = \sum_{\substack{p',\ell' \\ p'<p}} \sum_{d'\in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(u_{p',\ell'})$$

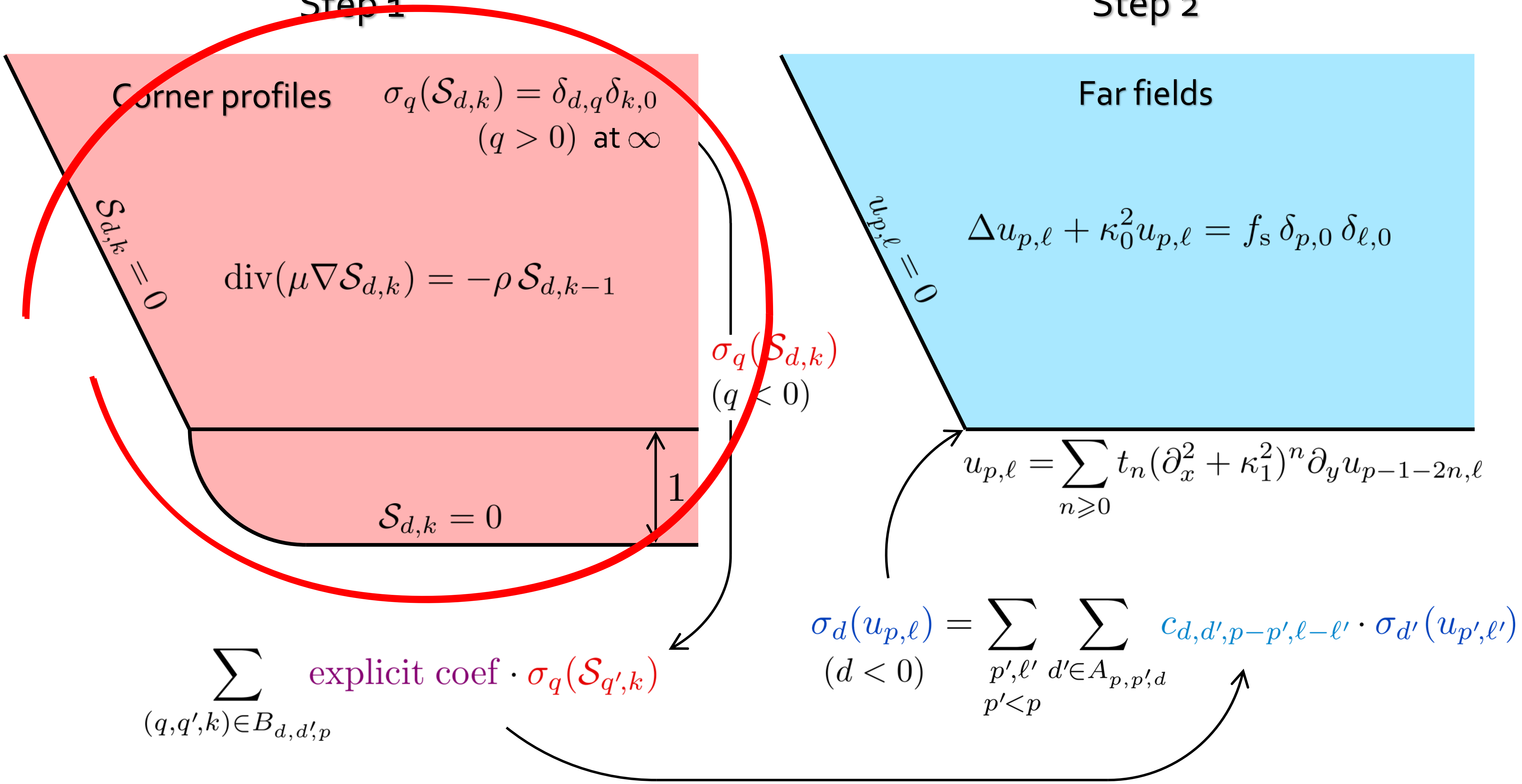
$(d < 0)$

→ Computer-assisted calculations on functions with explicit formulas (that intervene in $S_{p,\ell} \underset{r\to\infty}{\approx} \dots$ and $u_{p,\ell} \underset{r\to 0}{\approx} \dots$)

→ Exact, quick and memory cheap computations

$$\sum_{(q,q',k)\in B_{d,d',p}} \boxed{\text{explicit coef}} \cdot \sigma_q(\mathcal{S}_{q',k})$$

Algorithm



Step 1

Corner profiles $\sigma_q(\mathcal{S}_{d,k}) = \delta_{d,q}\delta_{k,0}$ $(q>0)$ at ∞

$\mathcal{S}_{d,k} = 0$

$$\operatorname{div}(\mu\nabla\mathcal{S}_{d,k}) = -\rho\,\mathcal{S}_{d,k-1}$$

$\mathcal{S}_{d,k} = 0$

1

$\sigma_q(\mathcal{S}_{d,k})$ $(q<0)$

$$\sum_{(q,q',k)\in B_{d,d',p}} \text{explicit coef}\cdot\sigma_q(\mathcal{S}_{q',k})$$

Step 2

Far fields

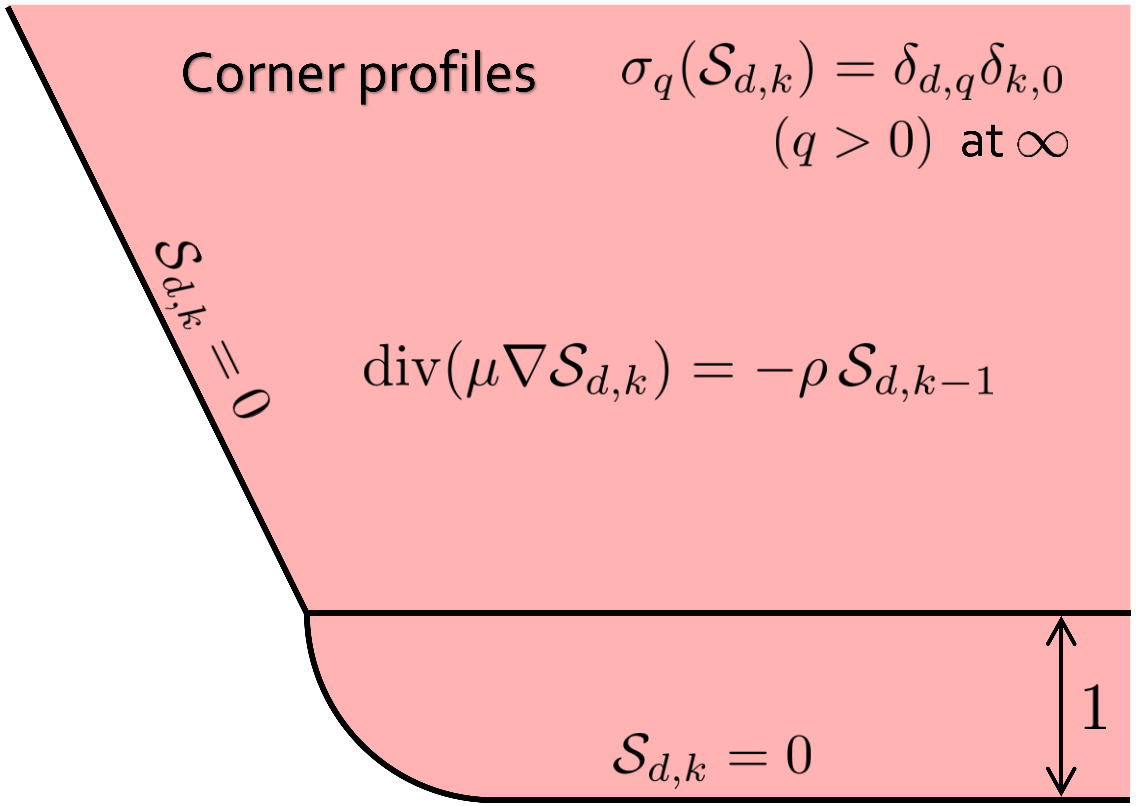
$u_{p,\ell} = 0$

$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0}$$

$$u_{p,\ell} = \sum_{n\geqslant 0} t_n(\partial_x^2+\kappa_1^2)^n\partial_y u_{p-1-2n,\ell}$$

$$\sigma_d(u_{p,\ell}) = \sum_{\substack{p',\ell'\\ p'<p}}\ \sum_{d'\in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(u_{p',\ell'}) \qquad (d<0)$$

Computation of $\mathcal{S}_{d,k}$ and $\sigma_q(\mathcal{S}_{d,k})$



Corner profiles $\sigma_q(\mathcal{S}_{d,k}) = \delta_{d,q}\delta_{k,0}$

$(q>0)$ at ∞

$\mathcal{S}_{d,k} = 0$

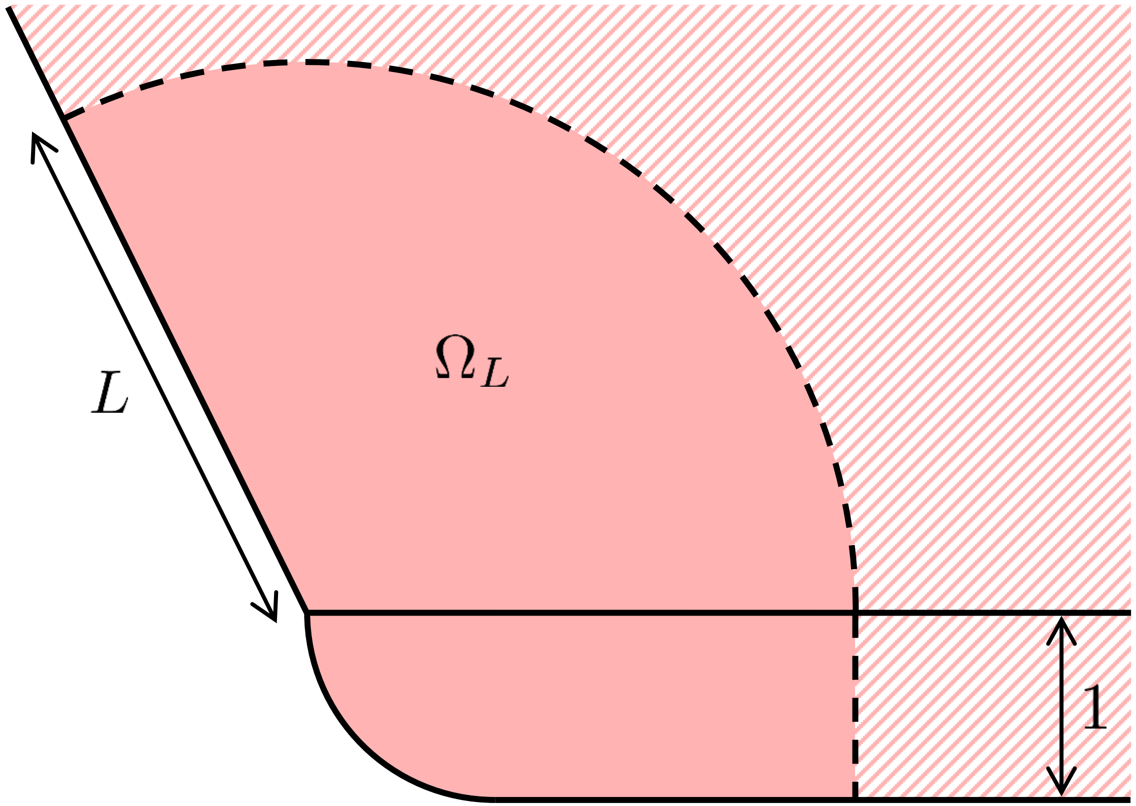
$$\operatorname{div}(\mu\nabla\mathcal{S}_{d,k}) = -\rho\,\mathcal{S}_{d,k-1}$$

$\mathcal{S}_{d,k} = 0$

$$\mathcal{S}_{d,k} = \text{known part} + \mathcal{S}^V_{d,k}$$

$$\mathcal{S}^V_{d,k} = \sum_{n=1}^{N} \sigma_{-n\frac{\pi}{\Theta}}(\mathcal{S}_{d,k})\,\varphi_n + O\big(r^{-(N+1)\frac{\pi}{\Theta}}\big)$$

Computation of $\mathcal{S}_{d,k}$ and $\sigma_q(\mathcal{S}_{d,k})$



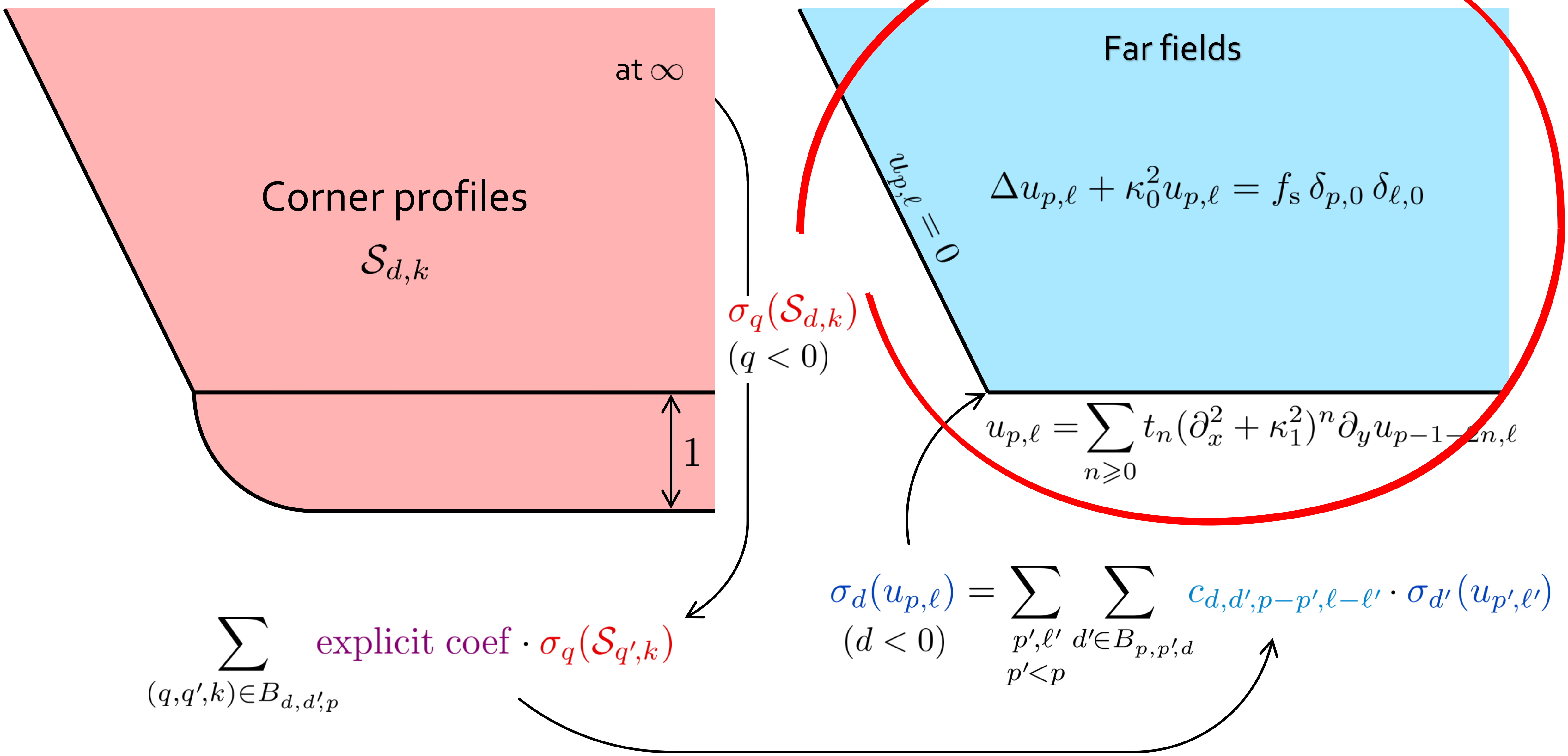
$$\mathcal{S}_{d,k} = \text{known part} + \mathcal{S}^V_{d,k}$$

$$\mathcal{S}^V_{d,k} = \sum_{n=1}^{N} \sigma_{-n\frac{\pi}{\Theta}}(\mathcal{S}_{d,k})\,\varphi_n + O\big(r^{-(N+1)\frac{\pi}{\Theta}}\big)$$

Enriched Galerkin method:

$$V_L = \{v \in H^1_0(\Omega) \mid v = 0 \text{ outside } \Omega_L\}$$
$$+ \operatorname{Span}(\varphi_1, \ldots, \varphi_N)$$

Algorithm



Step 1

at ∞

Corner profiles

$\mathcal{S}_{d,k}$

1

$\sigma_q(\mathcal{S}_{d,k})$

$(q<0)$

$$\sum_{(q,q',k)\in B_{d,d',p}} \text{explicit coef} \cdot \sigma_q(\mathcal{S}_{q',k})$$

Step 2

Far fields

$$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\, \delta_{p,0}\, \delta_{\ell,0}$$

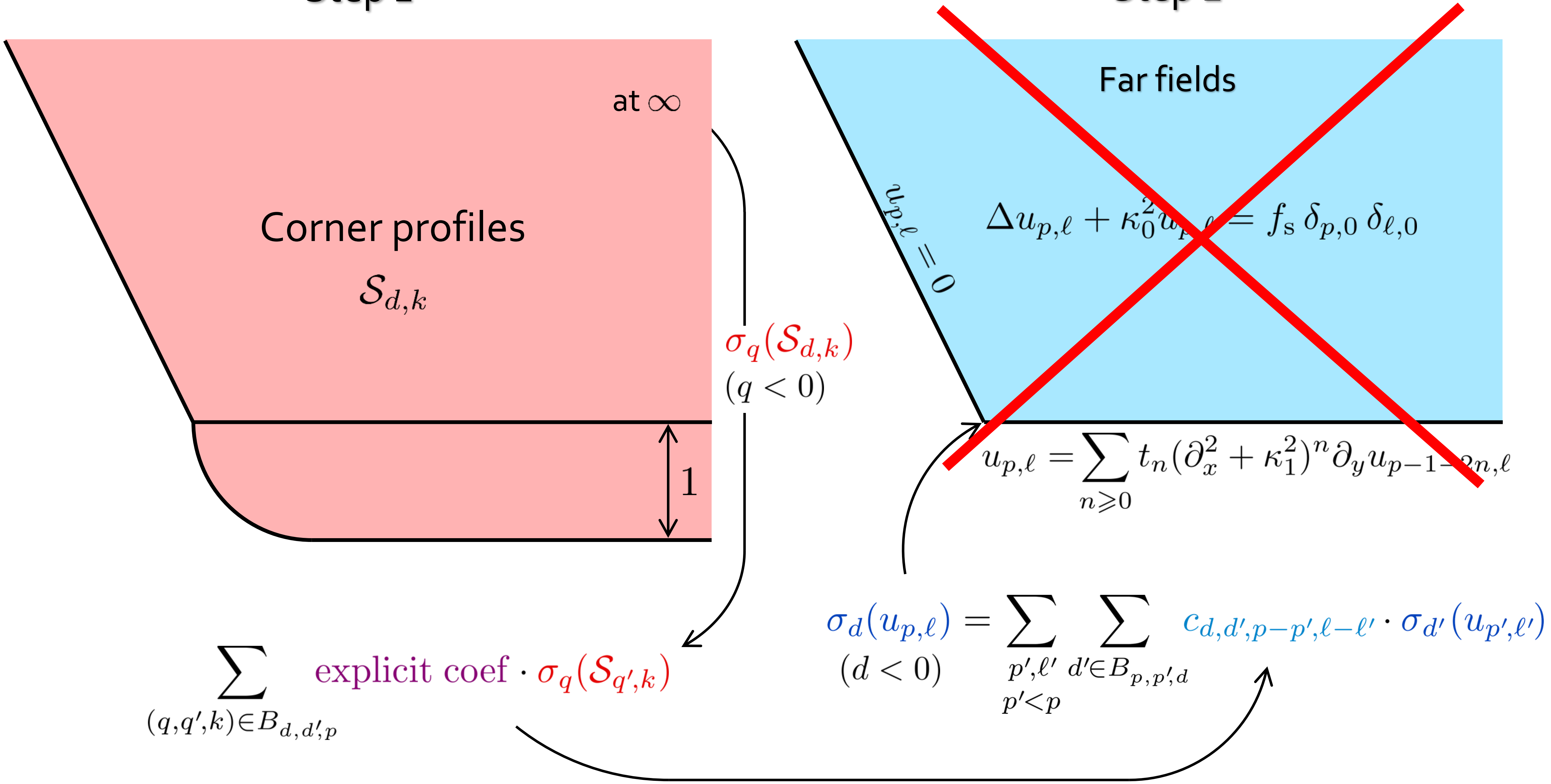
$u_{p,\ell} = 0$

$$u_{p,\ell} = \sum_{n\geqslant 0} t_n (\partial_x^2 + \kappa_1^2)^n \partial_y u_{p-1-2n,\ell}$$

$$\sigma_d(u_{p,\ell}) = \sum_{\substack{p',\ell' \\ p'<p}} \sum_{d'\in B_{p,p',d}} c_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(u_{p',\ell'})$$

$(d<0)$

Algorithm



Step 1

at ∞

Corner profiles

$\mathcal{S}_{d,k}$

$\sigma_q(\mathcal{S}_{d,k})$
$(q<0)$

$$\sum_{(q,q',k)\in B_{d,d',p}} \text{explicit coef} \cdot \sigma_q(\mathcal{S}_{q',k})$$

Step 2

~~Far fields~~

~~$\Delta u_{p,\ell} + \kappa_0^2 u_{p,\ell} = f_{\mathrm{s}}\, \delta_{p,0}\, \delta_{\ell,0}$~~

~~$u_{p,\ell} = 0$~~

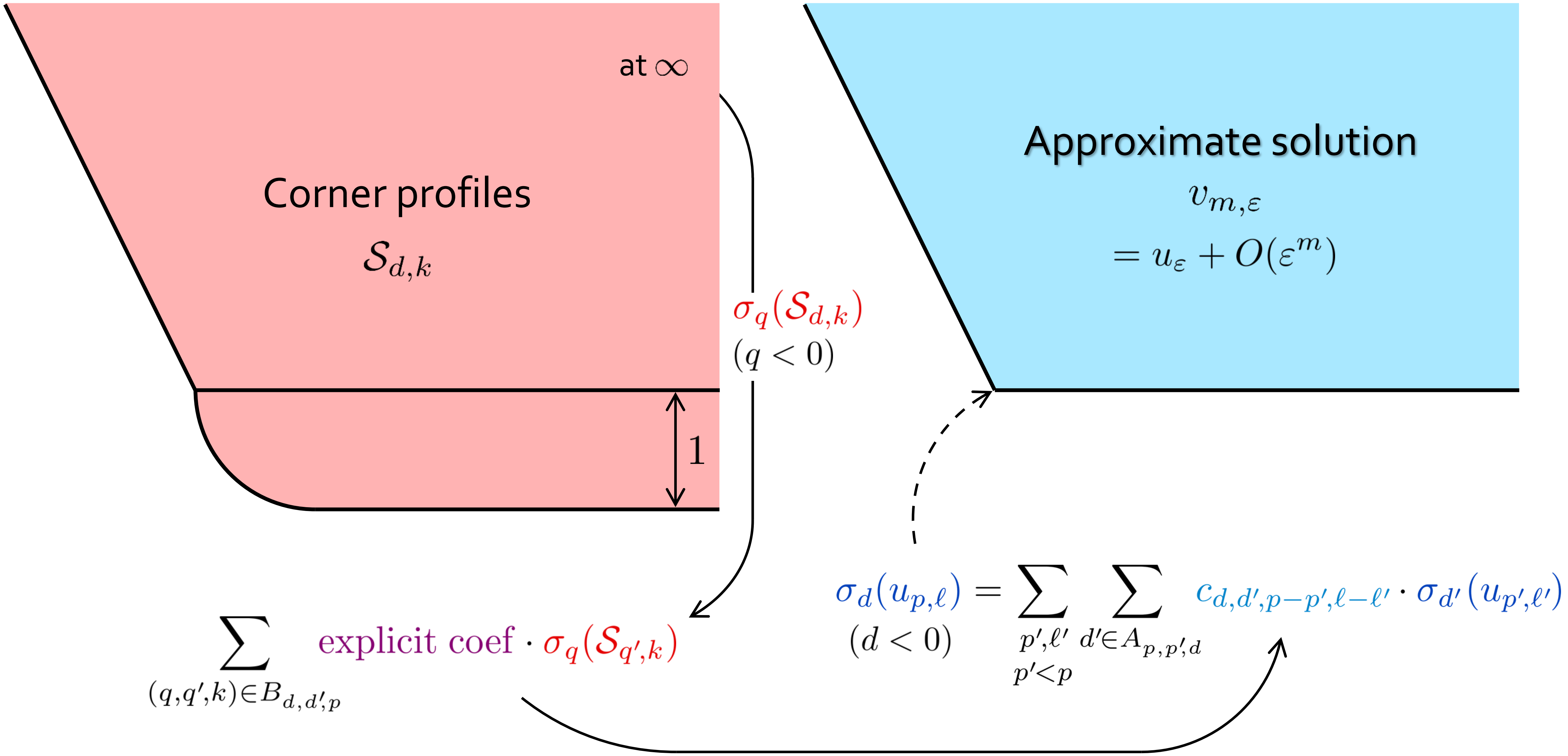
~~$u_{p,\ell} = \sum_{n \geqslant 0} t_n (\partial_x^2 + \kappa_1^2)^n \partial_y u_{p-1-2n,\ell}$~~

$$\sigma_d(u_{p,\ell}) = \sum_{\substack{p',\ell' \\ p'<p}} \sum_{d'\in B_{p,p',d}} c_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(u_{p',\ell'})$$
$(d<0)$

Effective models

Step 1

Step 2



Corner profiles

$\mathcal{S}_{d,k}$

$\sigma_q(\mathcal{S}_{d,k})$

$(q<0)$

$$\sum_{(q,q',k)\in B_{d,d',p}} \text{explicit coef}\cdot\sigma_q(\mathcal{S}_{q',k})$$

Approximate solution

$v_{m,\varepsilon}$

$= u_\varepsilon + O(\varepsilon^m)$

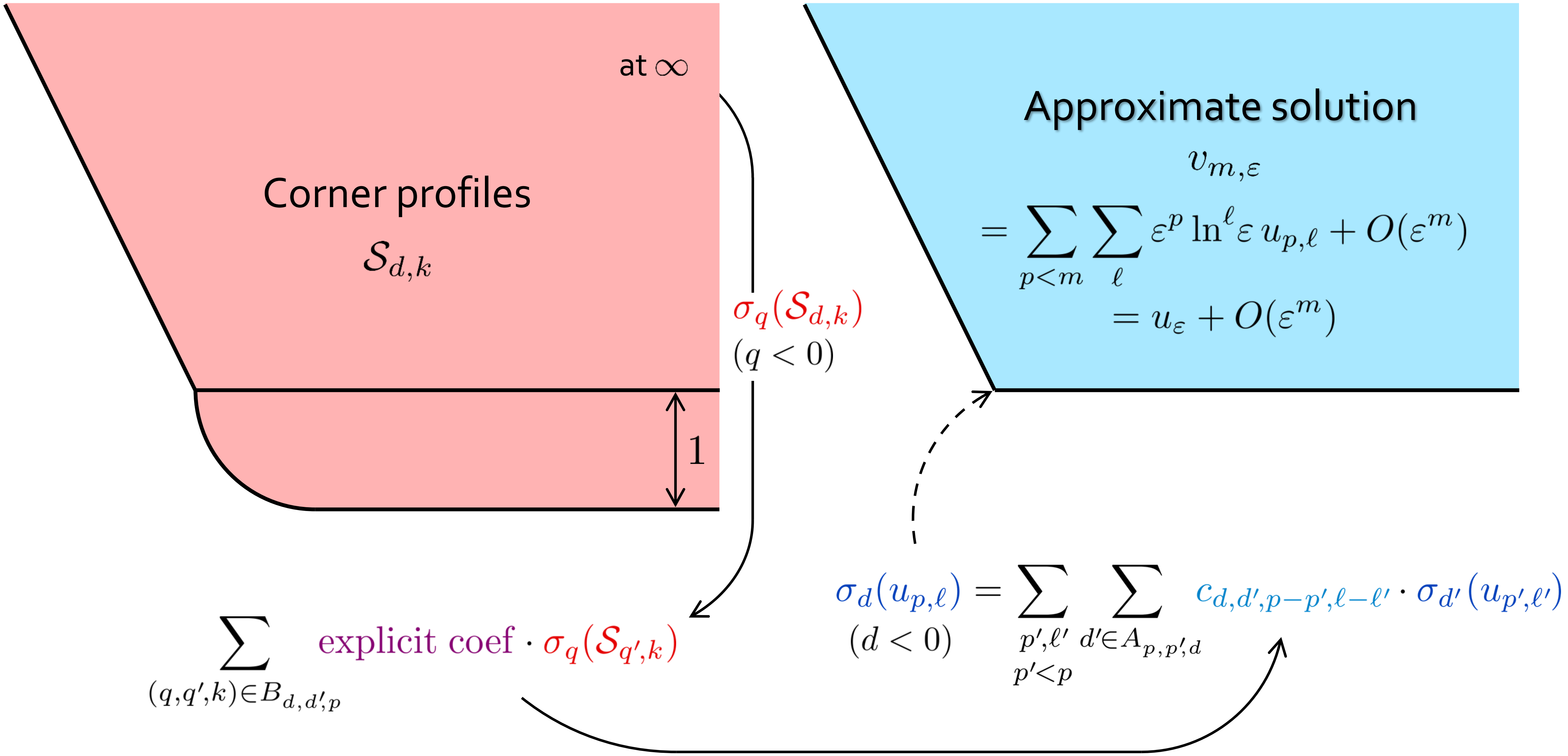
$$\sigma_d(u_{p,\ell}) = \sum_{\substack{p',\ell'\\ p'<p}} \sum_{d'\in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(u_{p',\ell'})$$

$(d<0)$

Effective models

Step 1

Step 2



Corner profiles

$\mathcal{S}_{d,k}$

$\sigma_q(\mathcal{S}_{d,k})$
$(q < 0)$

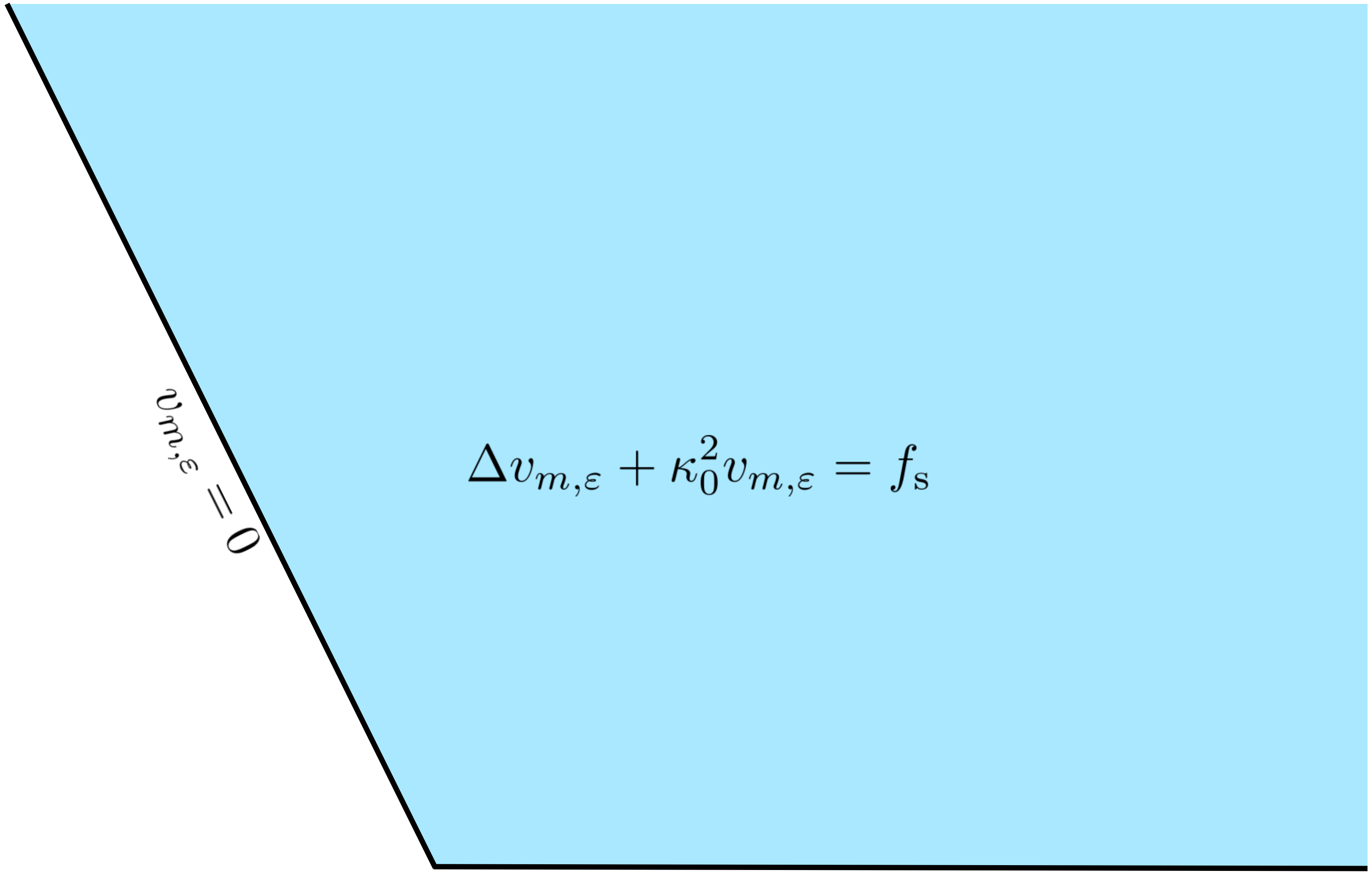
Approximate solution

$$v_{m,\varepsilon} = \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \, u_{p,\ell} + O(\varepsilon^m) = u_\varepsilon + O(\varepsilon^m)$$

$$\sum_{(q,q',k)\in B_{d,d',p}} \text{explicit coef} \cdot \sigma_q(\mathcal{S}_{q',k})$$

$$\sigma_d(u_{p,\ell}) = \sum_{\substack{p',\ell' \\ p'<p}} \sum_{d' \in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(u_{p',\ell'})$$
$(d < 0)$

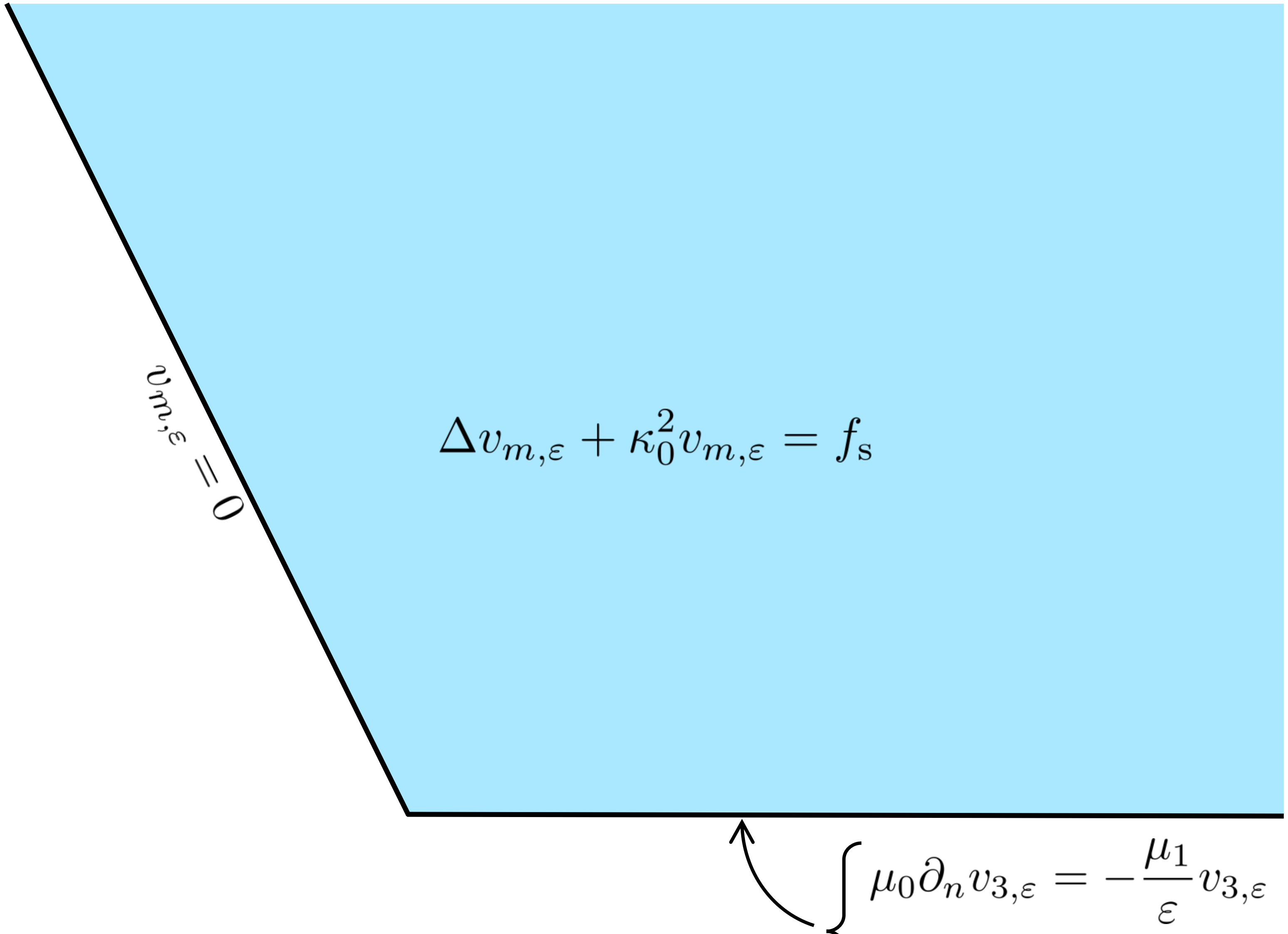
Effective models



$$v_{m,\varepsilon} = 0$$

$$\Delta v_{m,\varepsilon} + \kappa_0^2 v_{m,\varepsilon} = f_{\mathrm{s}}$$

Effective models

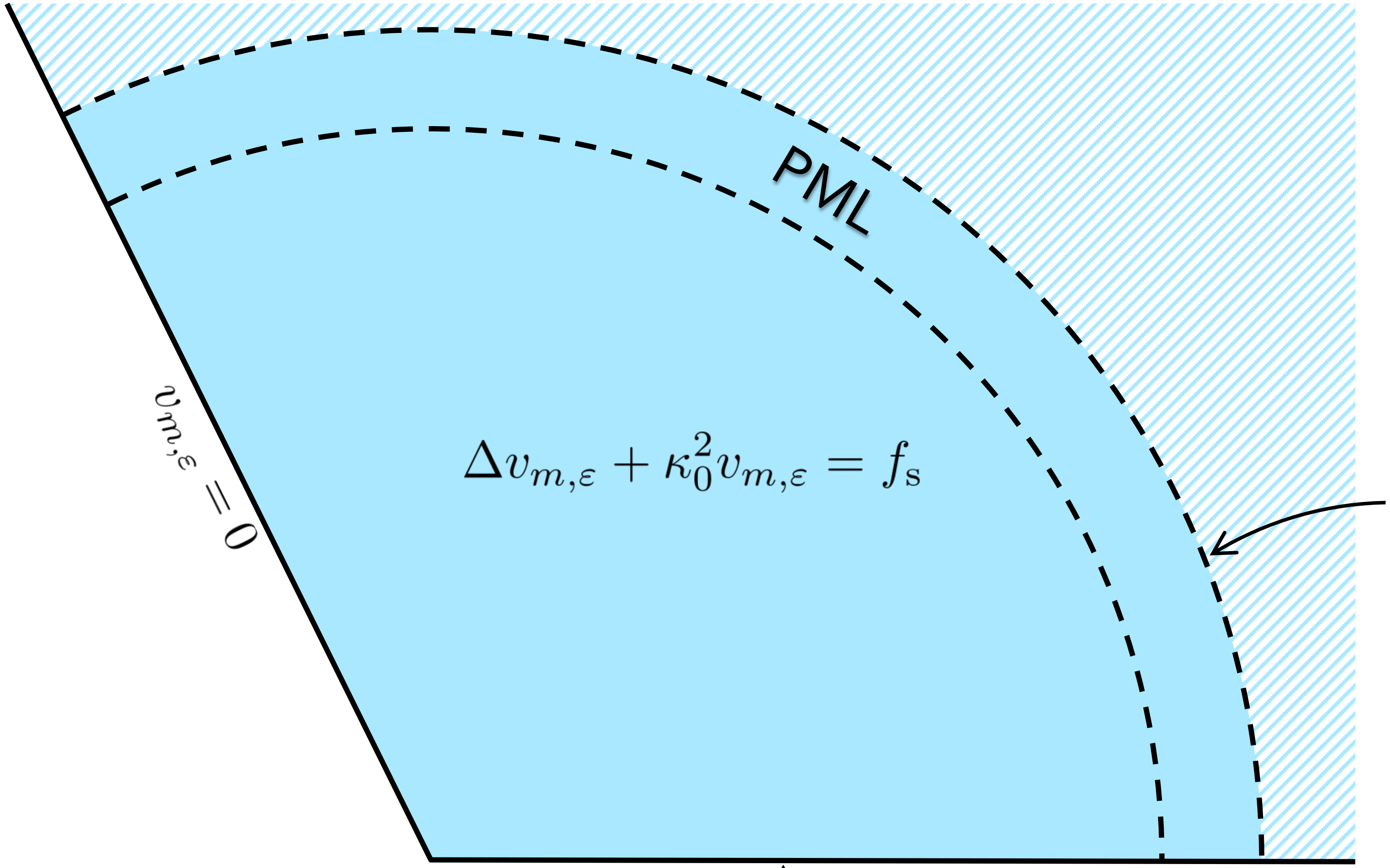


$$v_{m,\varepsilon} = 0$$

$$\Delta v_{m,\varepsilon} + \kappa_0^2 v_{m,\varepsilon} = f_{\mathrm{s}}$$

$$\begin{cases} \mu_0 \partial_n v_{3,\varepsilon} = -\dfrac{\mu_1}{\varepsilon} v_{3,\varepsilon} \\ \mu_0 \partial_n v_{5,\varepsilon} = -\dfrac{\mu_1}{\varepsilon} v_{5,\varepsilon} + \dfrac{\varepsilon \mu_1}{3} (\partial_x^2 + k_1^2) v_{5,\varepsilon} \end{cases}$$

Effective models



$$v_{m,\varepsilon} = 0$$

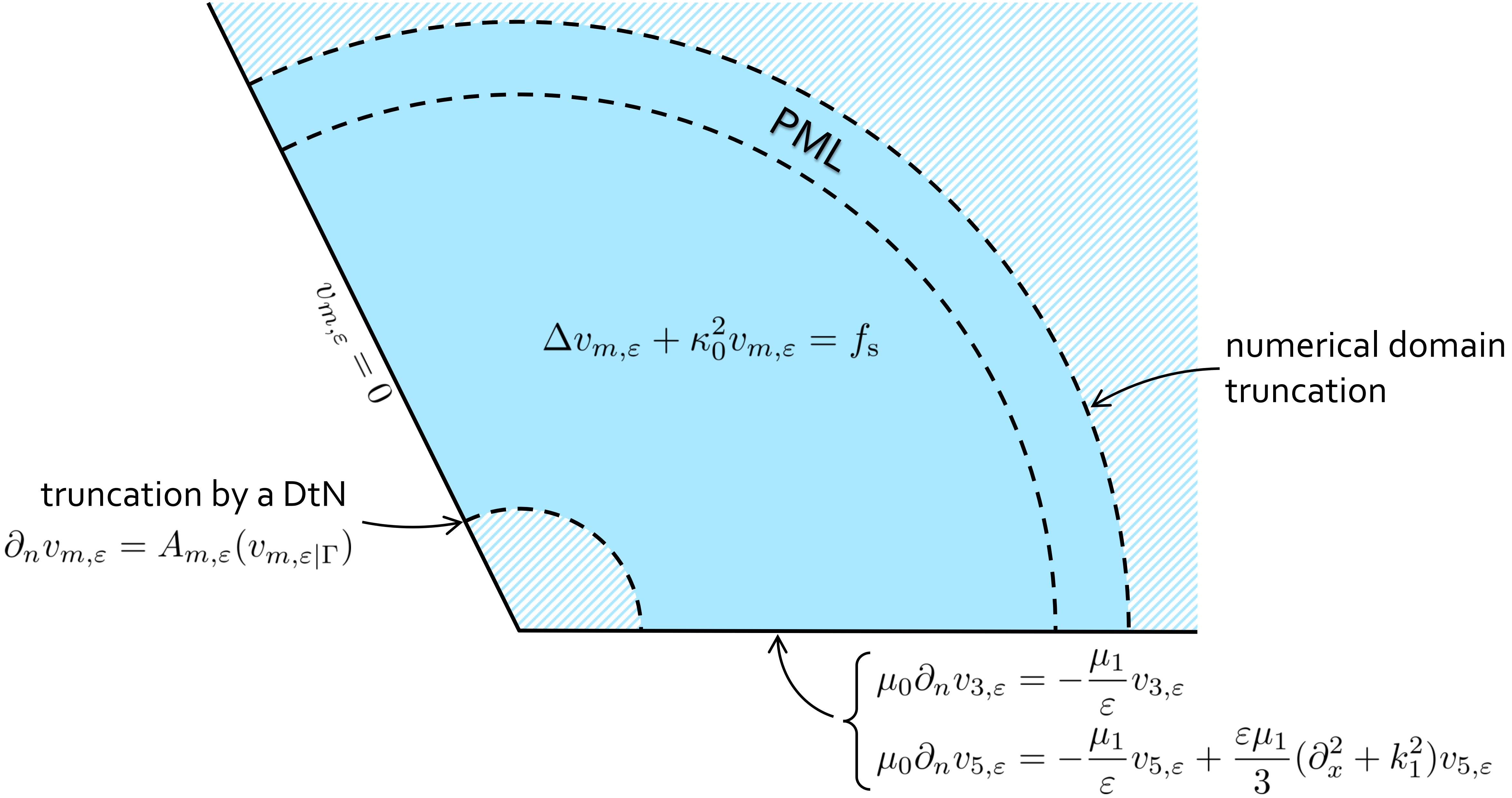
$$\Delta v_{m,\varepsilon} + \kappa_0^2 v_{m,\varepsilon} = f_{\mathrm{s}}$$

numerical domain truncation

$$\begin{cases} \mu_0 \partial_n v_{3,\varepsilon} = -\dfrac{\mu_1}{\varepsilon} v_{3,\varepsilon} \\ \mu_0 \partial_n v_{5,\varepsilon} = -\dfrac{\mu_1}{\varepsilon} v_{5,\varepsilon} + \dfrac{\varepsilon \mu_1}{3} (\partial_x^2 + k_1^2) v_{5,\varepsilon} \end{cases}$$

Previous attempt:
[Auvray, Vial 18, 19]

Effective models



PML

$$\Delta v_{m,\varepsilon} + \kappa_0^2 v_{m,\varepsilon} = f_{\mathrm{s}}$$

$v_{m,\varepsilon} = 0$

numerical domain truncation

truncation by a DtN
$\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

$$\begin{cases} \mu_0 \partial_n v_{3,\varepsilon} = -\dfrac{\mu_1}{\varepsilon} v_{3,\varepsilon} \\ \mu_0 \partial_n v_{5,\varepsilon} = -\dfrac{\mu_1}{\varepsilon} v_{5,\varepsilon} + \dfrac{\varepsilon \mu_1}{3} (\partial_x^2 + k_1^2) v_{5,\varepsilon} \end{cases}$$

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \, u_{p,\ell}$

Step 1: modal decomposition $\tilde{u}_{m,\varepsilon} = \sum (\text{unknown coef}) \times (\text{known function})$

Step 2: build $A_{m,\varepsilon}$ such that $A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon}) = \sum (\text{same coef}) \times \partial_n (\text{same function})$

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^{\ell}\varepsilon \, u_{p,\ell}$

<u>Step 1:</u> modal decomposition $\tilde{u}_{m,\varepsilon} = \sum (\text{unknown coef}) \times (\text{known function})$

Reminder: $u_{p,\ell} \underset{r\to 0}{\approx} \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(u_{p,\ell}) \, r^d \sin(d\theta) +$ explicit

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \, u_{p,\ell}$

<u>Step 1:</u> modal decomposition $\tilde{u}_{m,\varepsilon} = \sum (\text{unknown coef}) \times (\text{known function})$

Reminder: $u_{p,\ell} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(u_{p,\ell})\, r^d \sin(d\theta) +$ explicit

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

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<u>Step 1:</u> modal decomposition $\tilde{u}_{m,\varepsilon} = \sum (\text{unknown coef}) \times (\text{known function})$

Reminder: $u_{p,\ell} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(u_{p,\ell})\, r^d \sin(d\theta) +$ explicit

So $\tilde{u}_{m,\varepsilon} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \left(\sum_{p<m} \sum_{\ell} \varepsilon^p \ln^{\ell}\varepsilon \cdot \sigma_d(u_{p,\ell}) \right) \cdot \Phi_{m,\varepsilon,d}$ → $= r^d \sin(d\theta) + \ldots$

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \, u_{p,\ell}$

<u>Step 1:</u> modal decomposition $\tilde{u}_{m,\varepsilon} = \sum (\text{unknown coef}) \times (\text{known function})$

Reminder: $u_{p,\ell} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(u_{p,\ell})\, r^d \sin(d\theta) +$ explicit

So $$\tilde{u}_{m,\varepsilon} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \left(\sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \cdot \sigma_d(u_{p,\ell}) \right) \cdot \Phi_{m,\varepsilon,d} \longrightarrow = r^d \sin(d\theta) + \dots$$

using $$\underset{(d<0)}{\sigma_d(u_{p,\ell})} = \sum_{\substack{p',\ell' \\ p'<p}} \sum_{d' \in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(u_{p',\ell'})$$

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \, u_{p,\ell}$

<u>Step 1:</u> modal decomposition $\tilde{u}_{m,\varepsilon} = \sum (\text{unknown coef}) \times (\text{known function})$

Reminder: $u_{p,\ell} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(u_{p,\ell})\, r^d \sin(d\theta) +$ explicit

So
$$\begin{tikzcd}[column sep=small, row sep=small] \tilde{u}_{m,\varepsilon} = \displaystyle\sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \Big(\sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \cdot \sigma_d(u_{p,\ell}) \Big) \cdot \Phi_{m,\varepsilon,d} \arrow[dr, bend left] & \\ & = r^d \sin(d\theta) + \dots \\ = \displaystyle\sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} \Big(\sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \cdot \sigma_d(u_{p,\ell}) \Big) \cdot \Psi_{m,\varepsilon,d} \arrow[ur, bend right] & \end{tikzcd}$$

using $\underset{(d<0)}{\sigma_d(u_{p,\ell})} = \sum_{\substack{p',\ell' \\ p'<p}} \sum_{d' \in A_{p,p',d}} c_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(u_{p',\ell'})$

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^{\ell} \varepsilon \, u_{p,\ell}$

Step 1: modal decomposition $\tilde{u}_{m,\varepsilon} = \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} s_{m,\varepsilon,d} \, \Psi_{m,\varepsilon,d}$

Step 2: $(\Psi_{m,\varepsilon,d})_d$ is orthonormalized by Gram-Schmidt

We build maps $t_{m,\varepsilon,d} : \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} s_d \, \Psi_{m,\varepsilon,d} \mapsto s_d$

We define $A_{m,\varepsilon} : v \mapsto \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} t_{m,\varepsilon,d}(v) \cdot \partial_n \Psi_{m,\varepsilon,d}$

Construction of the DtN operator in $\partial_n v_{m,\varepsilon} = A_{m,\varepsilon}(v_{m,\varepsilon|\Gamma})$

We want $\partial_n \tilde{u}_{m,\varepsilon} = A_{m,\varepsilon}(\tilde{u}_{m,\varepsilon|\Gamma}) + O(\varepsilon^m)$ with $\tilde{u}_{m,\varepsilon} := \sum_{p<m} \sum_{\ell} \varepsilon^p \ln^\ell \varepsilon \, u_{p,\ell}$

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We define $A_{m,\varepsilon} : v \mapsto \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} t_{m,\varepsilon,d}(v) \cdot \partial_n \Psi_{m,\varepsilon,d}$

Only a finite number of d is considered → the DtN introduces an error $O\left(\left(\frac{R_{\mathrm{DtN}}}{R_{\mathrm{source}}} \right)^{N \frac{\pi}{\Theta}} \right)$

Effective models

$$= \begin{cases} 1 & \text{for } m = 3 \\ 4/3 & \text{for } m = 5 \end{cases}$$

<u>Theorem:</u> stability condition $\varepsilon \leqslant C\,\dfrac{(R_{\mathrm{DtN}})^a}{N^3\,[\ln R_{\mathrm{DtN}}]}$

polynomial

$\Rightarrow$ effective model well posed and uniformly stable w.r.t. ε

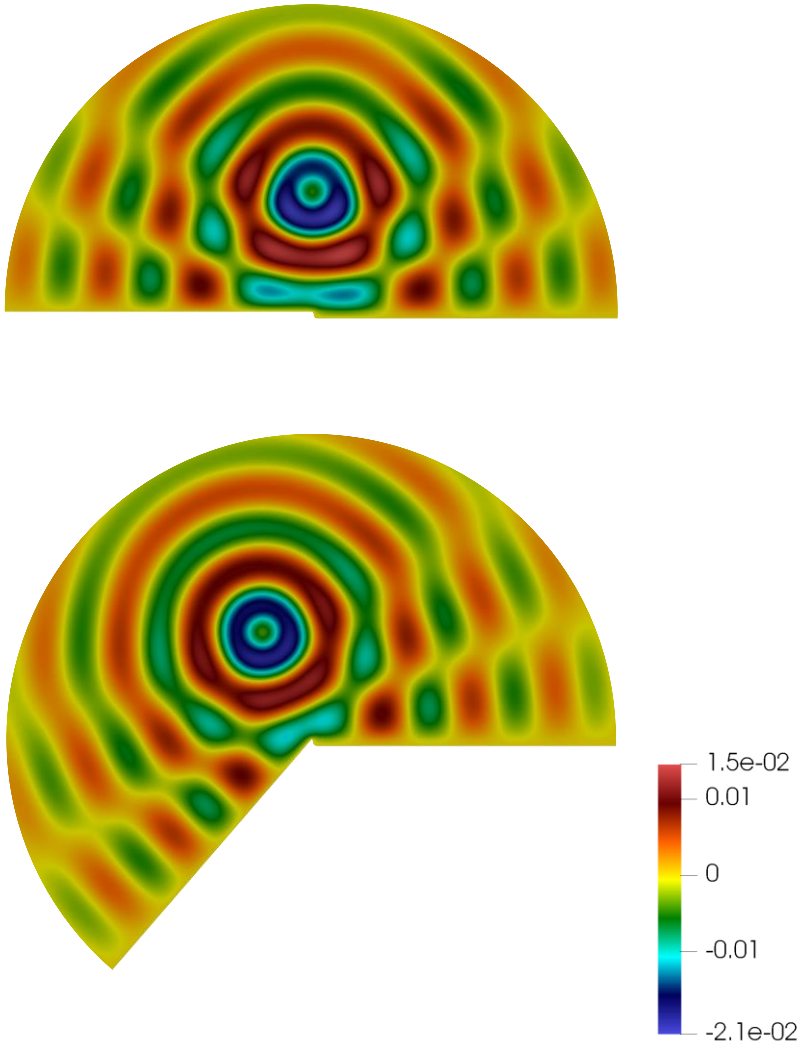
<u>Theorem:</u> $\|v_{m,\varepsilon} - u_\varepsilon\|_{H^1} = O(\varepsilon^m\,[\ln \varepsilon])$

with $N := \left\lceil m\,\frac{\Theta}{\pi}\log_2 \varepsilon \right\rceil$ and $R_{\mathrm{DtN}} \leqslant R_{\mathrm{source}}/2$

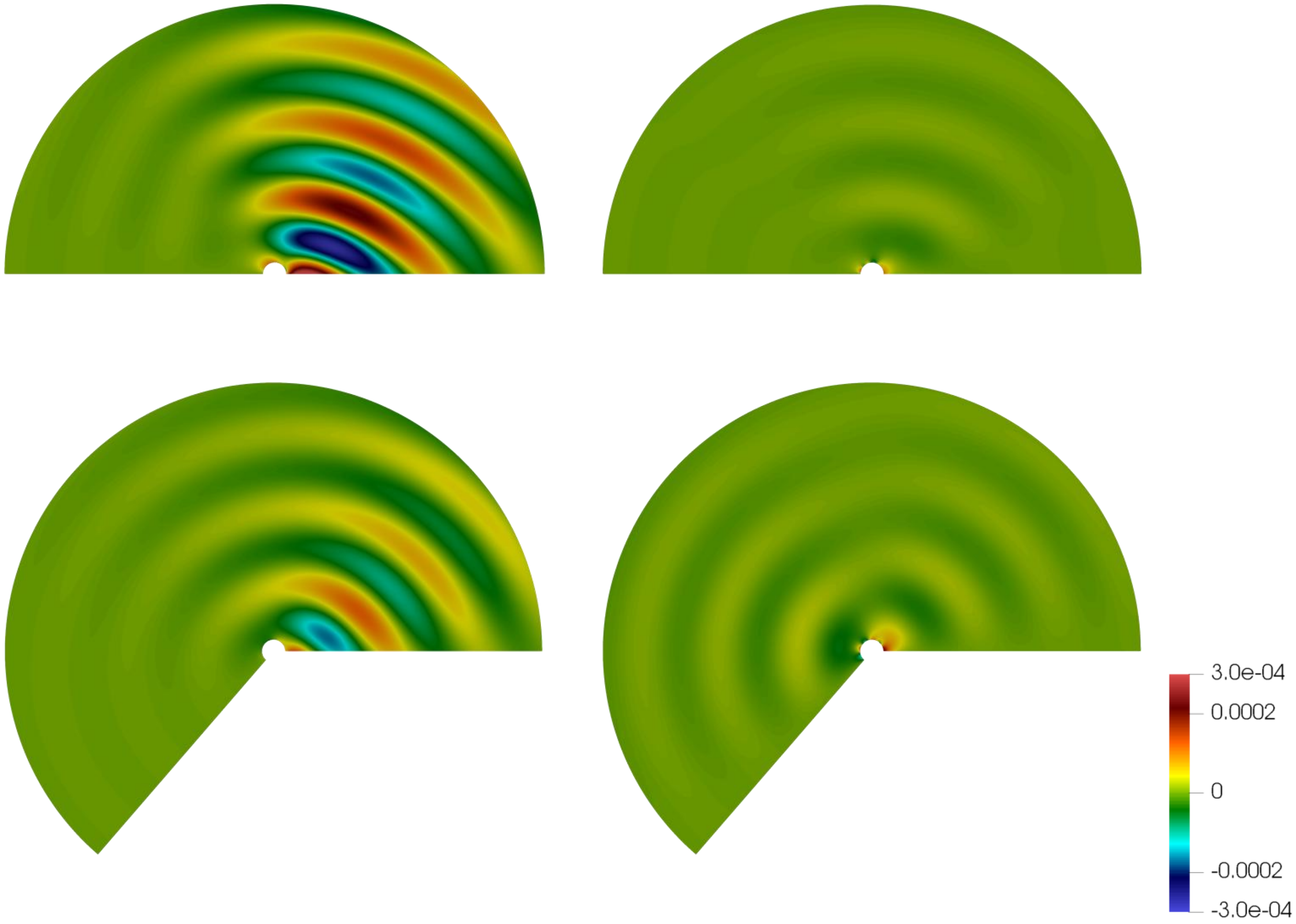
It comes from a more general estimate: $\|v_{m,\varepsilon} - u_\varepsilon\|_{H^1} \lesssim \left(\dfrac{\varepsilon}{R_{\mathrm{DtN}}}\right)^m [\ln \varepsilon]\,N^3 + \left(\dfrac{R_{\mathrm{DtN}}}{R_{\mathrm{source}}}\right)^{N\frac{\pi}{\Theta}}$

Numerical results

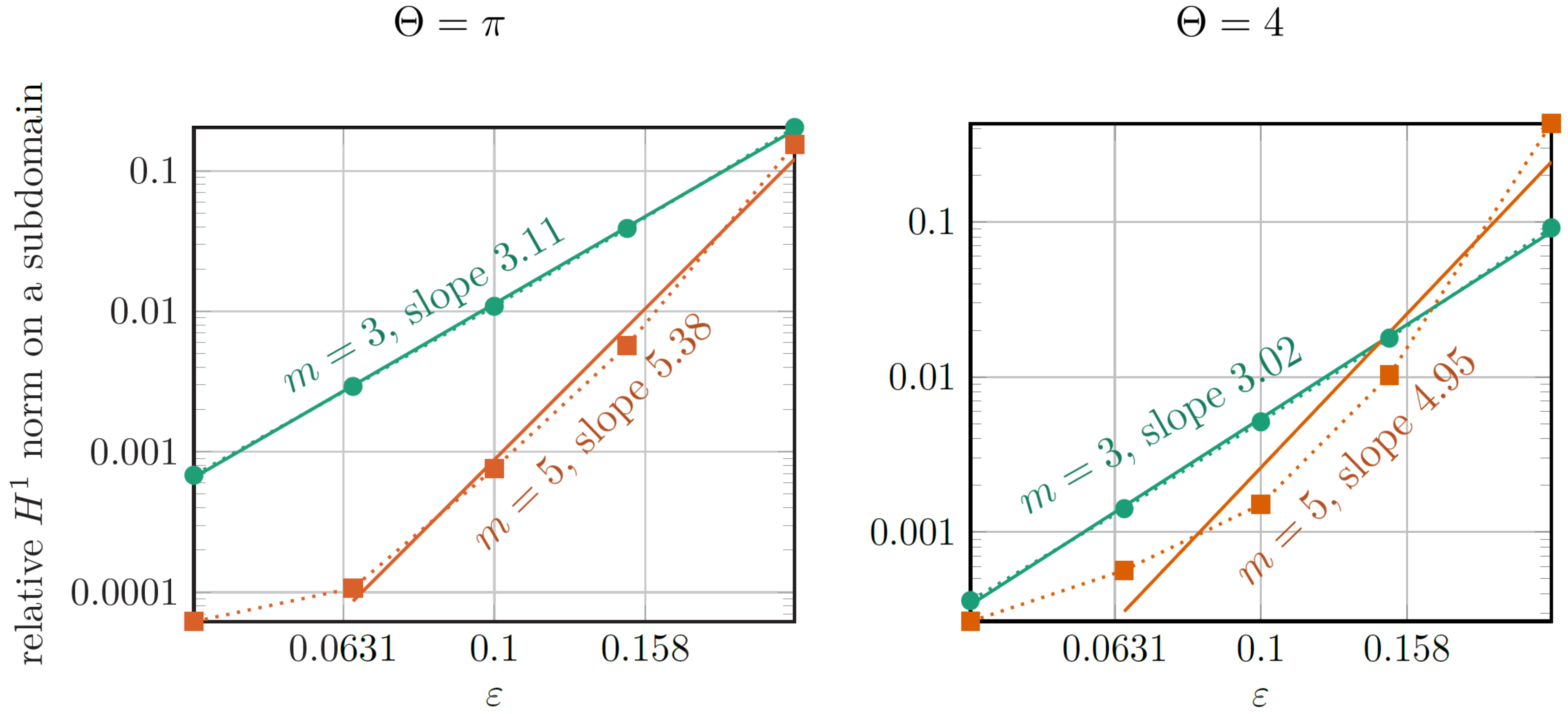
Exact solutions



Errors for $m = 3$ and 5



Numerical results



Short-term projects

Neumann boundary condition

Long-term projects

Periodic layers

Maxwell’s equations

Thank you for your attention !