Sur la structure des équations de la dynamique des poutres et une classification des problèmes associés

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Kinematical configuration of a beam-like structure

Material coordinates

$S \in [0, L]$ $\quad (\xi_1, \xi_2)$ local chart in $\mathcal{S}$

Configuration in the ambiant space

Placement :

$\varphi(S, t) = \mathrm{OG}$

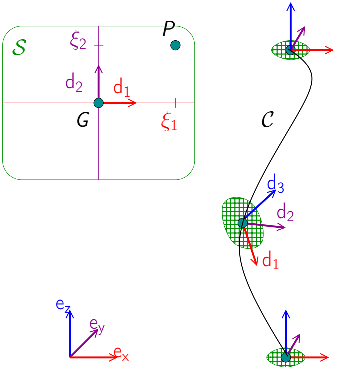
Rotation : $\mathbb{Q}(S, t) \in SO(3)$

$\mathrm{d_i}(S, t) = \mathbb{Q}(S, t)\mathrm{e_i}$

Place of a point P of the beam

$(\xi_1, \xi_2, S, t) \to \mathrm{OP} = \mathrm{OG} + \mathrm{GP}$

$\mathrm{GP} = \xi_1 \mathrm{d_1} + \xi_2 \mathrm{d_2}$



E.Cosserat, F.Cosserat, *Théorie des Corps déformables*, 1909
S.P.Timoshenko, J.C.Gere, *Theory of Elastic Stability*, 1961

$\mathcal{S}$ is **rigidly** transformed

$\mathrm{d_3}$ is **not** tangent to $\mathcal{C}$

G is the center of mass of $\mathcal{S}$

$\{\mathrm{d_i}\}$ is an inertial principal basis

Vectors on the tangent space TC

Speed v

$$\mathrm{v} := \frac{\partial \varphi}{\partial t}$$

Strain ε

$$\varepsilon := \frac{\partial \varphi}{\partial S} - \mathrm{d}_3$$

Spin $\boldsymbol{\omega}$

$$\frac{\partial \mathrm{d_i}}{\partial t} = \boldsymbol{\omega} \wedge \mathrm{d_i}$$

$$\boldsymbol{\omega} = j(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial t})$$

$$\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial t} \in so(3)$$

Curvature $\boldsymbol{\kappa}$

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$$\boldsymbol{\kappa} = j(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial S})$$

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Vectors on the tangent space TC

Speed v

$$\mathrm{v} := \frac{\partial \varphi}{\partial t}$$

$$\mathrm{v} = v_1 \mathrm{d}_1 + v_2 \mathrm{d}_2 + v_3 \mathrm{d}_3$$

Strain ε

$$\boldsymbol{\varepsilon} := \frac{\partial \varphi}{\partial S} - \mathrm{d}_3$$

$$\boldsymbol{\varepsilon} = \varepsilon_1 \mathrm{d}_1 + \varepsilon_2 \mathrm{d}_2 + \varepsilon_3 \mathrm{d}_3$$

Spin ω

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Curvature κ

$$\frac{\partial \mathrm{d_i}}{\partial S} = \boldsymbol{\kappa} \wedge \mathrm{d_i}$$

$$\boldsymbol{\kappa} = j(\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial S})$$

$$\mathbb{Q}^T \frac{\partial \mathbb{Q}}{\partial S} \in so(3)$$

$$\boldsymbol{\kappa} = \kappa_1 \mathrm{d}_1 + \kappa_2 \mathrm{d}_2 + \kappa_3 \mathrm{d}_3$$

Associated moments and energy

Momentum $\mathrm{p} = \rho A \mathrm{v}$

$$\mathrm{p} := \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & \rho A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbb{A}\mathrm{v}$$

Force N

$$\mathrm{N} := \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \mathbb{G}\boldsymbol{\varepsilon}$$

Moment of momentum $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} := \begin{bmatrix} \rho I_1 & 0 & 0 \\ 0 & \rho I_2 & 0 \\ 0 & 0 & \rho I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \mathbb{J}\boldsymbol{\omega}$$

Torque M

$$\mathrm{M} := \begin{bmatrix} EI_1 & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & GI_3 \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} = \mathbb{H}\boldsymbol{\kappa}$$

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Kinetic energy density

$$T(\mathrm{v}, \boldsymbol{\omega}) = \frac{1}{2}\left(\mathrm{v}\mathbb{A}\mathrm{v} + \boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega}\right)$$

Deformation energy density

$$U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = \frac{1}{2}\left(\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} + \boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa}\right)$$

Associated moments and energy

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$$\text{Lagrangian} \quad \mathcal{L}(\mathrm{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \int_0^L \frac{1}{2}\mathrm{v}\mathbb{A}\mathrm{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa} \;\, \mathrm{d}S$$

$$\text{Hamiltonian} \quad \mathcal{H}(\mathrm{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) := \int_0^L \frac{1}{2}\mathrm{v}\mathbb{A}\mathrm{v} + \frac{1}{2}\boldsymbol{\omega}\mathbb{J}\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\varepsilon}\mathbb{G}\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\kappa}\mathbb{H}\boldsymbol{\kappa} \;\, \mathrm{d}S$$

Application for (time) derivation of a quadratic form

$$f = \frac{1}{2}\mathrm{u}\mathbb{X}\mathrm{u}, \qquad \mathbb{X}^T = \mathbb{X}$$

Derivation of a vector

$$\frac{\partial \mathrm{u}}{\partial t} = \frac{\partial u_i}{\partial t}\mathrm{d_i} + \boldsymbol{\omega} \wedge \mathrm{u}$$

$$\frac{\partial f}{\partial t} = \frac{\partial u_i}{\partial t}\mathbb{X}_{ij} u_j = \left(\frac{\partial \mathrm{u}}{\partial t} - \boldsymbol{\omega} \wedge \mathrm{u}\right) \cdot (\mathbb{X}\mathrm{u})$$

$$\frac{\partial f}{\partial t} = \frac{\partial \mathrm{u}}{\partial t}\mathbb{X}\mathrm{u} - (\mathrm{u} \wedge (\mathbb{X}\mathrm{u})) \cdot \boldsymbol{\omega}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathrm{u}\mathbb{X}\mathrm{u}\right) \neq \frac{\partial \mathrm{u}}{\partial t}\mathbb{X}\mathrm{u}$$

$\frac{\partial \mathrm{u}}{\partial t} - \boldsymbol{\omega} \wedge \mathrm{u}$ is a corotational derivative

Equilibrium

$$\frac{\partial \mathsf{N}}{\partial S} = \frac{\partial\, \mathbb{A}\mathsf{v}}{\partial t} \qquad (1)$$

$$\frac{\partial \mathsf{M}}{\partial S} + \frac{\partial \varphi}{\partial S} \wedge \mathsf{N} = \frac{\partial\, \mathbb{J}\boldsymbol{\omega}}{\partial t} \qquad (2)$$

$$\mathsf{N} := \mathbb{G}\boldsymbol{\varepsilon}$$

$$\mathsf{M} := \mathbb{H}\boldsymbol{\kappa}$$

$$\mathsf{v} := \frac{\partial \varphi}{\partial t} \qquad \boldsymbol{\omega} := j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial t}) \qquad \boldsymbol{\kappa} := j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial S}) \qquad \boldsymbol{\varepsilon} := \frac{\partial \varphi}{\partial S} - \mathsf{d}_3$$

Equilibrium

$$\frac{\partial\ \mathbb{G}\boldsymbol{\varepsilon}}{\partial S} = \frac{\partial\ \mathbb{A}\mathsf{v}}{\partial t} \qquad (1)$$

$$\frac{\partial\ \mathbb{H}\boldsymbol{\kappa}}{\partial S} + (\boldsymbol{\varepsilon} + \mathsf{d}_3) \wedge \mathbb{G}\boldsymbol{\varepsilon} = \frac{\partial\ \mathbb{J}\boldsymbol{\omega}}{\partial t} \qquad (2)$$

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Equilibrium

$$\frac{\partial\, \mathbb{G}\varepsilon}{\partial S} = \frac{\partial\, \mathbb{A}\mathrm{v}}{\partial t} \quad (1)$$

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- $\varphi(S,t)$ and $\mathbb{Q}(S,t)$ are not used as explicite unknowns
- Four vectors are the unknowns v, $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$, ε.
- Ill-posed problem with 12 unknown components and 6 scalar equations
- first order, non-linear, partial differential equations.

Equilibrium

$$\frac{\partial\, \mathbb{G}\varepsilon}{\partial S} = \frac{\partial\, \mathbb{A}\mathrm{v}}{\partial t} \qquad (1)$$

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Additional *closure* relations

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} \qquad \frac{\partial \varepsilon}{\partial t} = \frac{\partial \mathrm{v}}{\partial S} - \boldsymbol{\omega} \wedge \mathrm{d}_3$$

obtained from $\frac{\partial}{\partial t}\frac{\partial \mathrm{d}_i}{\partial S} = \frac{\partial}{\partial S}\frac{\partial \mathrm{d}_i}{\partial t}$ $\qquad$ $\frac{\partial}{\partial t}\frac{\partial \varphi}{\partial S} = \frac{\partial}{\partial S}\frac{\partial \varphi}{\partial t}$

Two systems

First system: unknowns v, $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$, $\boldsymbol{\varepsilon}$

$$\frac{\partial\, \mathbb{G}\boldsymbol{\varepsilon}}{\partial S} = \frac{\partial\, \mathbb{A}\mathrm{v}}{\partial t} \tag{3}$$

$$\frac{\partial\, \mathbb{H}\boldsymbol{\kappa}}{\partial S} + (\boldsymbol{\varepsilon} + \mathrm{d}_3) \wedge (\mathbb{G}\boldsymbol{\varepsilon}) = \frac{\partial\, \mathbb{J}\boldsymbol{\omega}}{\partial t} \tag{4}$$

$$\frac{\partial\, \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} = \frac{\partial\, \boldsymbol{\kappa}}{\partial t} \tag{5}$$

$$\frac{\partial\, \mathrm{v}}{\partial S} - \boldsymbol{\omega} \wedge \mathrm{d}_3 = \frac{\partial\, \boldsymbol{\varepsilon}}{\partial t} \tag{6}$$

Second system: unknowns $\mathbb{Q}$, $\mathrm{d_i}$ and φ

$$\boldsymbol{\omega} := j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial t}) \qquad \boldsymbol{\kappa} := j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial S}) \qquad \mathrm{v} := \frac{\partial \varphi}{\partial t} \qquad \boldsymbol{\varepsilon} := \frac{\partial \varphi}{\partial S} - \mathrm{d}_3$$

- One second order PDE $\Rightarrow$ Two first order PDE

!! Initial and boundary conditions

Numerical point of view

$$\frac{\partial\, \mathbb{G}\boldsymbol{\varepsilon}}{\partial S} = \frac{\partial\, \mathbb{A}\mathrm{v}}{\partial t} \tag{7}$$

$$\frac{\partial\, \mathbb{H}\boldsymbol{\kappa}}{\partial S} + (\boldsymbol{\varepsilon} + \mathrm{d}_3) \wedge (\mathbb{G}\boldsymbol{\varepsilon}) = \frac{\partial\, \mathbb{J}\boldsymbol{\omega}}{\partial t} \tag{8}$$

$$\frac{\partial\, \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} = \frac{\partial\, \boldsymbol{\kappa}}{\partial t} \tag{9}$$

$$\frac{\partial\, \mathrm{v}}{\partial S} - \boldsymbol{\omega} \wedge \mathrm{d}_3 = \frac{\partial\, \boldsymbol{\varepsilon}}{\partial t} \tag{10}$$

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

with non-linearities induced by the skew-matrices:

$$\mathbb{W} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \mathbb{K} = \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix} \quad \mathbb{E} = \begin{bmatrix} 0 & -(\varepsilon_3 + 1) & \varepsilon_2 \\ \varepsilon_3 + 1 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix}$$

Three point of views

$$\frac{\partial\,\mathbb{G}\boldsymbol{\varepsilon}}{\partial S} = \frac{\partial\,\mathbb{A}\mathrm{v}}{\partial t}$$ Newton's laws

$$\frac{\partial\,\mathbb{H}\boldsymbol{\kappa}}{\partial S} + (\boldsymbol{\varepsilon} + \mathrm{d}_3)\wedge(\mathbb{G}\boldsymbol{\varepsilon}) = \frac{\partial\,\mathbb{J}\boldsymbol{\omega}}{\partial t}$$ Rotational analogues of Newton's laws

$$\frac{\partial\,\mathrm{v}}{\partial S} - \boldsymbol{\omega}\wedge\mathrm{d}_3 = \frac{\partial\,\boldsymbol{\varepsilon}}{\partial t}$$ compatibility of translations

$$\frac{\partial\,\boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega}\wedge\boldsymbol{\kappa} = \frac{\partial\,\boldsymbol{\kappa}}{\partial t}$$ compatibility of rotations

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix}\begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

$$\mathcal{D}\,u' + \mathcal{Y}(u)\,u = \mathcal{M}\,\dot{u}$$

$\mathcal{D}$ Symetric Constant

$\mathcal{Y}(u)$ Skew-symetric Non-linear

$\mathcal{M}$ Diagonal Constant

Determining the placement

Position $\varphi(S, t)$

One has to solve

$$\underline{\text{either}} \qquad \frac{\partial \varphi}{\partial S} = \varepsilon + d_3 \qquad \underline{\text{or}} \qquad \frac{\partial \varphi}{\partial t} = v.$$

As $\{v, \omega, \varepsilon, \kappa\}$ are supposed to be known, these relations write:

$$\varphi' + \kappa \wedge \varphi = \varepsilon + d_3 \qquad \text{or} \qquad \dot{\varphi} + \omega \wedge \varphi = v.$$

The left-side case can be solved if a BC $\varphi(0, t)$ is given, and the right-side case can be solved if an IC $\varphi(S, 0)$ is prescribed.

Orientation $\mathbb{Q}(S, t)$ or equivalently $\{d_i\}$

One has to solve

$$\underline{\text{either}} \qquad \frac{\partial \mathbb{Q}}{\partial t} - \mathbb{Q}\mathbb{W} = 0 \qquad \underline{\text{or}} \qquad \frac{\partial \mathbb{Q}}{\partial S} - \mathbb{Q}\mathbb{K} = 0$$

The left-side case can be solved if a BC $\{d_i(0, t)\}$ is given, and the right-side case can be solved if an IC $\{d_i(S, 0)\}$ is prescribed.

Motivation

- *"The thread model does not exist"* P.Seppecher, Ecole d'été d'Oleron.

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- ... but some counterexample
  - Tension/compression of bar : linear, dynamical

$$EAu_3'' = \rho A \ddot{u}_3, \qquad u_3 \ : \ \text{longitudinal displacement}$$

  - String : ∼linear, dynamical

$$Fu_1'' = \rho A \ddot{u}_1, \qquad F = EAu_3', \qquad u_1 \ : \ \text{transverse displacement}$$

  - Catenary : non-linear, static
  - Rope jumped : non-linear, quasi-static

- ... and intersting phenomena
  - Non-unicity of the solution
  - Non regularity of the solution
  - Zero-Euler's critical load

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- ... and intersting phenomena
  - Non-unicity of the solution
  - Non regularity of the solution
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- Boundary condition plays a foundamental role

$$\|\varphi(L) - \varphi(0)\| \sim L \qquad \|\varphi(L) - \varphi(0)\| < L \qquad \|\varphi(L) - \varphi(0)\| > L$$

Motivation

The simple string equation (plane+linear)

$$EA\varepsilon_0 u'' = \rho A\ddot{u}$$

A difficult string equation (plane+non-linear)

$$\left(EAu_x'\left(1 - \frac{1}{\sqrt{u_x'^2 + (1+\varepsilon_0+u_z')^2}}\right)\right)' = \rho A\ddot{u}_x \qquad (11)$$

$$\left(EA\left(u_z' - \frac{1+\varepsilon_0+u_z'}{\sqrt{u_x'^2 + (1+\varepsilon_0+u_z')^2}}\right)\right)' = \rho A\ddot{u}_z \qquad (12)$$

Hypotheses

Circular cross-section $\Leftrightarrow I_1 = I_2$

No shear or bending rigidity $\Leftrightarrow$ Only longitudinal or torsional rigidity

$$GA \to 0, \quad EA \neq 0, \qquad EI_1 \to 0, \quad EI_2 \to 0, \quad GI_3 \neq 0$$

$$\mathbb{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & EA \end{bmatrix} \qquad \mathbb{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_3 \end{bmatrix}$$

Application to standard structural mechanics

$$\frac{GA}{EA} \ll 1 \qquad \frac{EI_1}{GI_3} \ll 1 \qquad \text{but} \quad I_3 = I_1 + I_2 \qquad \Rightarrow \qquad \frac{1}{2} \ll \frac{G}{A} \ll 1$$

Thread model exists, if one consider that

- it's an effective model for a microstructured material
- it's not only a 1D-manifold

Rq a thread is composed of an arrangement of filaments !! yes !!!

Degeneracy

$$\mathbb{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & EA \end{bmatrix} \qquad \mathbb{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{G} & 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{G}\mathbb{K} & \mathbb{G}\mathbb{E} & -\mathbb{G}\mathbb{W} & 0 \\ 0 & \mathbb{H}\mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

- Problem
  - Lines 7,8,11 and 12 of this algebraic problem are $0 = 0$
  - $\Rightarrow$ No possibility to express some compatibility condition

Degeneracy

$$\mathbb{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & EA \end{bmatrix} \qquad \mathbb{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{K} & \mathbb{E} & -\mathbb{W} & 0 \\ 0 & \mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

- Problem
    - Lines 7,8,11 and 12 of this algebraic problem are $0 = 0$

    ⇒ No possibility to express some compatibility condition
- First alternative
    - Do not multiply compatibility condition by $\mathbb{G}$ and $\mathbb{H}$

    ⇒ The structure of the problem is broken :
    $\mathcal{D}$ is no more symetric, $\mathcal{Y}(u)$ is no more skew-symetric,

Degeneracy

$$\mathbb{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & EA \end{bmatrix} \qquad \mathbb{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & GI_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \mathbb{G} & 0 \\ 0 & 0 & 0 & \mathbb{H} \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \end{bmatrix} \begin{bmatrix} v' \\ \omega' \\ \varepsilon' \\ \kappa' \end{bmatrix} + \begin{bmatrix} -\mathbb{W}\mathbb{A} & 0 & \mathbb{K}\mathbb{G} & 0 \\ 0 & -\mathbb{W}\mathbb{J} & \mathbb{E}\mathbb{G} & \mathbb{K}\mathbb{H} \\ \mathbb{K} & \mathbb{E} & -\mathbb{W} & 0 \\ 0 & \mathbb{K} & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \\ \varepsilon \\ \kappa \end{bmatrix} = \begin{bmatrix} \mathbb{A} & 0 & 0 & 0 \\ 0 & \mathbb{J} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \\ \dot{\varepsilon} \\ \dot{\kappa} \end{bmatrix}$$

- Problem
  - Lines 7,8,11 and 12 of this algebraic problem are $0 = 0$

  $\Rightarrow$ No possibility to express some compatibility condition
- First alternative
  - Do not multiply compatibility condition by $\mathbb{G}$ and $\mathbb{H}$

  $\Rightarrow$ The structure of the problem is broken :
  $\mathcal{D}$ is no more symetric, $\mathcal{Y}(u)$ is no more skew-symetric,
- Second alternative
  - Consider $EI_1 = \epsilon$, $EI_2 = \epsilon$, $GA = \epsilon$, with $\epsilon \to 0$

  $\Rightarrow$ Critic for the numerical purpose

$$\lim_{\epsilon \to 0} u_\epsilon \stackrel{?}{=} u_0$$

Bi-dimensional analysis in the plane $(d_1, d_3) = (e_x, e_z)$, $d_2 = e_y$

$$u = \begin{bmatrix} v_1 \\ v_3 \\ \omega_2 \\ \varepsilon_1 \\ \varepsilon_3 \\ \kappa_2 \end{bmatrix} \qquad \hat{\mathcal{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & EA & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathcal{D}}\, u' + \hat{\mathcal{Y}}(u) u = \hat{\mathcal{M}}\, \dot{u} \qquad \hat{\mathcal{Y}}(u) = \begin{bmatrix} 0 & -\rho A \omega_2 & 0 & 0 & 0 & EA\kappa_2 & 0 \\ \rho A \omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -EA\varepsilon_1 & 0 \\ 0 & \kappa_2 & -1-\varepsilon_3 & 0 & 0 & 0 & 0 \\ -\kappa_2 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathcal{M}} = \begin{bmatrix} \rho A & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Reorganisation of the plane problem

The problem is cast into two subproblem.

- The first preserve a standard transport equation structure

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EA \\ 0 & 0 & 0 & 0 \\ 0 & EA & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1' \\ v_3' \\ \omega_2' \\ \varepsilon_3' \end{bmatrix} + \begin{bmatrix} 0 & -\rho A\omega_2 & 0 & EA\kappa_2 \\ \rho A\omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -EA\varepsilon_1 \\ -EA\kappa_2 & 0 & EA\varepsilon_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ \omega_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \rho A & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 \\ 0 & 0 & \rho I_3 & 0 \\ 0 & 0 & 0 & EA \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{\omega}_2 \\ \dot{\varepsilon}_3 \end{bmatrix}$$

- The second is composed of the two last compatibility equation

$$\begin{aligned} v_1' + \kappa_2 v_3 - (1+\varepsilon_3)\omega_2 &= \dot{\varepsilon}_1 \\ \omega_2' &= \dot{\kappa}_2 \end{aligned}$$

Rq These two problems are coupled

Euler-Bernoulli hypothesis

- No shear $\varepsilon_1 \to 0$
- The conservation of angular momentum is neglected (like for string)

The problem is cast into two subproblem.

- The first preserve a standard transport equation structure

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & EA \\ 0 & EA & 0 \end{bmatrix} \begin{bmatrix} v_1' \\ v_3' \\ \varepsilon_3' \end{bmatrix} + \begin{bmatrix} 0 & -\rho A \omega_2 & EA\kappa_2 \\ \rho A \omega_2 & 0 & 0 \\ -EA\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \rho A & 0 & 0 \\ 0 & \rho A & 0 \\ 0 & 0 & EA \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_3 \\ \dot{\varepsilon}_3 \end{bmatrix}$$

- The second is composed of the two last compatibility equation

$$v_1' + \kappa_2 v_3 - (1 + \varepsilon_3)\omega_2 = 0$$
$$\omega_2' = \dot{\kappa}_2$$

⇒ The problem persists : additional kinematical hyptheses looks not to be a solution to preserve the structure of the problem.

Paradigm shift

Important remark

The loose of symetry of matrices $\mathcal{Y}$ and $\mathcal{D}$ is related to

Compatibility conditions:

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial S} + \boldsymbol{\omega} \wedge \boldsymbol{\kappa} \qquad \frac{\partial \boldsymbol{\varepsilon}}{\partial t} = \frac{\partial \mathrm{v}}{\partial S} - \boldsymbol{\omega} \wedge \mathrm{d}_3$$

obtained from

$$\frac{\partial}{\partial t}\frac{\partial \mathbb{Q}}{\partial S} = \frac{\partial}{\partial S}\frac{\partial \mathbb{Q}}{\partial t} \qquad \frac{\partial}{\partial t}\frac{\partial \boldsymbol{\varphi}}{\partial S} = \frac{\partial}{\partial S}\frac{\partial \boldsymbol{\varphi}}{\partial t}$$

If the variable are not (only) vectors $\{\mathrm{v}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}\} \in TC$ but (also) kinematical quantities $\{\boldsymbol{\varphi}, \mathbb{Q}\} \in C$, the compatibility condition are imediately satisfied!

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Ben justement, c'est ce que l'on a fait avec Oscar !!!

Timoshenko beam under finite and dynamic transformations: Lagrangian coordinates and Hamiltonian structures, Communications in Analysis and Mechanics, 17(4), 2025

First formulation

$\{p, \sigma, \varepsilon, \kappa\}$

$$H(p,\sigma,\varepsilon,\kappa) = \int_0^L \frac{1}{2} p \mathbb{A}^{-1} p + \frac{1}{2}\sigma \mathbb{J}^{-1}\sigma + \frac{1}{2}\varepsilon \mathbb{G} \varepsilon + \frac{1}{2}\kappa \mathbb{H} \kappa \;\, \mathrm{d}S$$

Poisson bracket, for any f and g functions of the variables $\{p, \sigma, \varepsilon, \kappa\}$:

$$\begin{aligned}\{f,g\} = \int_0^L & < \frac{\partial f}{\partial p}, \frac{\partial}{\partial S}\left(\frac{\partial g}{\partial \varepsilon}\right) > - < \frac{\partial g}{\partial p}, \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial \varepsilon}\right) > \\ & + < \frac{\partial f}{\partial \sigma}, \frac{\partial}{\partial S}\left(\frac{\partial g}{\partial \kappa}\right) > - < \frac{\partial g}{\partial \sigma}, \frac{\partial}{\partial S}\left(\frac{\partial f}{\partial \kappa}\right) > \\ & + < \kappa, \frac{\partial g}{\partial \varepsilon} \wedge \frac{\partial f}{\partial p} - \frac{\partial f}{\partial \varepsilon} \wedge \frac{\partial g}{\partial p} > \\ & + < \sigma, \frac{\partial g}{\partial \sigma} \wedge \frac{\partial f}{\partial \sigma} > + < p, \frac{\partial g}{\partial \sigma} \wedge \frac{\partial f}{\partial p} - \frac{\partial f}{\partial \sigma} \wedge \frac{\partial g}{\partial p} > \\ & + < \varepsilon + d_3, \frac{\partial g}{\partial \varepsilon} \wedge \frac{\partial f}{\partial \sigma} - \frac{\partial f}{\partial \varepsilon} \wedge \frac{\partial g}{\partial \sigma} > \\ & + < \kappa, \frac{\partial g}{\partial \kappa} \wedge \frac{\partial f}{\partial \sigma} - \frac{\partial f}{\partial \kappa} \wedge \frac{\partial g}{\partial \sigma} > \;\, \mathrm{d}S.\end{aligned}$$

Second formulation

$\{\varphi, \mathbb{Q}, \mathrm{p}, \Sigma\}$

$$H(\varphi, \mathbb{Q}, \mathrm{p}, \Sigma) = \frac{1}{2}\int_0^L < \mathrm{p}, \mathbb{A}^{-1}\mathrm{p} > + < j(\mathbb{Q}^{-1}\Sigma), \mathbb{J}^{-1}j(\mathbb{Q}^{-1}\Sigma) >$$
$$+ < \frac{\partial \varphi}{\partial S} - \mathrm{d}_3, \mathbb{G}(\frac{\partial \varphi}{\partial S} - \mathrm{d}_3) >$$
$$+ < j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial S}), \mathbb{H}j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial S}) > \ \mathrm{d}S.$$

Poisson bracket, for any f and g functions of the variables $\{\varphi, \mathbb{Q}, \mathrm{p}, \Sigma\}$:

$$\{f, g\} = \int_0^L < \frac{\partial f}{\partial \varphi}, \frac{\partial g}{\partial \mathrm{p}} > - < \frac{\partial g}{\partial \varphi}, \frac{\partial f}{\partial \mathrm{p}} >$$
$$+ \ll \frac{\partial f}{\partial \mathbb{Q}}, \frac{\partial g}{\partial \Sigma} \gg - \ll \frac{\partial g}{\partial \mathbb{Q}}, \frac{\partial f}{\partial \Sigma} \gg \ \mathrm{d}S.$$

where

$$\Sigma = \mathbb{Q}j^{-1}\left(\mathbb{J}j(\mathbb{Q}^{-1}\delta\mathbb{Q})\right)$$

Third formulation

$\{\varphi, \mathrm{p}, \mathbb{Q}, \boldsymbol{\sigma}\}$

$$H(\varphi, \mathrm{p}, \mathbb{Q}, \boldsymbol{\sigma}) = \frac{1}{2}\int_0^L < \mathrm{p}, \mathbb{A}^{-1}\mathrm{p} > + < \boldsymbol{\sigma}, \mathbb{J}^{-1}\boldsymbol{\sigma} > + < (\frac{\partial \varphi}{\partial S} - \mathrm{d}_3), \mathbb{G}(\frac{\partial \varphi}{\partial S} - \mathrm{d}_3) > + < j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial S}), \mathbb{H}j(\mathbb{Q}^{-1}\frac{\partial \mathbb{Q}}{\partial S}) > \,\mathrm{d}S$$

Poisson bracket, for any f and g functions of the variables $\{\varphi, \mathrm{p}, \mathbb{Q}, \boldsymbol{\sigma}\}$:

$$\{f, g\} = \int_0^L < \frac{\partial f}{\partial \varphi}, \frac{\partial g}{\partial \mathrm{p}} > - < \frac{\partial g}{\partial \varphi}, \frac{\partial f}{\partial \mathrm{p}} > + \ll \frac{\partial f}{\partial \mathbb{Q}}, \mathbb{Q}j^{-1}(\frac{\partial g}{\partial \boldsymbol{\sigma}}) \gg - \ll \frac{\partial g}{\partial \mathbb{Q}}, \mathbb{Q}j^{-1}(\frac{\partial f}{\partial \boldsymbol{\sigma}}) \gg \mathrm{d}S$$

Conclusion

- **Le modèle de fil existe**, il y'en a même deux (avec ou sans cisallement). Il suffit de le considérer comme une dégénérescence d'une variété matérielle unidimensionnelle particulière : un fibré des repères.
- Un fil est un **milieu microstructuré**
- La dégénérescence induit une **perte de structure** du problème...
- Cette étude n'est **qu'un début**.

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- Un fil est un **milieu microstructuré**
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La vie ne tient qu'à un fil !