A shock-capturing numerical scheme for a non-conservative self-organized hydrodynamics model

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In this work, we numerically approximate the weak solutions of the following non-linear evolution PDE system :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{(1a)} \\ \rho(\partial_t u + cu\partial_x u) + \lambda(1-u^2)\partial_x \rho = 0, & \text{(1b)} \\ \rho(\partial_t v + cu\partial_x v) - \lambda uv\partial_x \rho = 0, & \text{(1c)} \\ \sqrt{u^2+v^2} = 1. & \text{(1d)} \end{cases}$$

This system depicts a collective self-organized motion where $\rho > 0$ is the density of individuals, $\Omega = (u, v)^T$ the speed of the flow, and $0 < c < 1$ and $\lambda > 0$ two given constants. It is called the Self-Organized Hydrodynamics (SOH) model and is the macroscopic limit of the Vicsek model [4].
System (1) contains a geometric constraint, and is hyperbolic and non-conservative. To overcome these issues, we first reformulate the model as a 2×2 PDE system without its constraint, for a well-prepared initial velocity.
Next, as shock wave solutions are not well-defined for non-conservative systems, we describe them as limits of traveling wave solutions [3] of the associated viscous model derived in [1]. This definition allows us to exhibit a way to describe shock waves. The conditions obtained are called generalized Rankine-Hugoniot conditions, and enable us to construct exact solutions to the Riemann problem.
In addition, a Godunov-type scheme is derived in order to approximate the solutions of (1). We then add a viscous correction to this scheme, since shock wave solutions are defined thanks to the viscous SOH system.
Finally, the resulting numerical scheme is compared to the splitting method from [2] and the exact solution. These results show the relevance of this new numerical scheme to capture properly shock wave solutions of the SOH model (1).

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