Well-posedness of a stochastic Navier–Stokes system with dynamically coupled subgrid scales

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Fully developed turbulent geophysical flow are hard to compute due to the extremely wide spectrum of forcing scale, from the solar-driven motions (10 000 km) to the dissipation scale (1 mm). In that perspective, probabilistic models are an interesting approach for it provides a precise description of the small scale with a stochastic process. The large scale is described by a dynamic equation involving both fluid velocity and pressure.
A recent framework has been proposed in [1]. It consists in two coupled variational principles incorporating stochastic transport constraints. This gives rise to the following coupled stochastic partial differential system :

$$\begin{cases} d_t\boldsymbol{u} + \nu_1 \Delta \boldsymbol{u} + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}dt + \nabla\cdot\sum_{i=0}^{\infty}\boldsymbol{\xi}_i\boldsymbol{\xi}_i^{\intercal}\nabla\boldsymbol{u}]dt \\ \qquad\qquad = \sum_{i=0}^{\infty}\left(\boldsymbol{\xi}_i d\beta_i\cdot\nabla\right)\boldsymbol{u} - \nabla(dp - dp_\sigma), \\ \partial_t\boldsymbol{\xi}_i + \nu_2\Delta^s\boldsymbol{\xi}_i + \left(\boldsymbol{\xi}_i\cdot\nabla\right)\boldsymbol{u} + \left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{\xi}_i = -\nabla q, \qquad i\in\mathbb{N}. \end{cases} \tag{1}$$

The first equation describes the large scale dynamic while the second corresponds to the dynamic of the noise correlation functions. In this talk, I will present a proof of well-posedness result of so-called martingale and pathwise solutions in dimension 2 and 3. We state the main result :

Théorème 1. *The following statements hold :*

1. ***Existence*** $d = 2$: *Let* $d = 2$ *and* $s = 1$*. For any initial condition* $(\boldsymbol{u}_0, \boldsymbol{\xi}_0) \in V$*, such that* $\sum_{i=0}^{\infty}\left\|\boldsymbol{\xi}_{i,0}\right\|_{L^2}^2 < +\infty$*, there exists a martingale solution to* (1)*.*
2. ***Uniqueness*** $d = 2$: *Suppose now that* $s > 1$*, and consider the stochastic basis* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, \boldsymbol{W}_t\right)$. *There exists at most one pathwise solution, that is unique up to indistinguishability.*
3. ***Existence*** $d = 3$: *Let* $d = 3$ *and* $s = 3/2$*. For any initial conditions* $(\boldsymbol{u}_0, \boldsymbol{\xi}_0) \in V$*, such that* $\sum_{i=0}^{\infty}\left\|\boldsymbol{\xi}_{i,0}\right\|_{L^2}^2 < +\infty$*, there exists a martingale solution to* (1)

[1] A. Debussche, É. Mémin. *Variational principles for fully coupled stochastic fluid dynamics across scales.* Physica D : Nonlinear Phenomena, **481**, 134777, 2025. doi :10.1016/j.physd.2025.134777.

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