CANUM 2026

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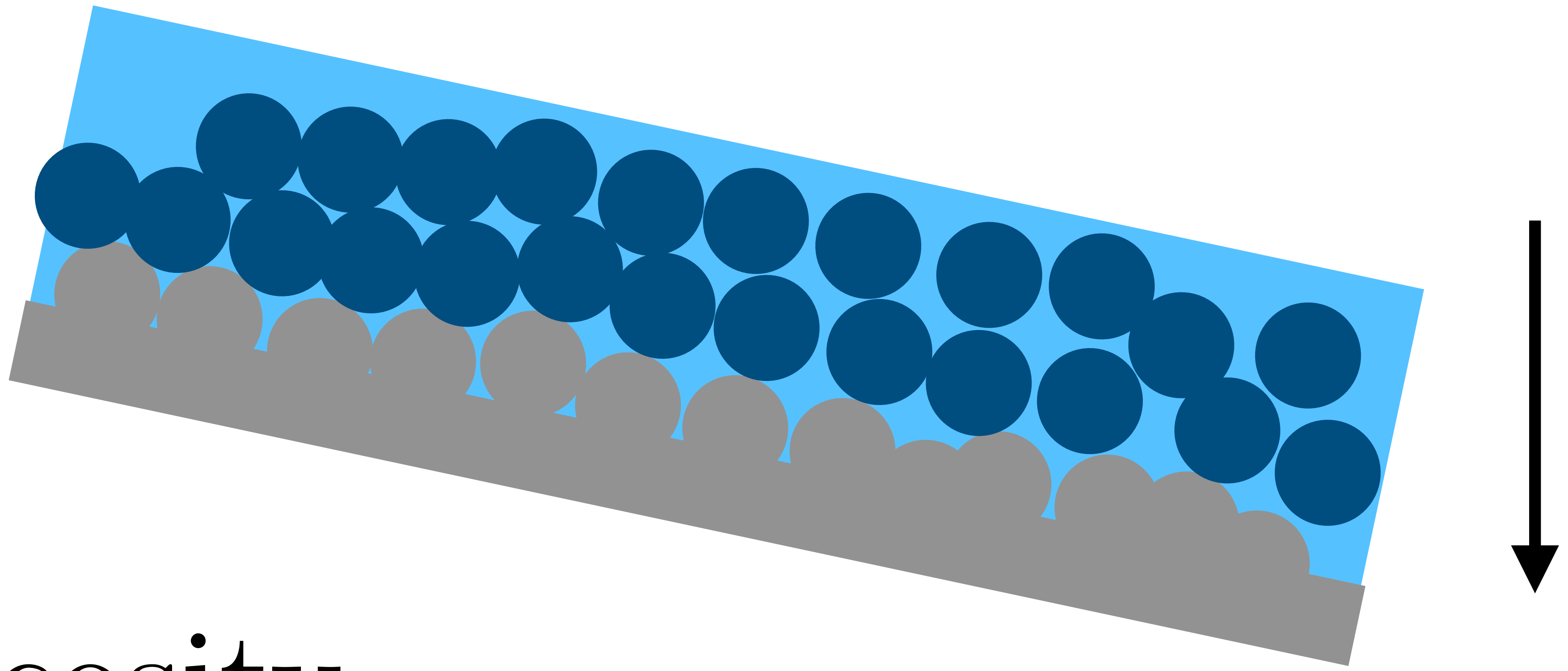
Context

A Mudflow provocated by rain

Particles in a viscous fluid



$g = \text{Gravitational Force}$

$\eta = \text{Viscosity}$

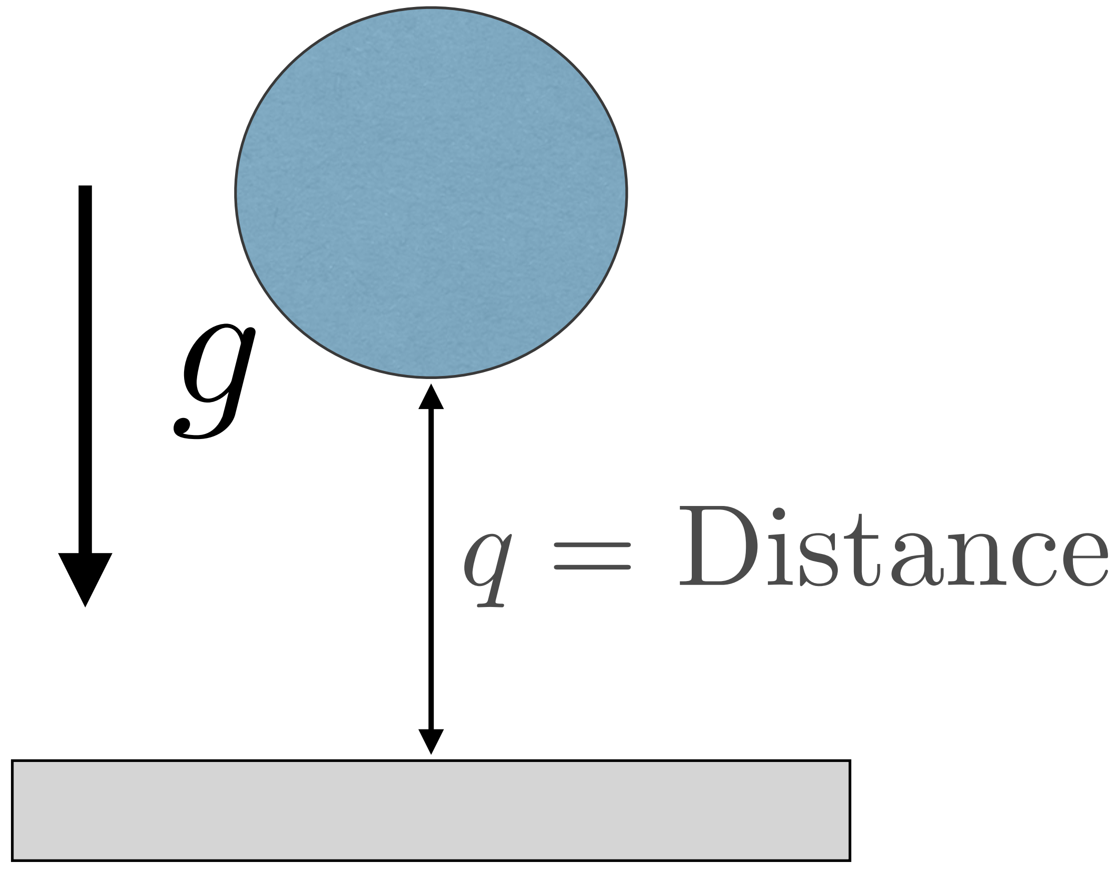
$\mu = \text{Friction Coefficient}$

Forces affecting macroscopic behavior :

- Lubrication
- Solid Contact
- Frictional Contact

A stiff problem

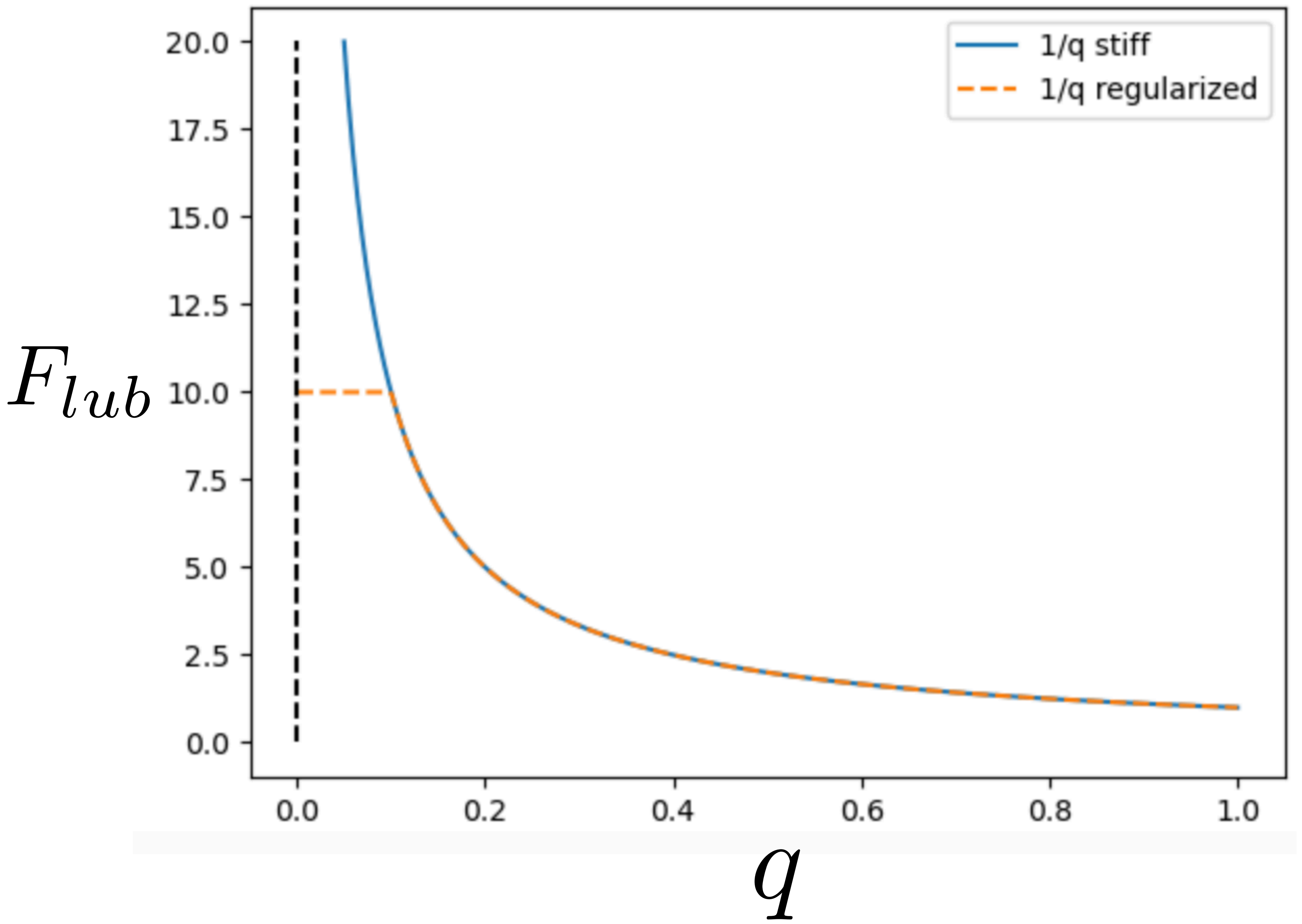
Lubrication force



$$F_{lub}(q) = -6\pi\eta r^2 \frac{\dot{q}}{q}$$

Two methods

- DEM : Explicit Force
  - No Lubrication
  - Regularization



- Contact Dynamics: No Explicit Force

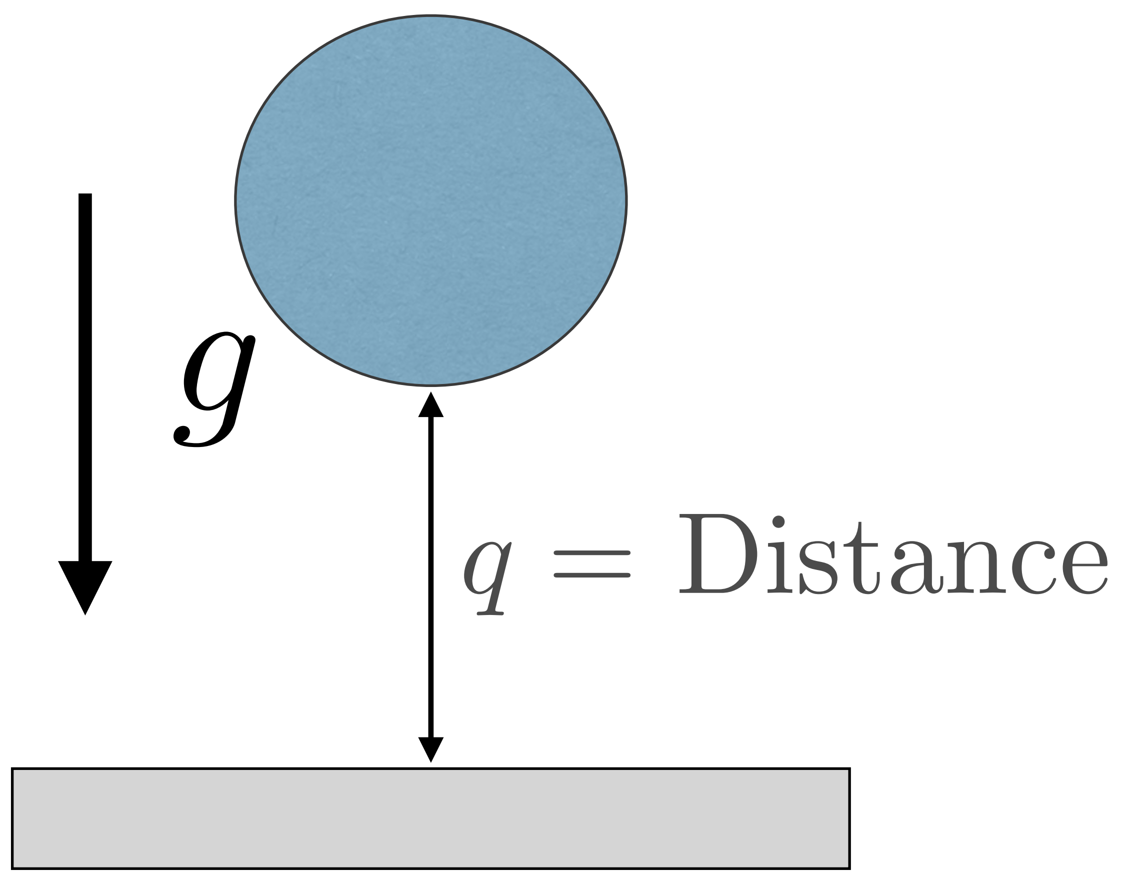
$\Longrightarrow$

- Taking lubrication into account until contact
- Numerical Scheme Without Stiffness

- Cox, R. G. (1974). The motion of suspended particles almost in contact. *International Journal of Multiphase Flow*

Modeling lubrication: the gluey contact model

Lubrication force



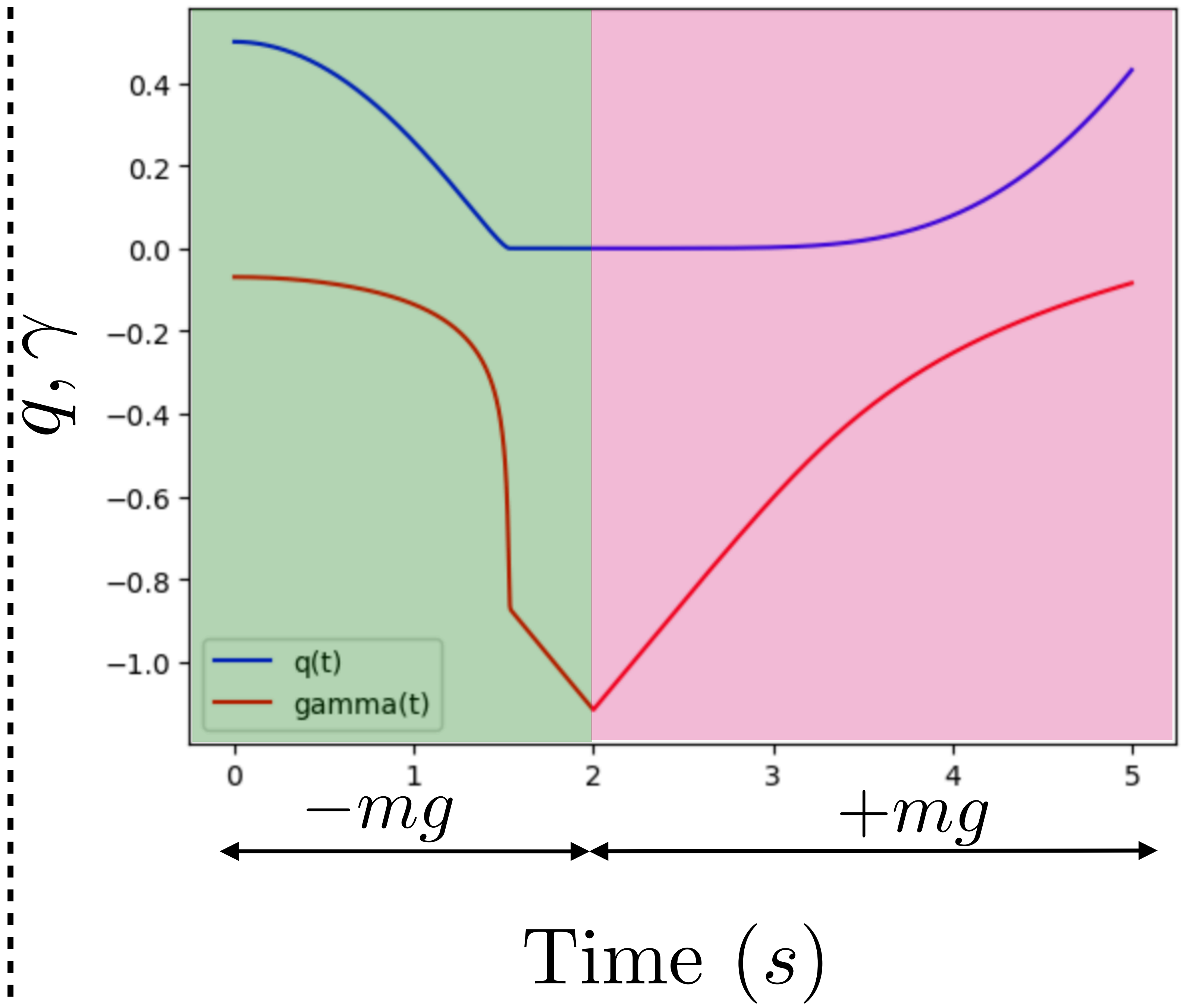
$$F_{lub}(q) = -6\pi\eta r^2 \frac{\dot{q}}{q}$$

$$m\ddot{q} = \begin{cases} -mg - 6\pi\eta r^2 \frac{\dot{q}}{q} & \text{if } t < 2 \\ +mg - 6\pi\eta r^2 \frac{\dot{q}}{q} & \text{if } t \geq 2 \end{cases}$$

Pushing phase

Pulling phase

Solving Differential Equation



$$\gamma = \eta ln(q)$$

• Cox, R. G. (1974). The motion of suspended particles almost in contact. *International Journal of Multiphase Flow*

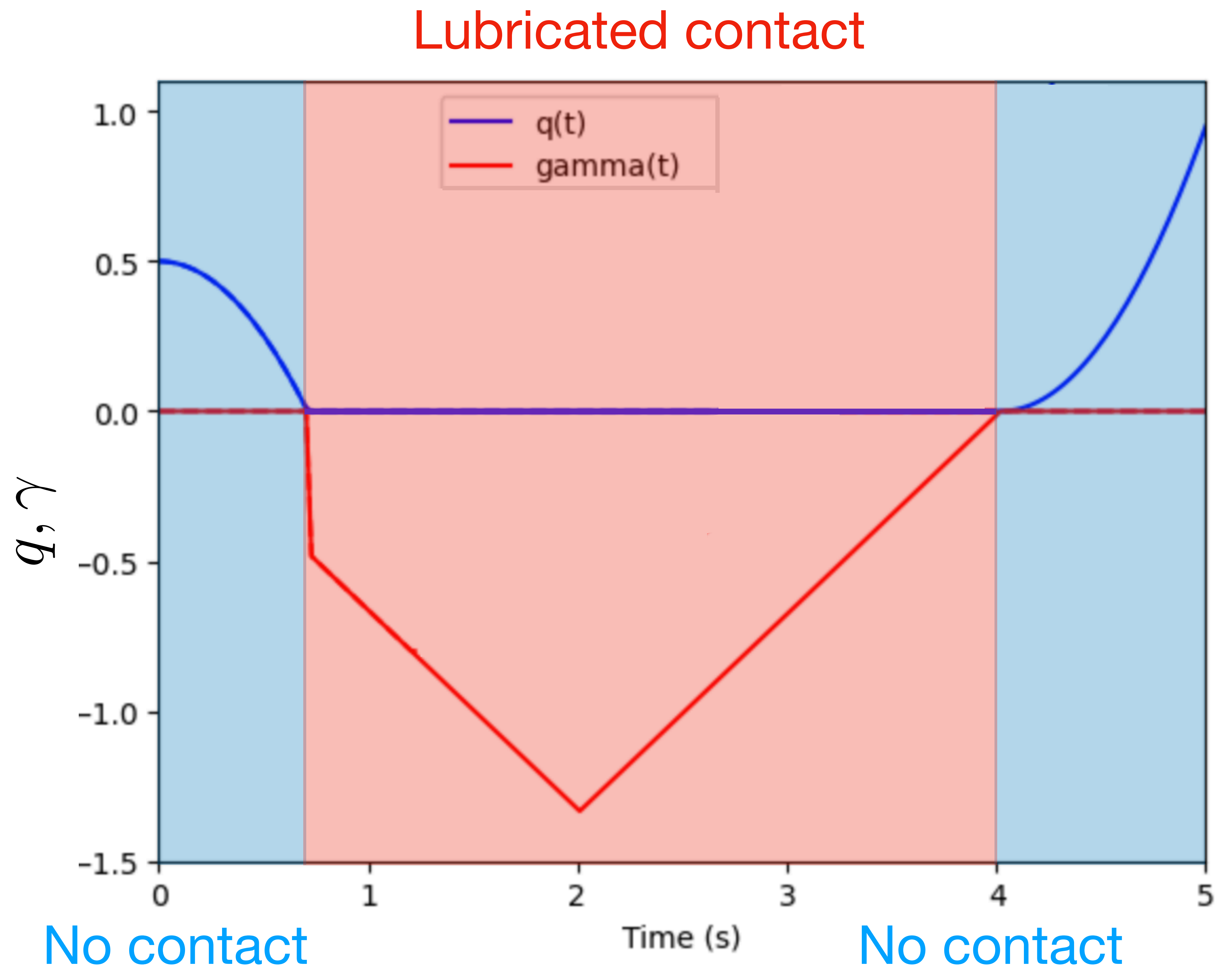
Modeling lubrication: the gluey contact model

Impact Law

$$\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$$

$$C_{q,\gamma} = \begin{cases} \{0\} & \text{si} \quad \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} \quad \gamma^- = 0 \\ & \qquad\quad q = 0 \\ \mathbb{R} & \text{sinon} \end{cases}$$

Adapted model with solid contact



Modeling lubrication: the gluey contact model

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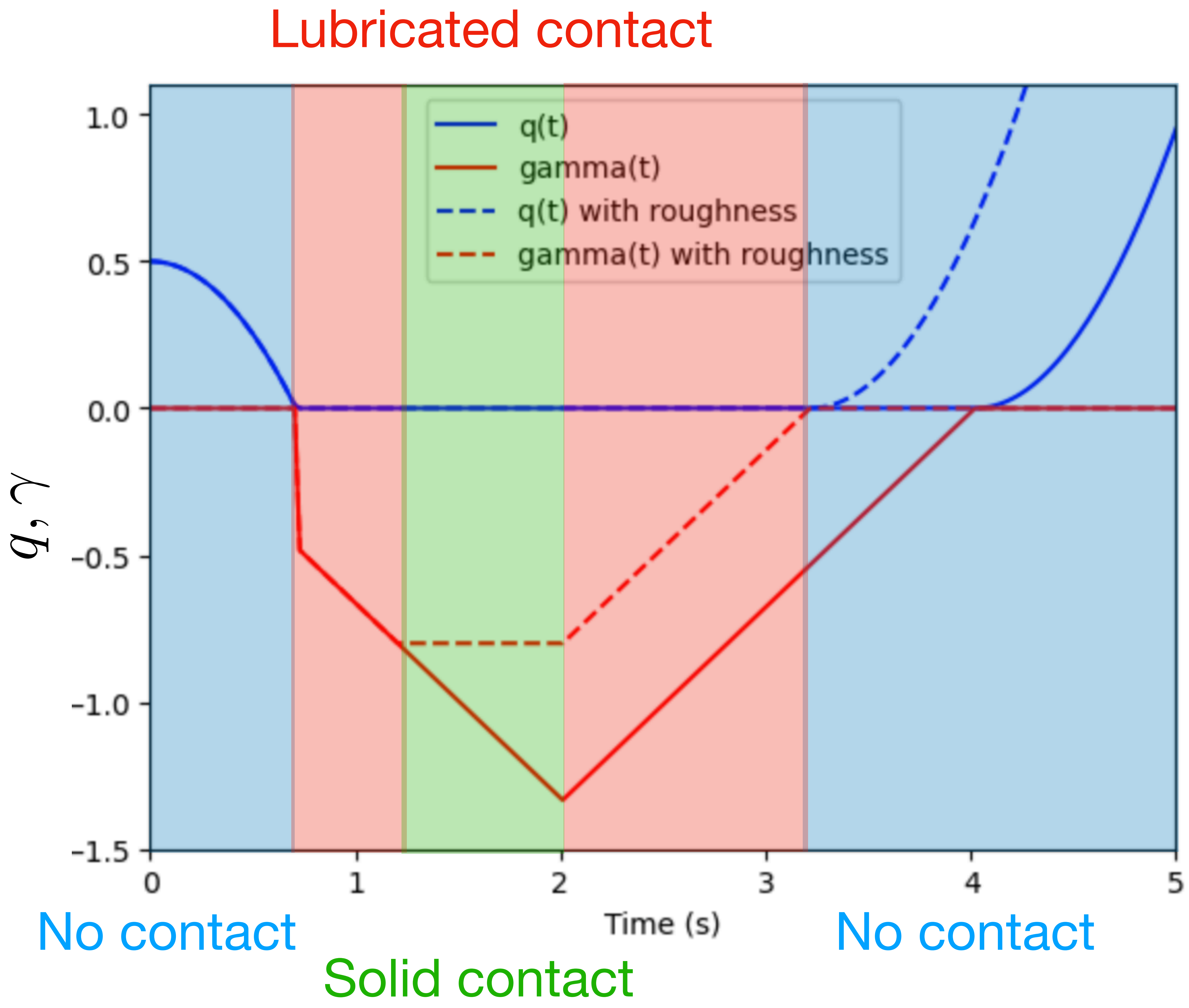

Modeling lubrication: the gluey contact model

Impact Law

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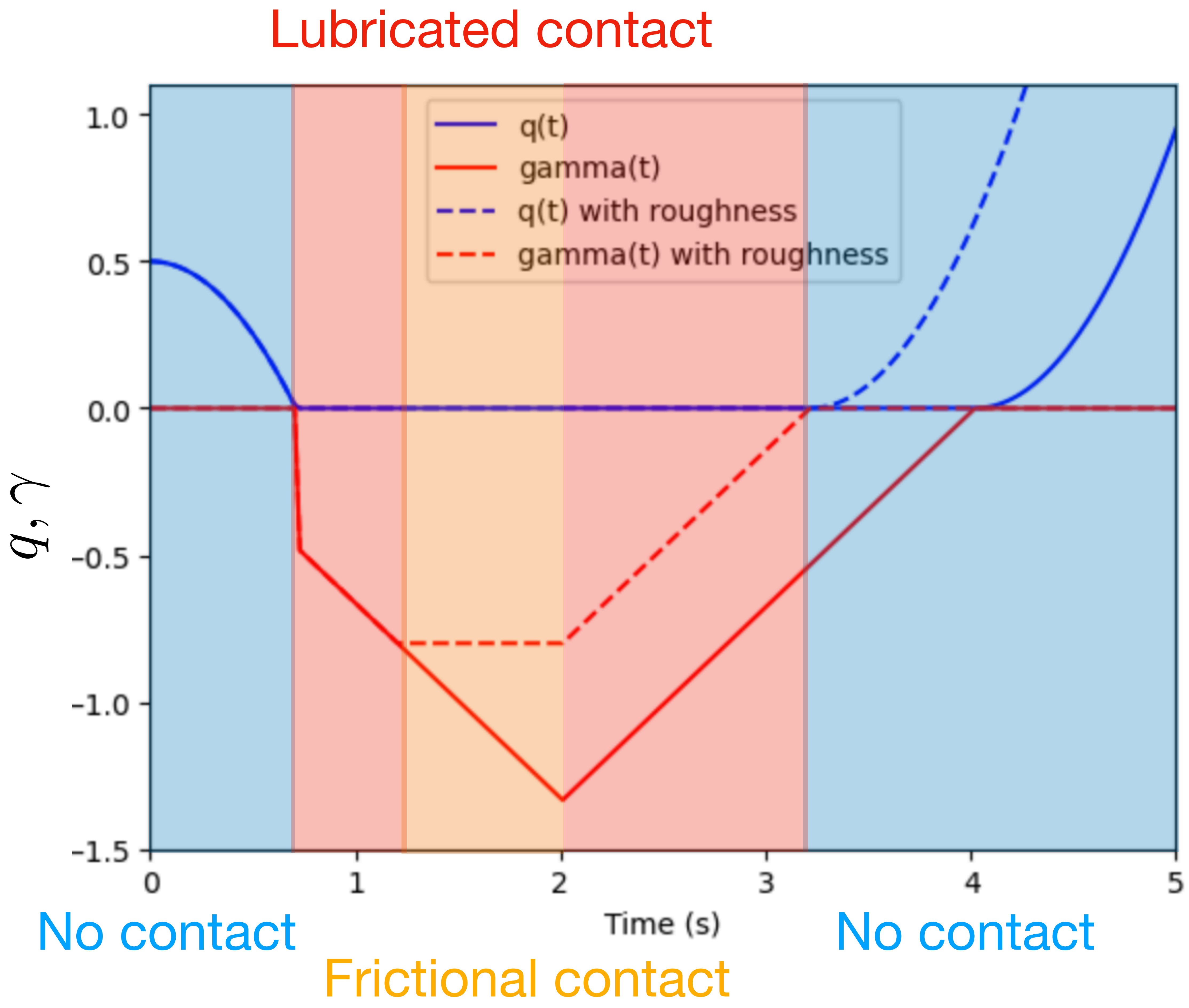
Adapted model with solid contact



Discretizing the continuous model : Viscosity Without Friction

A discrete minimisation problem $\dot{q}^+ = P_{C_{q,\gamma}} \dot{q}^-$

$$\min_{\mathbf{u} \in K^k} J(\mathbf{u})$$

$$J(\mathbf{u}) = \frac{1}{2}|\mathbf{u} - \mathbf{U}^{k+1}|^2, \; \mathbf{U}^{k+1} = \mathbf{u}^k + \Delta t \frac{1}{m}\mathbf{F}^k$$

$$K^k = \mathbf{u} \in \mathbb{R}^2 \text{ s t } \left| \begin{array}{l} -D - \Delta t \mathbf{u_n} \leq 0 \\ D + \Delta t \mathbf{u_n} \leq 0, \text{ si } \gamma^k < 0 \end{array} \right.$$

$\Longrightarrow$

A dual problem

$$\min_{(f^+, f^-) \in K} E(f^+, f^-)$$

$$\mathbf{u_f} = \mathbf{U}^{k+1} + \Delta t \tfrac{1}{m}(f^+ - f^-).\mathbf{n}$$

$$E(f^+, f^-) = (f^+ - f^-)[(D^k + \Delta t(\tfrac{\mathbf{u_f}_n + \mathbf{U_n}}{2})] + cst$$

$$K = \{(f^+, f^-), \; f^+ \geq 0, \; f^- \geq 0\}$$

Convergence results have been established

Discretizing the continuous model

$\Delta t = \text{Time Step}$

A unified formulation of the problem

$$\underset{\mathbf{f}\in K}{min}\ E(\mathbf{f}) = f_n[D^k + \Delta t\frac{\mathbf{U}_f+\mathbf{U}}{2}\cdot\mathbf{n}] + \Delta t\mathbf{f}_\mathbf{T}\cdot\frac{\mathbf{U}_f+\mathbf{U}}{2}$$

Inelastic

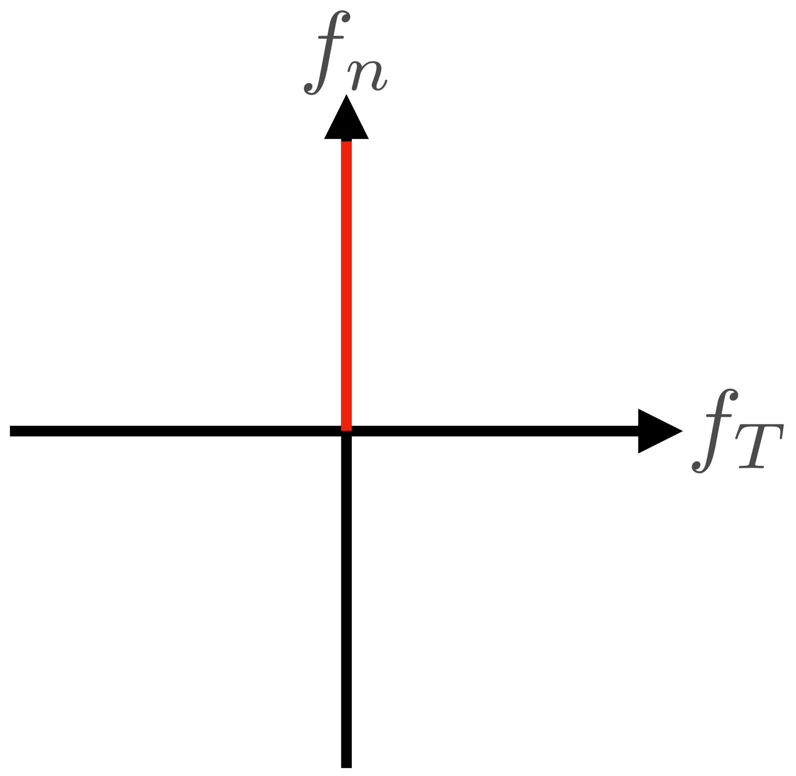
$$K = \{\mathbf{f} = f^+\mathbf{n},\ f^+\in\mathbb{R}^+\}$$

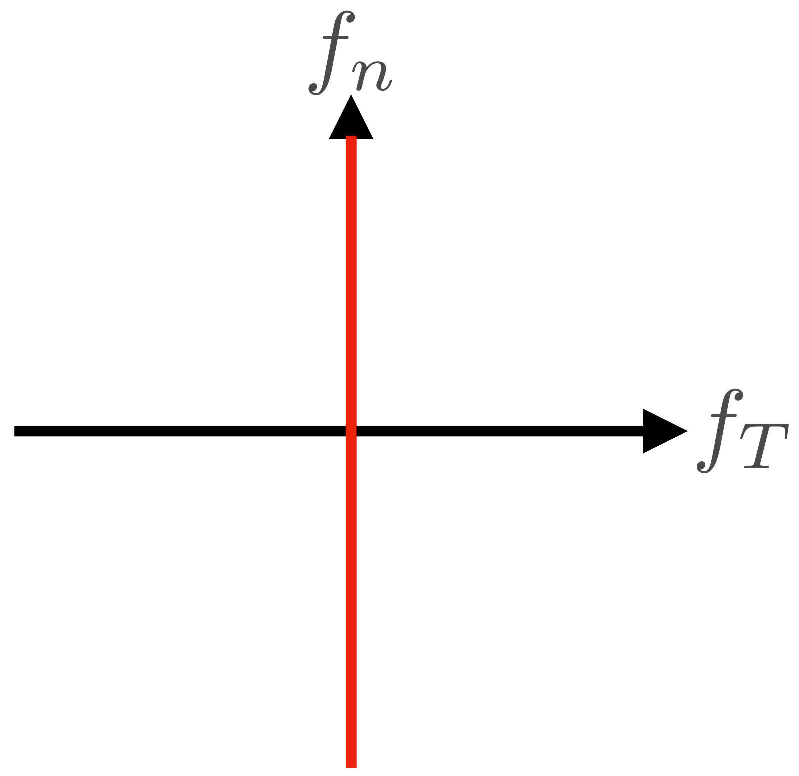
Lubrication

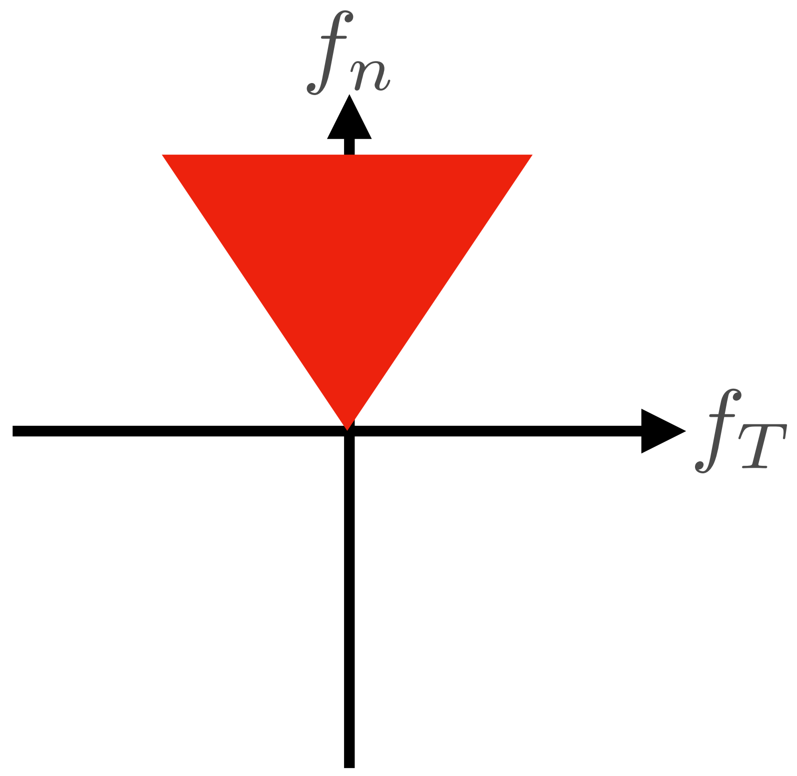
$$K = \{\mathbf{f} = (f^+ - f^-)\mathbf{n},\ (f^+, f^-)\in\mathbb{R}^{+2}\}$$

Friction

$$K = \{\mathbf{f} = (f_n)\mathbf{n} + \mathbf{f}_\mathbf{T},\ |\mathbf{f}_\mathbf{T}|\leq\mu f_n\}$$







Coulomb Friction

Solved using a Projected Gradient Method

An Issue With the Frictional Formulation

The dual problem for friction

$$\min_{(f_n, \mathbf{f_T}) \in K^k} E(f_n, \mathbf{f_T})$$

$$\mathbf{u_f} = \mathbf{U}^{k+1} + \Delta t \tfrac{1}{m}(f_n.\mathbf{n} + \mathbf{f_T})$$

$$E(f_n, \mathbf{f_T}) = f_n[(D^k + \Delta t(\tfrac{\mathbf{u_{fn}} + \mathbf{U}_n}{2})] + \langle \mathbf{f_T}, \Delta t \tfrac{\mathbf{u_{fT}} + \mathbf{U}_T}{2}) \rangle$$

$$K^k = \{(f_n, \mathbf{f_T}),\ \mu f_n \geq |\mathbf{f_T}|\}$$

$$\Longrightarrow$$

The primal problem

$$\min_{\mathbf{u} \in K^k} J(\mathbf{u})$$

$$J(\mathbf{u}) = \frac{1}{2}|\mathbf{u} - \mathbf{U}^{k+1}|^2,\ \mathbf{U}^{k+1} = \mathbf{u}^k + \Delta t \frac{1}{m}\mathbf{F}^k$$

$$K^k \quad = \{\mathbf{u} \in \mathbb{R}^2,\ D + \Delta t \mathbf{u_n} \geq \mu \Delta t |\mathbf{u_t}|\}$$

- D.E. Stewart. Convergence of a Time-Stepping Scheme for Rigid-Body Dynamics and Resolution of Painleve's Problem. Archive for Rational Mechanics and Analysis
- V. Acary, F. Cadoux, C. Lemarechal, and J. Malick. A formulation of the linear discrete coulomb friction problem via convex optimization. ZAMM-Journal of Applied Mathematics and Mechanics

An Issue With the Frictional Formulation

The dual problem for friction with fixed point

$$\min_{(f_n,\mathbf{f_T})\in K^k} E(f_n,\mathbf{f_T})$$

$$\mathbf{u_f} = \mathbf{U}^{k+1} + \Delta t \tfrac{1}{m}(f_n.\mathbf{n} + \mathbf{f_T})$$

$$E(f_n,\mathbf{f_T}) = f_n[(D^k + s + \Delta t(\tfrac{\mathbf{u_{fn}}+\mathbf{U}_n}{2})] + \langle \mathbf{f_T}, \Delta t \tfrac{\mathbf{u_{fT}}+\mathbf{U}_T}{2})\rangle$$

$$K^k = \{(f_n,\mathbf{f_T}),\ \mu f_n \geq |\mathbf{f_T}|\}$$

$$F(s) = \mu \Delta t |\mathbf{u_T^s}|$$

$\Longrightarrow$

The primal problem with fixed point

$$\min_{\mathbf{u}\in K_s^k} J(\mathbf{u})$$

$$J(\mathbf{u}) = \frac{1}{2}|\mathbf{u} - \mathbf{U}^{k+1}|^2,\ \mathbf{U}^{k+1} = \mathbf{u}^k + \Delta t \frac{1}{m}\mathbf{F}^k$$

$$K_s^k = \{\mathbf{u} \in \mathbb{R}^2,\ D + \Delta t \mathbf{u_n} \geq \mu \Delta t |\mathbf{u_t}| - s\}$$

$$F(s) = \mu \Delta t |\mathbf{u_T^s}|$$

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Discretizing the continuous model

$\Delta t = \text{Time Step}$

A unified formulation of the problem

$$\min_{\mathbf{f} \in K} E(\mathbf{f}) = f_n[D^k + \Delta t \mathbf{U}_f \cdot \mathbf{n}] + \Delta t \mathbf{f_T} \cdot \mathbf{U}_f$$

Inelastic

$$K = \{\mathbf{f} = f^+ \mathbf{n},\ f^+ \in \mathbb{R}^+\}$$

Lubrication

$$K = \{\mathbf{f} = (f^+ - f^-)\mathbf{n},\ (f^+, f^-) \in \mathbb{R}^{+2}\}$$

Friction

$$K = \{\mathbf{f} = (f_n)\mathbf{n} + \mathbf{f_T},\ |\mathbf{f_T}| \leq \mu f_n\}$$

Lubrication + Friction

Coupling

Coupling Lubrication And Friction

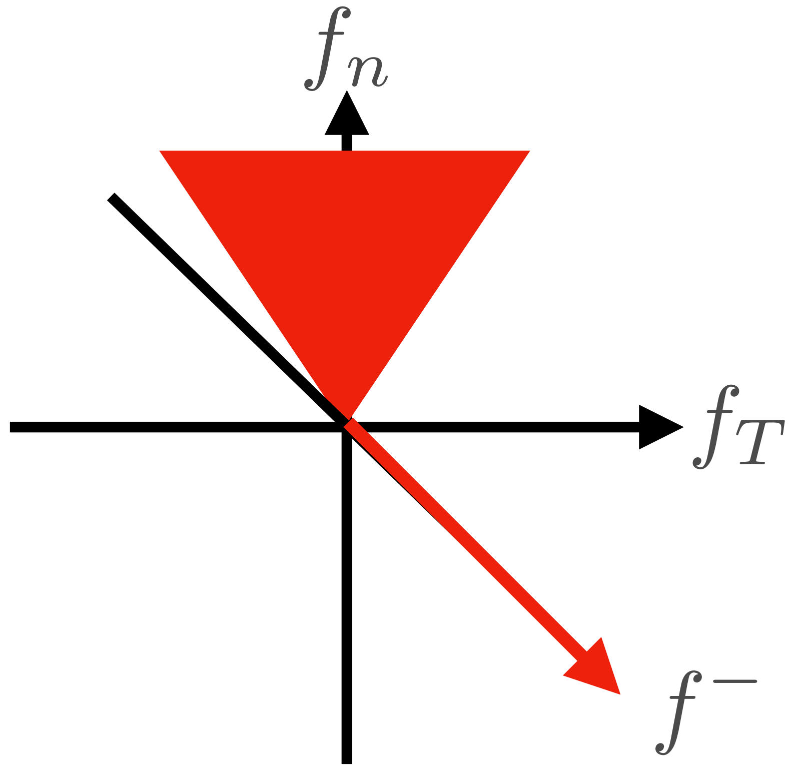
$\Delta t = \text{Time Step}$

A unified formulation of the problem

$$\min_{\mathbf{f}\in K} E(\mathbf{f}) = (f_n - f_-)[D^k + \Delta t \frac{\mathbf{U}_f + \mathbf{U}}{2} \cdot \mathbf{n}] + \Delta t \mathbf{f}_\mathbf{T} \cdot \frac{\mathbf{U}_f + \mathbf{U}}{2} + f_n s$$

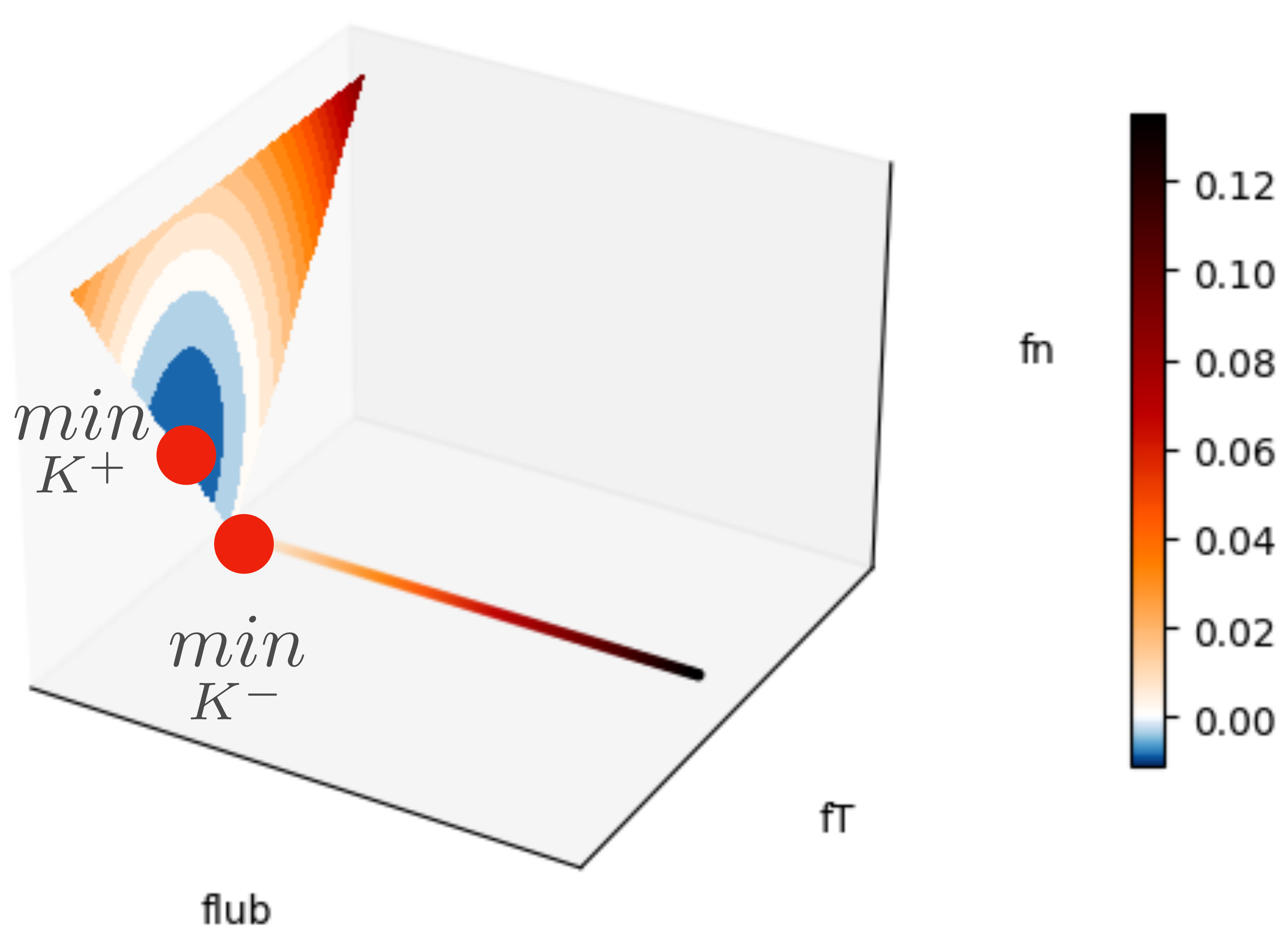
Lubrication + Friction

$$K = \{\mathbf{f} = (f_n - f^-)\mathbf{n} + \mathbf{f}_\mathbf{T},\ |\mathbf{f}_\mathbf{T}| \leq \mu f_n,\ f^- \geq 0,\ f^- f_n = 0\}$$

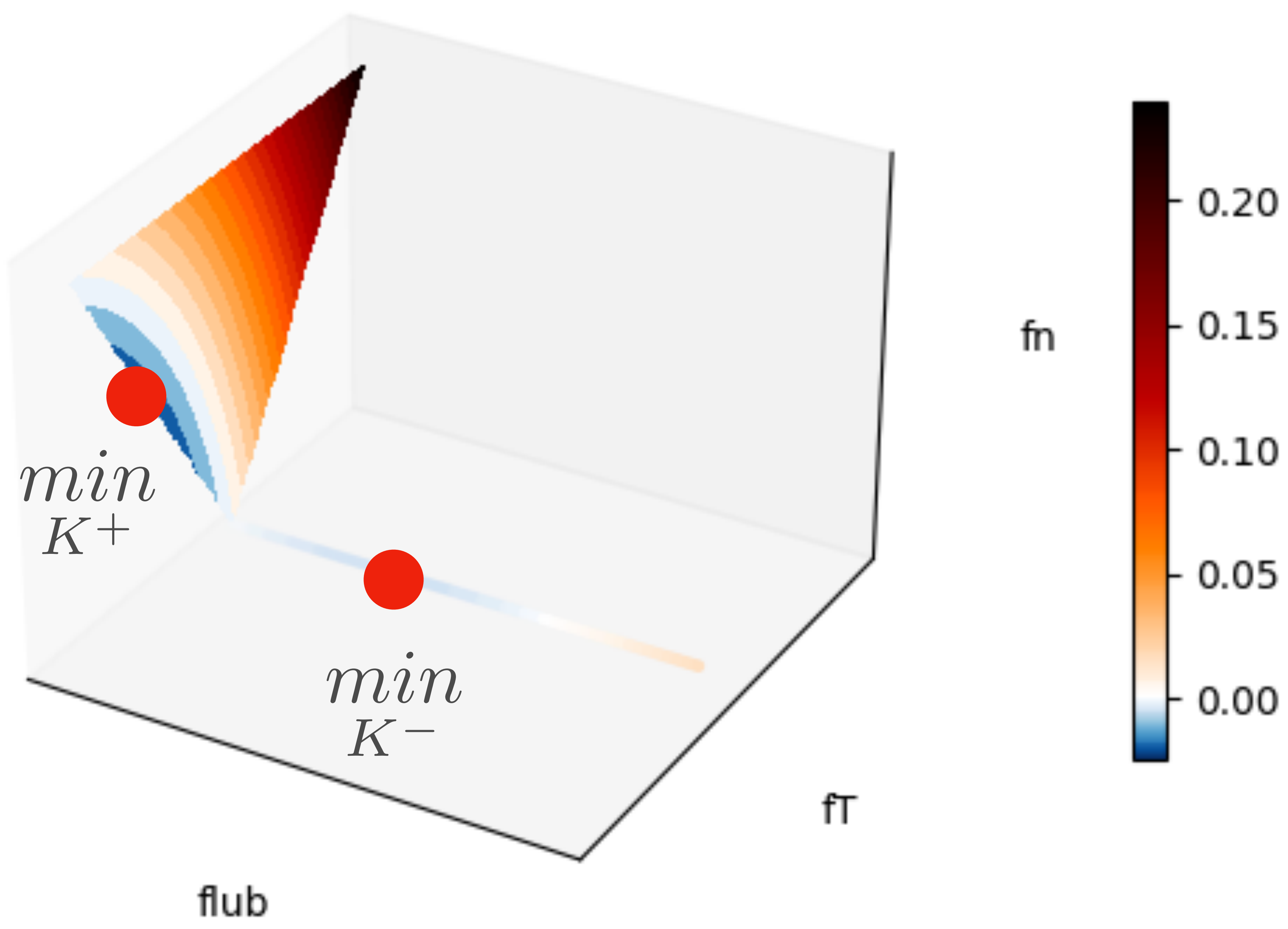


Convergence of the gradient algorithm : Idea of The Proof

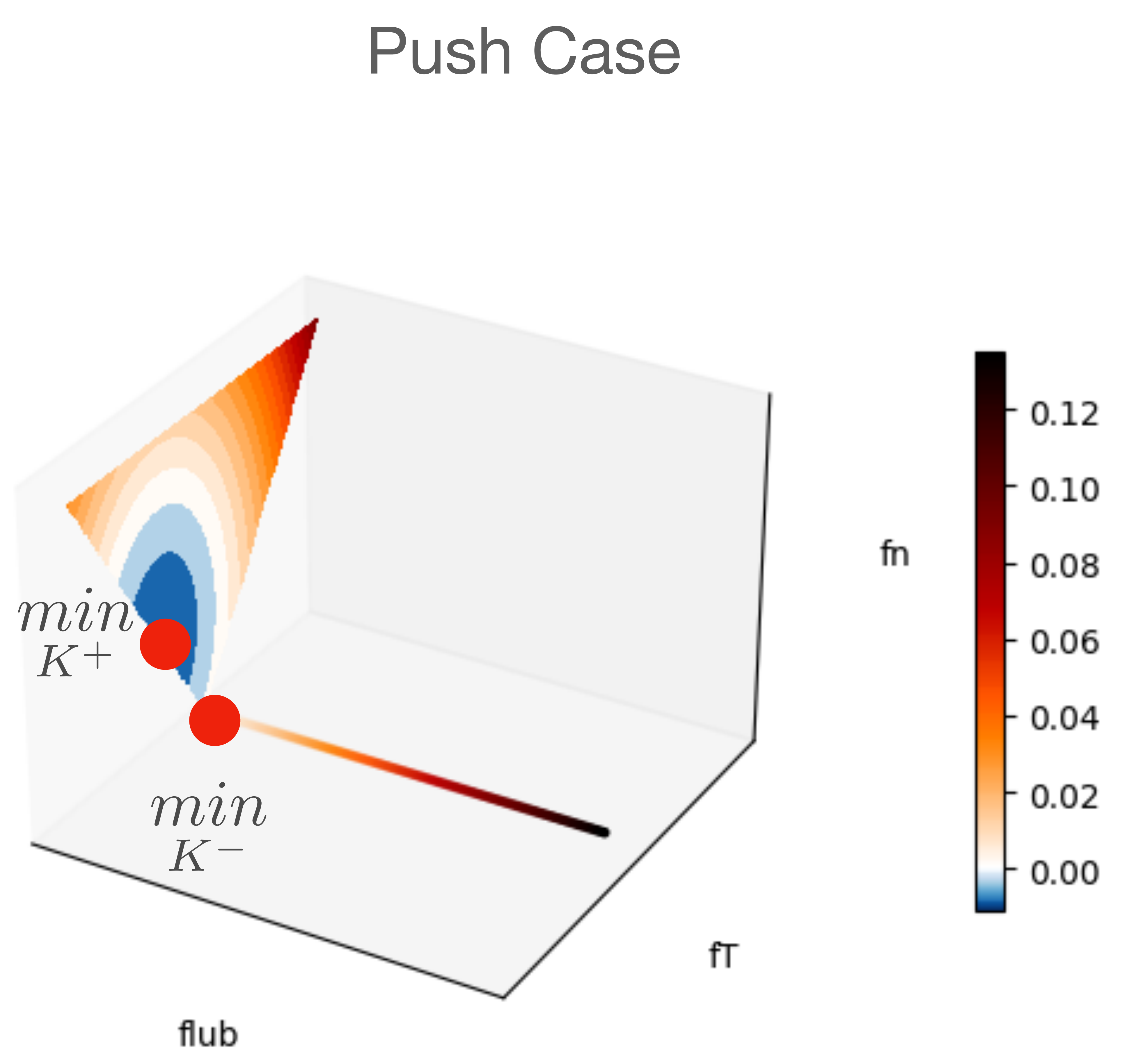
Push Case

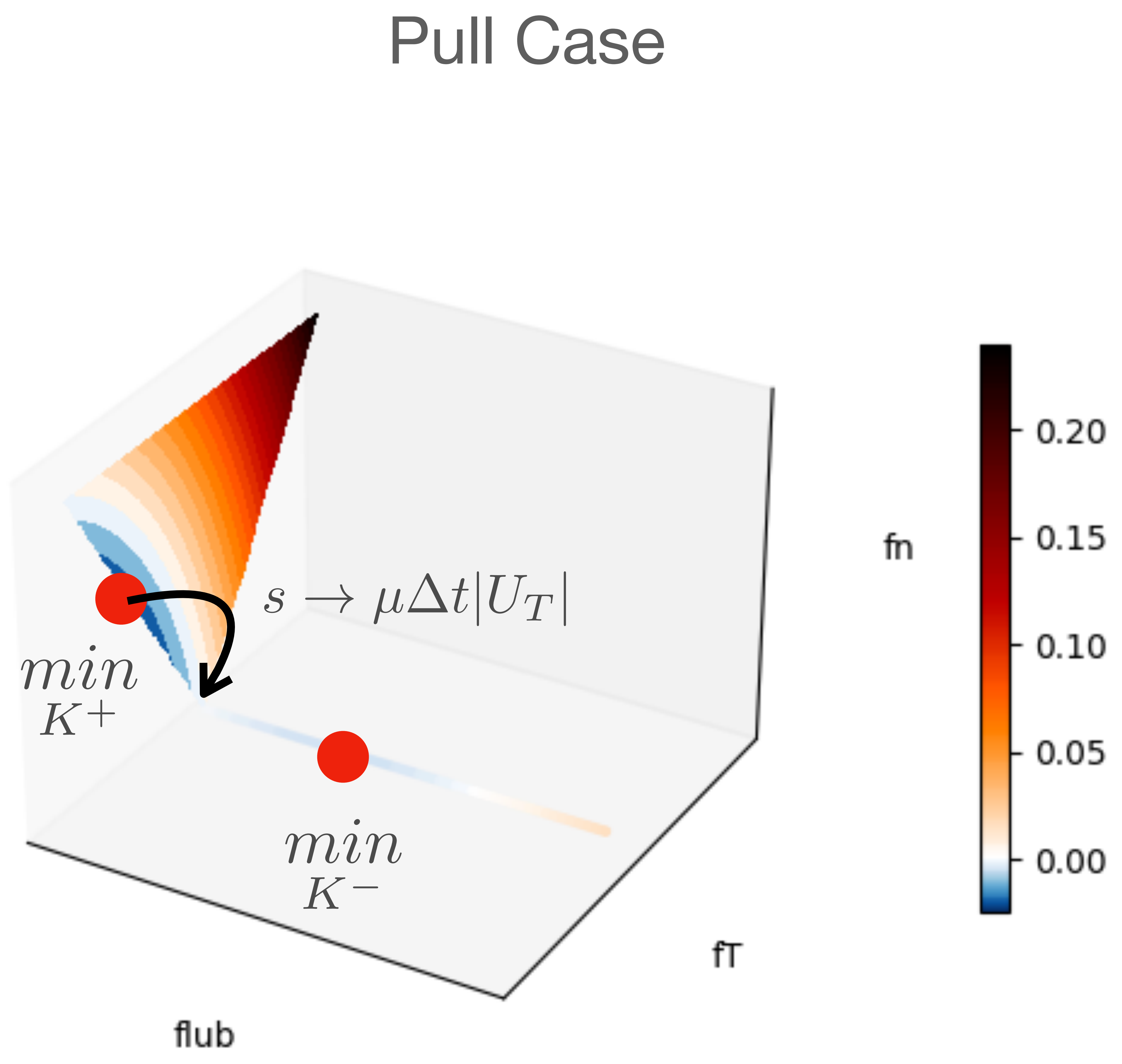


Pull Case



Convergence of the gradient algorithm : Idea of The Proof





Convergence of the gradient algorithm : Idea of The Proof

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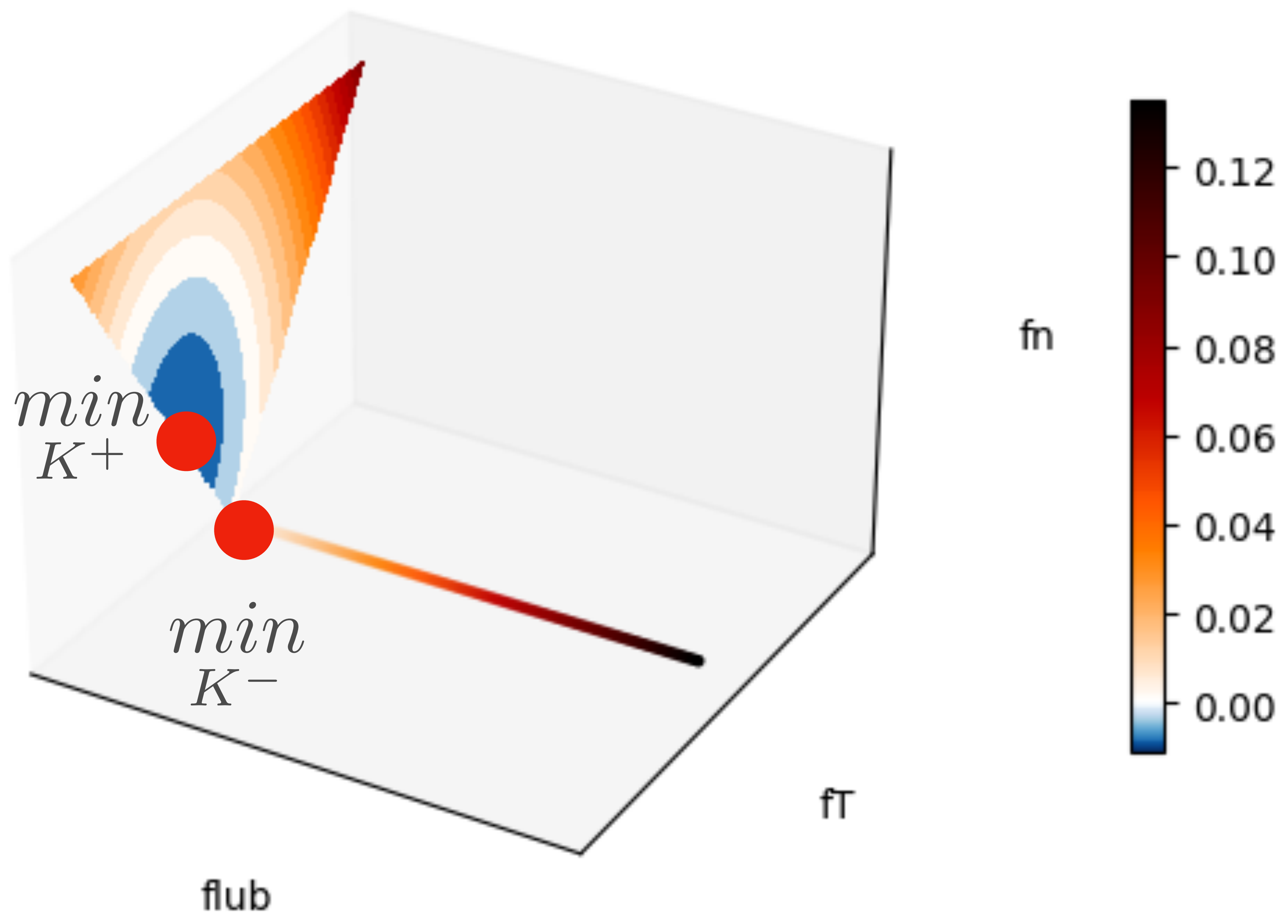
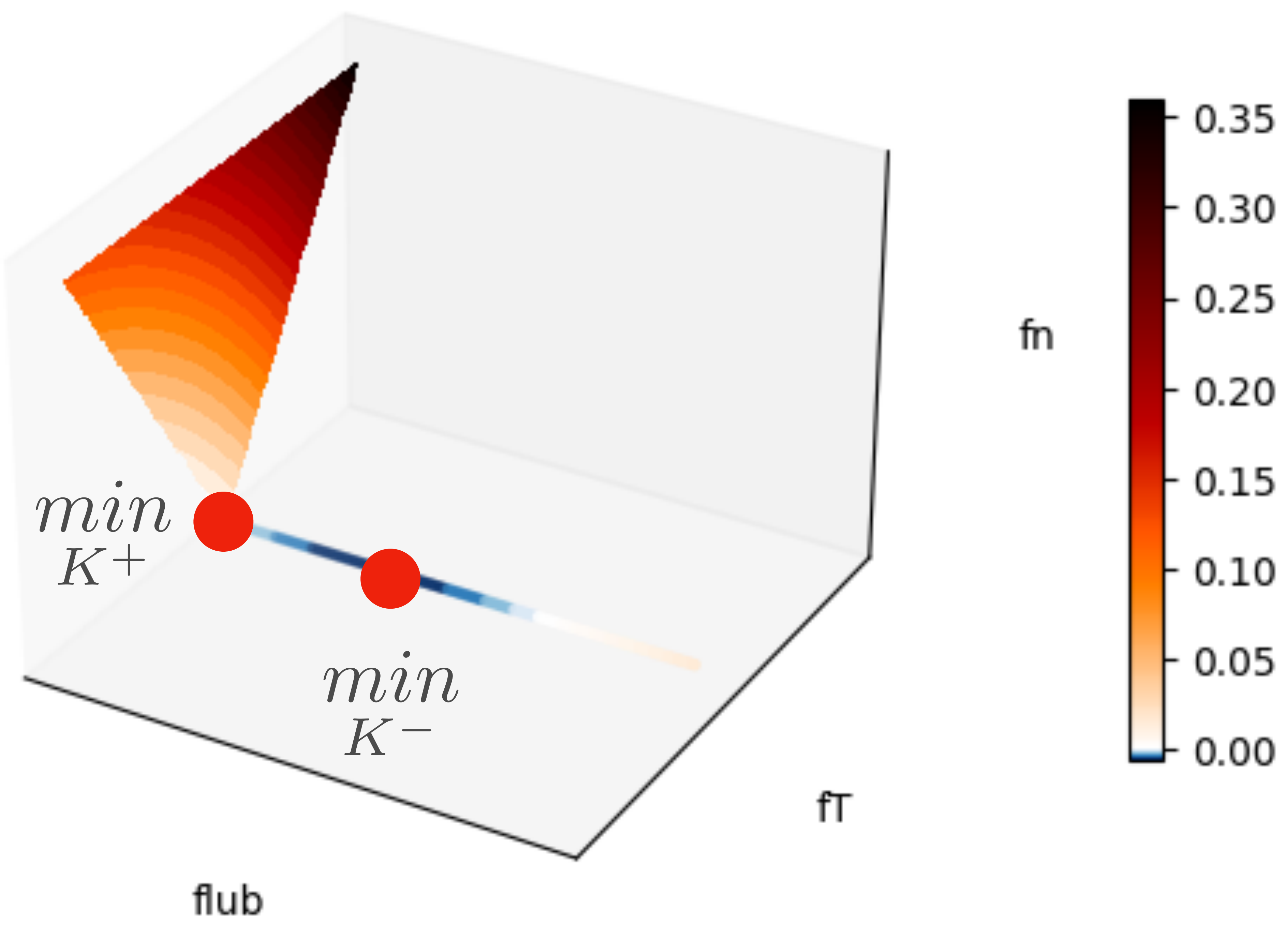
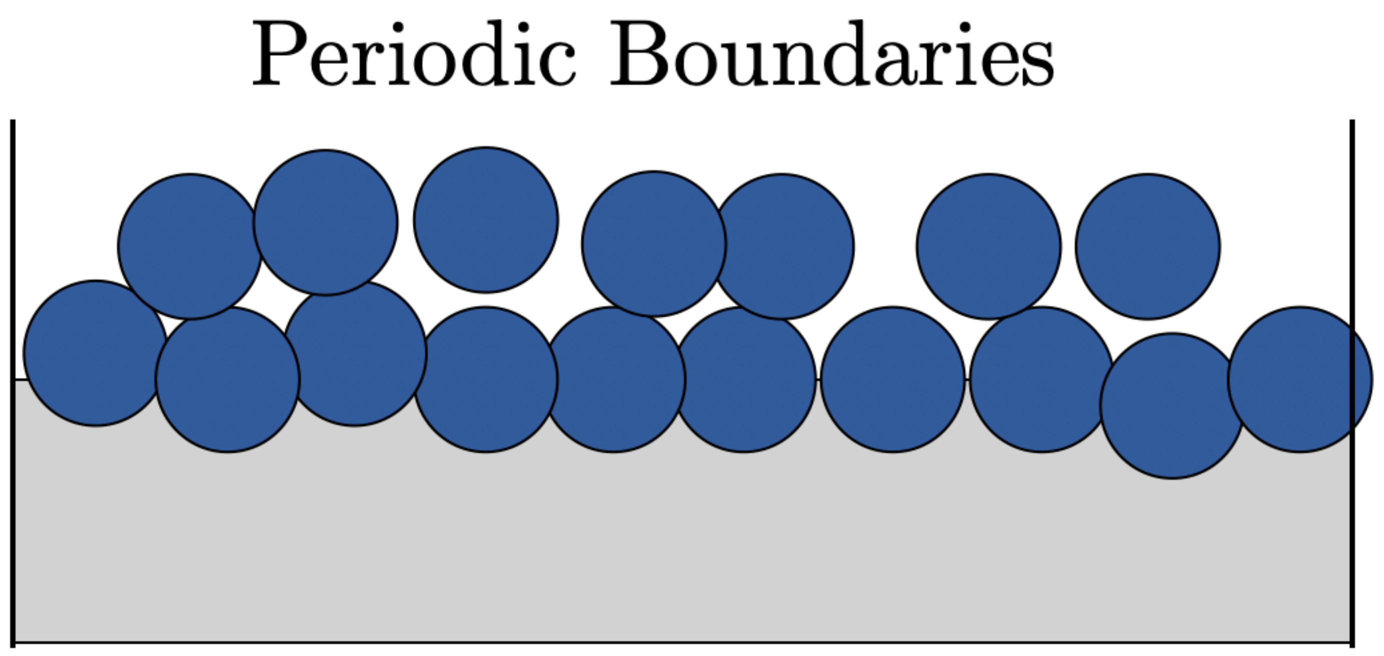


Pull Case (fixed point convergence)

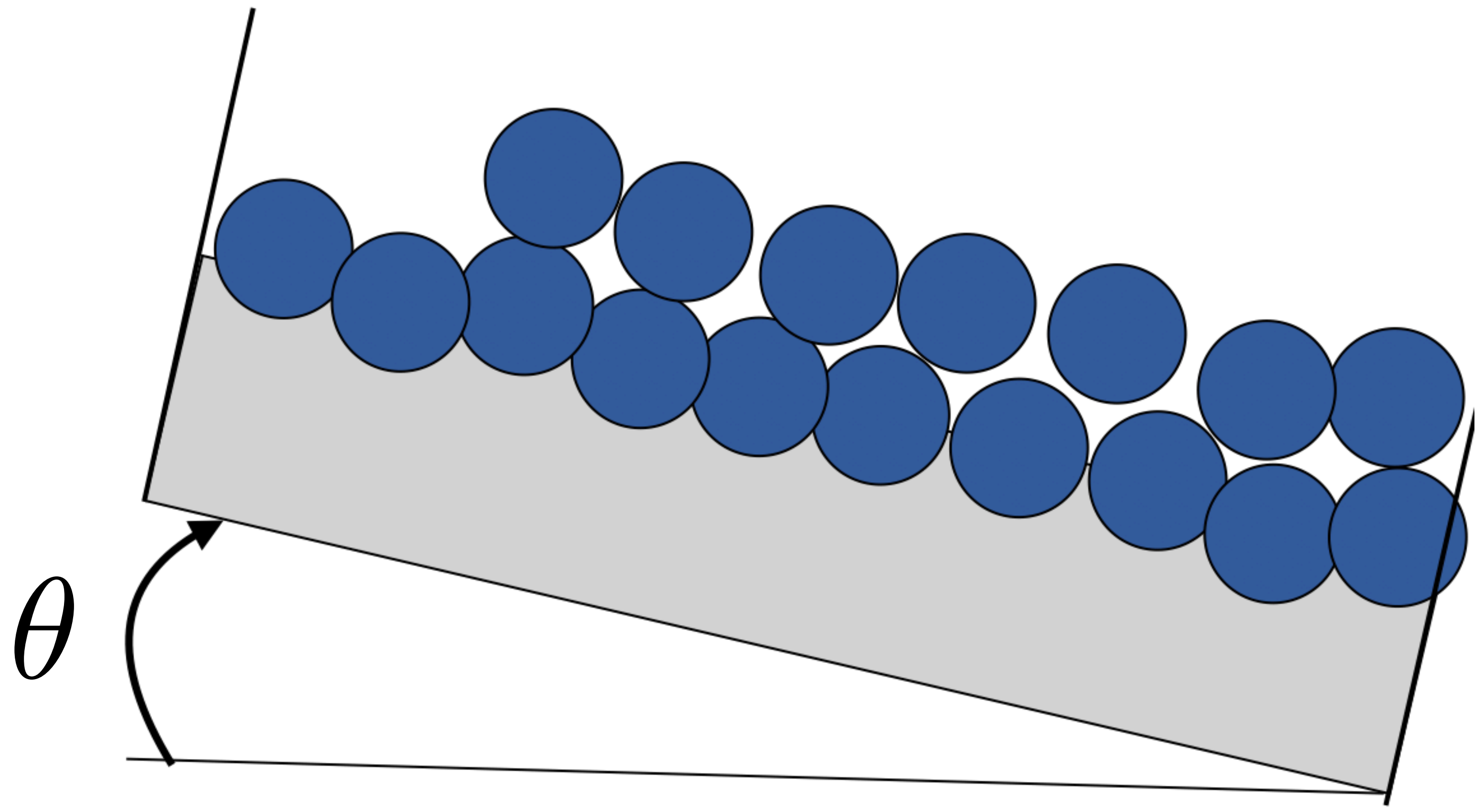


2D Numerical simulations and validation

Step 1 : Sedimentation of grains on a plane



Step 2 : Change angle and measure of stationary flow



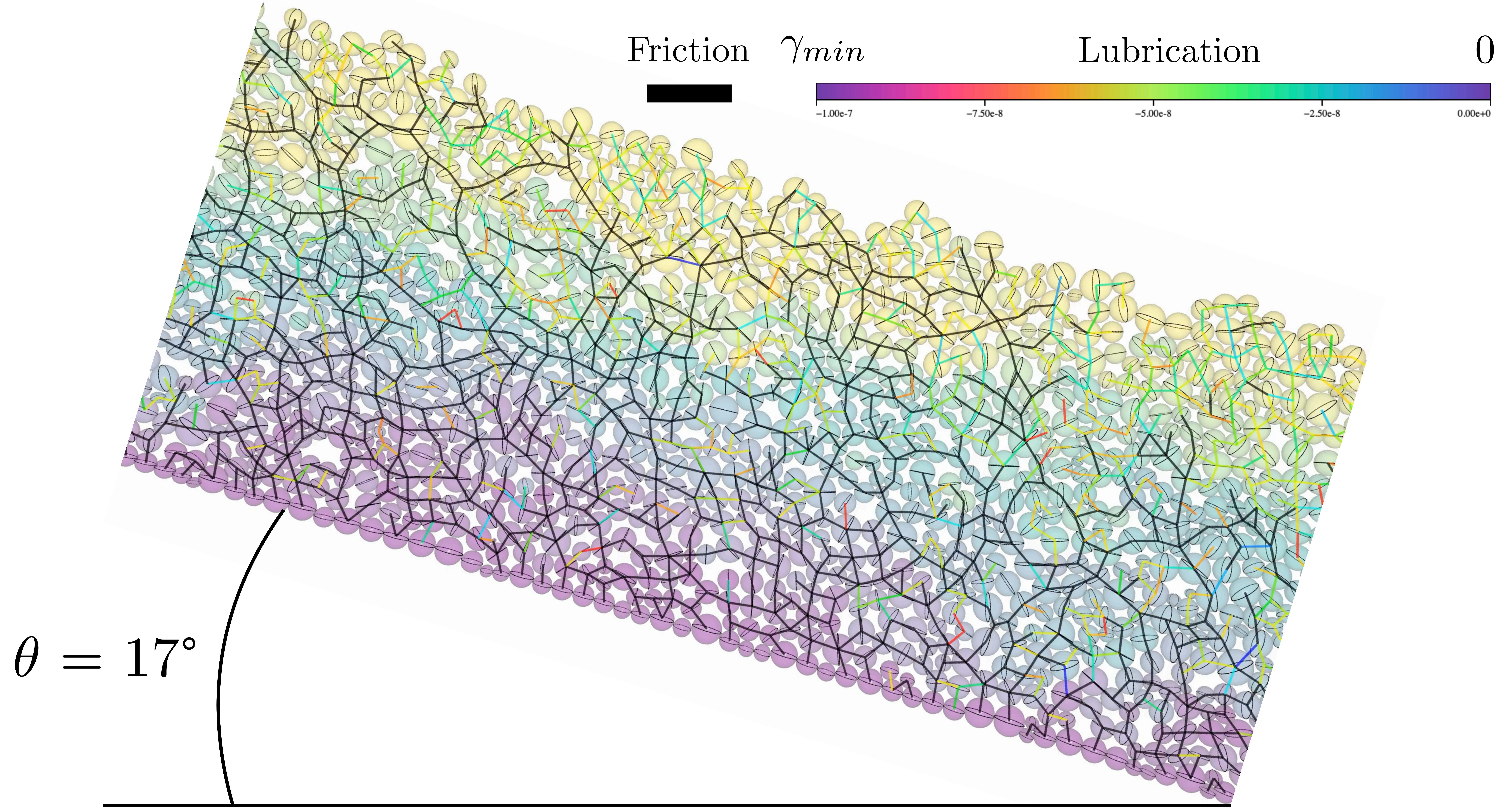
Numerical Parameters :

$\text{Particles Radius} = 1mm \pm 20\%$ $\text{Roughness} = 1\mu m$ $\rho_p = 2500kg.m^{-3}$ $\mu = 0.3$

$\eta = 10^{-3} Pa.s$ $\rho_f = 1500kg.m^{-3}$ $\theta = 15°\text{-}28°$

Simulations done with Scopi : https://github.com/hpc-maths/scopi

Numerical simulations and validation



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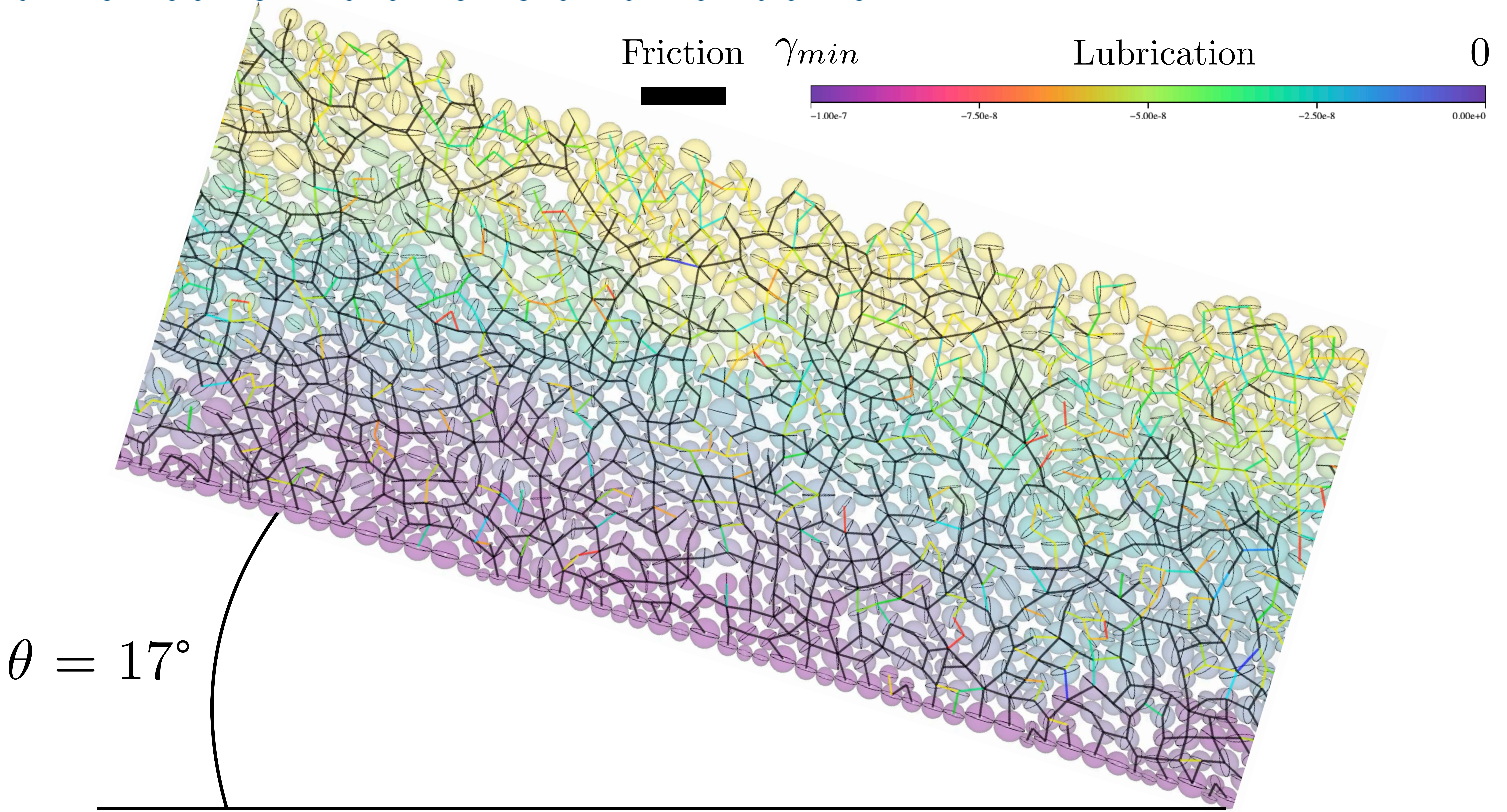

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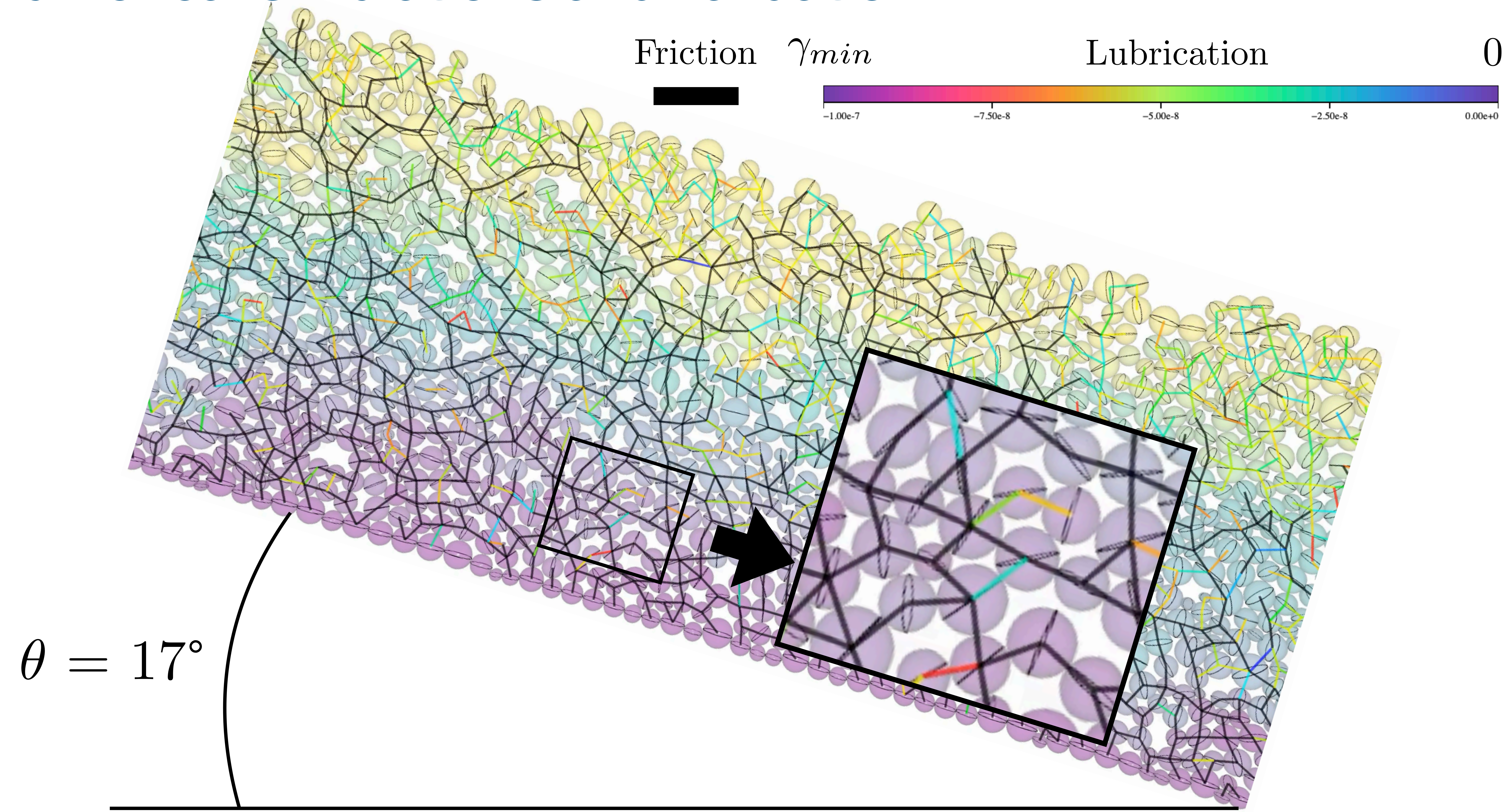

Conclusion and Future Direction

- Current Achievements
  - Development of a model integrating:
    Lubrication forces
    Solid contact
    Frictional interactions
  - Validation of the model and numerical algorithm

- Short-Term Perspectives
  - Study of the transition between lubricated and frictional regimes
  - Study of the number of frictional contacts
  - Local analysis of different physical parameters
  - Investigation of hysteresis effects on the flow angle

Modeling lubrication: the gluey contact model

$$m\ddot{q} = mf_y + \lambda$$

$$\dot{q}^+ = P_{C_{q,\gamma}}\dot{q}^-$$

$$C_{q,\gamma} = \left| \begin{array}{lll} \{0\} & \text{si} & \gamma^- < 0 \\ \mathbb{R}^+ & \text{si} & \gamma^- = 0 \\ & & q = 0 \\ \mathbb{R} & \text{sinon} & \end{array} \right.$$

$$\text{supp}(\lambda) \subset \{t, q(t) = 0\}$$

$$\text{supp}(\delta) \subset \{t, \gamma(t^+) = \gamma_{min}\}$$

$$\dot{\gamma} = -\lambda + \delta$$

$$q \geq 0, \gamma_{min} \leq \gamma \leq 0, \delta \geq 0$$

Adapted model with solid contact